SOLVING OF PORTFOLIO OPTIMIZATION PROBLEMS WITH “MATHEMATICA”

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ABSTRACT:
Optimization models play an increasingly role in financial decisions. Portfolio optimization problems are based on mean-variance models for returns and for risk-neutral density estimation. The mathematical portfolio optimization problems are the quadratic or linear parametrical programming sometimes with integer variables. This paper analyzes the mathematical models and optimization techniques for some classes of portfolio optimization problems by using the computing system “Mathematica”.

KEYWORDS: Markowitz, portfolio optimization, absolute deviation, portfolio diversification, efficient frontier, Sharpe ratio, minimax model.

1. INTRODUCTION

Conception of an optimal portfolio of assets was first time mentioned by Louis Bacheliers in his doctoral thesis which was defended in 1900 in Paris. Unfortunately, this thesis exactly like the theory of optimization created by L. Kantorovich and T. Kupmans the Nobel Prize winners in economy were less common among financial managers. They managed to use primary skills of actuarial mathematics, elementary concepts of share fare value (price). The modern portfolio theory was firstly reviewed in the work written by Markowitz [5] and Sharpe [7], who were awarded Nobel Prize in Economics in 1990. This theory is seems to be of high importance. If you make an inquiry about "portfolio theory" and "portfolio optimization" using the search engine Google.com you will be given about 13, 5 mln links for the first one and about 2, 2 mln links for the second one.

However, the demand for such techniques among Belarusian bankers was earlier small. Now the situation has changed: a small margin of interest makes banks manage assets using modern optimizing methods.

As a rule the direct methods to obtain the optimal risk portfolio are complicated [3]. Therefore the modern tools for solving portfolio optimization problems are very significant, as far as allows to simplify and to reduce calculations. This paper considers using computing system “Mathematica” [10] for solving some classes of portfolio optimization problems.

2. THE STANDARD MARKOWITZ PORTFOLIO MODEL AND IT’S APPROACHES

Let's suppose that investor has the possibility to choose from the variety of different financial assets like securities, bonds and investment projects. The main point is to define investment portfolio $x = (x_1, \ldots, x_n)$, where x_j is proportion of the asset. Then the budget constraint is

$$\sum_{j=1}^{n} x_j = 1,\ x_j \geq 0,\ j = \overline{1,n}. \tag{1}$$

It is valuable to say, that absolute weightings of assets could be included in the Markowitz. For instance, by K we denote the investor's initial capital. Then the budget constraint (1) might be replaced for:

$$\sum_{j=1}^{n} K_j \bar{x}_j = K,\ x_j \geq 0,\ j = \overline{1,n}, \tag{1.1}$$

where K_j is the price of asset j. If all assets are infinitely divisible replaced variables

$$x_j = \frac{K_j \bar{x}_j}{K},$$

we get budget constraint (1).

Markowitz portfolio model [6] assumes to use two criterions: portfolio expected return and portfolio volatility (measure of risk adjusted). Important to add that theory uses the historical parameter, volatility, as a proxy for risk, while return is an expectation on the future.

The return $R(x)$ of the portfolio x is the component-weighted expected the return R_i of the constituent assets. The expected return of an asset is a probability-weighted average of the return in all scenarios. Calling p_t the probability of scenario t and r_{jt} the return in scenario t, we may write the expected return as

$$r_j = E(R_j) = \sum_{t=1}^{T} p_t r_{jt} .$$

It's assumed that all scenario t (historical) are equal probability in the future, then $p_t = 1/T$ and $r_j = \sum_{t=1}^{T} r_{jt} / T$ (see example 1).

The function of the expected return of the portfolio is needed to be maximized:

$$r(x) = E(R(x)) = \sum_{j=1}^{n} x_j r_j \to \max . \tag{2}$$

If we suppose that $r_1 \geq \ldots \geq r_n$ then optimal solution of the problem (1), (2) is $x_{opt} = (1,0,\ldots,0)$, i.e. all capital should invest in the most profitable asset (greedy solution). Clearly, it is very risky. That is why investors add (upper bound constraint) $x_i \leq u_i$, $j = \overline{1,n}$ to budget constraints. In this case greedy solution has following form

$$x_{opt} = \left(u_1, \ldots, u_k, \left(1 - \sum_{j=1}^{k} u_j \right), 0, \ldots, 0 \right),$$

where $\sum_{j=1}^{k} u_j \leq 1$ and $\sum_{j=1}^{k+1} u_j \geq 1$ and stays optimal. It is possible further to add constraints for diversification of risks. However, Markowitz proposed other approach.

Some authors use fuzzy numbers to represent the future return of assets that approximated as fuzzy numbers the expected return and risk are evaluated by interval-valued means [3].

One of the best-known measures of risk is standard deviation of expected returns. Let's σ_{ii} is covariance of the returns i and j, i.e. $\sigma_{ij} = \frac{1}{T} \sum_{t=1}^{T} (r_{it} - r_i)(r_{jt} - r_j)$.

Markowitz derived the general formula for the standard deviation of the portfolio (risk of the portfolio) as follows:

$$\sigma(x) = \sqrt{E(R(x) - r(x))^2} = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \sigma_{ij} x_j \to \min . \tag{3}$$

The variance of all asset's returns is the expected value of the squared deviations from the expected return:

$$\sigma^2 = \sum_{t=1}^{T} p_t (r_t - E(r))^2 .$$

Remark that the covariance matrix $\sigma = (\sigma_{ij})_{n \times n}$ is positively semi-definite and consequently $\sigma(x)$ and $\sigma^2(x)$ are convex functions. That is why standard Markowitz portfolio model (1) – (3) is bi-criteria optimization problem with linear (2) and convex quadratic (3) objective functions.

In some occasions standard deviation could be substituted for k-order target risk:

$$\sigma(x) = E\left[(R(x) - r(x))^k \right]^{1/k} .$$

Let’s apply Markowitz's model to the problem of the optimization portfolio of blue chips, Hi-Tech corporation's shares, real estate and Treasure bonds. The annual times series for the return are given below for each asset between six years.

Example1. Portfolio problem with four assets.

Average annual percentage

$j \setminus t$		r_{jt}						$r_j = E(R_j)$
		1	2	3	4	5	6	
Blue chips	x_1	18,24	12,12	15,23	5,26	2,62	10,42	10,6483
Hi-Tech shares	x_2	12,24	19,16	35,07	23,46	-10,62	-7,43	11,98
Real estate Market	x_3	8,23'	8,96	8,35	9,16	8,05	7,29	8,34
Treasury Bonds	x_4	8,12	8,26	8,34	9,01	9,11	8,95	8,6317

Average annual percentage r_{jt} is specified

$$r_{jt} = \frac{P_{jt+1} - P_{jt}}{P_{jt}},$$

where P_{jt} is asset price j at instant time t.

The return and covariance matrixes can be easily find in the "Mathematica" system by using build-in functions Mean and Covariance. The covariance matrix is:

$$\sigma = \begin{pmatrix} 29.0552 & 40.3909 & -0.2879 & -1.9532 \\ 40.3909 & 267.344 & 6.8337 & -3.6970 \\ -0.2879 & 6.8337 & 0.3759 & -0.0566 \\ -1.9532 & -3.6970 & -0.0566 & 0.1597 \end{pmatrix}.$$

The *first* approach leads to the task of minimizing the variance of the portfolio (1) return given a lower bound on the expected portfolio return

$$r(x) \geq k, \tag{4.1}$$

i.e. under all possible portfolios x, consider only those which satisfy the constraints, in particular those which return at least an expected return of k. Then among those portfolios determine the one with the smallest return variance. Problem (1), (3), (4.1) is quadratic optimization problem with a positive semi-definite objective matrix σ:

$$\sigma^2(x) = 29.0552\,x_1^2 + 80.7818\,x_1\,x_2 - 0.5758\,x_1\,x_3 - 3.9064\,x_1\,x_4 +$$
$$+267.344\,x_2^2 + 13.6677\,x_2\,x_3 + 7.3940\,x_2\,x_4 + 0.3759\,x_3^2 - 0.1133\,x_3\,x_4 + 0.1597\,x_4^2 \rightarrow \min,$$
$$10{,}6483\,x_1 + 11{,}98\,x_2 + 8{,}34\,x_3 + 8{,}6317\,x_4 \geq k,$$
$$x_1 + x_2 + x_3 + x_4 = 1,\ x_j \geq 0,\ j = \overline{1,4}.$$

This problem can be solved by using standard quadratic programming algorithms or in a very efficient way by using the computing system "Mathematica" and it's build-in function Minimize. Setting in the problem (1), (3), (4.1) for portfolio optimization and solving it for guaranteed return k =10.7%, we get the optimal portfolio (x_1 =0.9523, x_2 = 0.0437, x_3 =0, x_4 =0.0040) with risk $\sigma(x)$= 5.4959% (one of the corner portfolio).

The *second* approach we consider the task of maximizing the mean of the portfolio return $r(x)$ under a given upper bound k for the variance $\sigma(x)$:

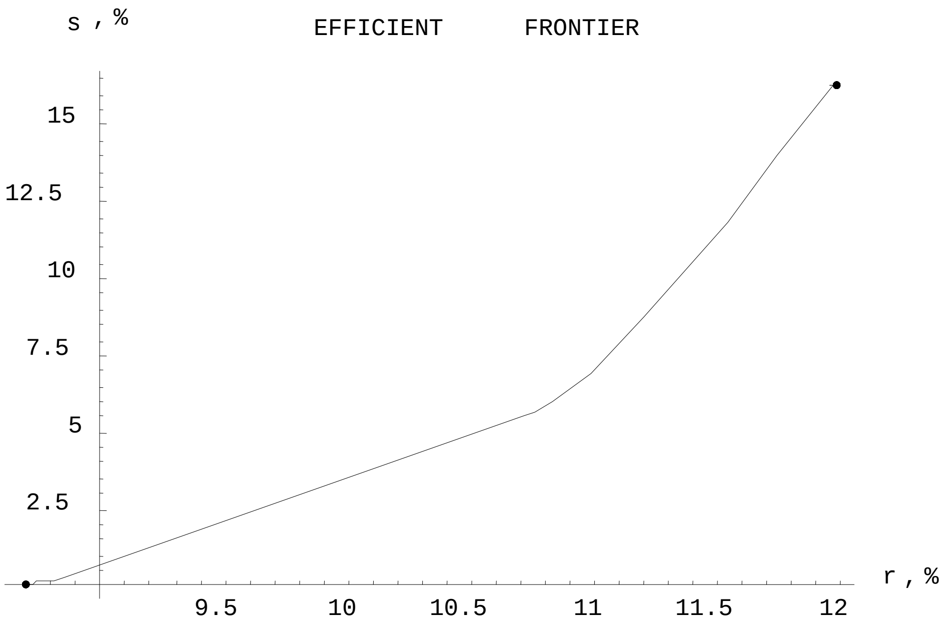
$$\sigma(x) \leq k. \tag{4.2}$$

Problem (1), (2), (4.2) is a linear parametric programming with an additional convex quadratic constraint (4.2) and parameter k.

This problem can be also efficiently solved by using the "Mathematica" system and it's build-in function Maximize. Setting in the problem (1), (3), (4.2) for portfolio optimization and solving it for as example k =1%, we get the optimal portfolio (x_1=0.2189, x_2 = 0.0114, x_3 =0, x_4 =0.7697) with return $r(x)$ = 9.1103%.

A portfolio x is efficient (Pareto optimal) if and only if no other feasible portfolio that improves at least one of the two optimization criteria without worsening the other. An efficient portfolio is the portfolio of risky assets that gives the lowest variance of return of all portfolios having the same expected return. Alternatively we may say that an efficient portfolio has the highest expected return of all portfolios having the same variance. The efficient frontier sur-plane (r,σ) is the image $(r(x),\sigma(x))$ of all efficient portfolios x. Let's plot the efficient frontier by using the build-in function ParametricPlot in "Mathematica" system:

Fig.1. The efficient frontier



While choosing an efficient portfolio we could apply for weighting objective function approach. The *third* approach is based on using the Carlin theorem of coincidence Pareto-optimal solutions in (1) – (3) in optimal solutions in the one-criterion parametric optimization with parameter k:

$$kr(x)-(1-k)\sigma(x)\to \max . \qquad (4)$$

Here the parameter k $(0\le k\le 1)$ shows investor's risk. This problem can be also easily solved by using build-in function Maximize in the system "Mathematica".

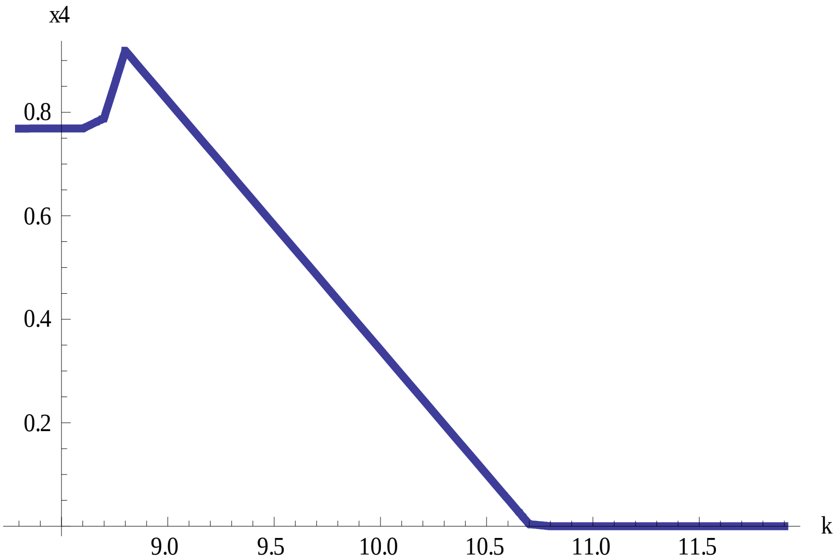
The lower k =0 the less risk we apply for the model, investor is more conservative. Minimal risk is 0.0884% with portfolio (x_1= 0.0537, x_2= 0, x_3 = 0.1776, x_4 = 0.7687) and return 8.687% (another corner portfolio).

If k =1 investor must accept risk in order to receive higher returns. Maximal risk is 16.3507% with portfolio (x_1= 0, x_2 = 1, x_3 = 0, x_4 = 0) and return 11.98%.

This algorithm for parametric quadratic programming solves the problem (1), (4) for all k in the interval $[0;1]$. Starting from one point on the efficient portfolio the algorithm computes a sequence of so called corner portfolios $x_{opt}=(x_{opt\,1},\ldots,x_{opt\,m})$. These corner portfolios define all efficient portfolio are convex combinations of the two adjacent corner portfolios: if x'_{opt} and x''_{opt} are adjacent corner portfolios with expected returns $r(x'_{opt})$ and $r(x''_{opt})$, $r(x'_{opt})\le r(x''_{opt})$ then for every $r(x_{opt})=\lambda\cdot r(x'_{opt})+(1-\lambda)\cdot r(x''_{opt})$ the efficient portfolio x_{opt} is calculated as $x_{opt}=\lambda x'_{opt}+(1-\lambda)x''_{opt}$, $0\le\lambda\le 1$.

For instance, find corner portfolios for Treasury bonds (x_4) with the portfolio return $k\in[8.5;\;\;11.9]$ by using build-in function Evaluate in the "Mathematica" system:

Fig.2. The corner portfolios for Treasury bonds



Corner portfolios for other assets can be find by the same way. There are three corner portfolios: for returns k_1=8.687%, k_2=8.8% and k_3=10.7%. Solving the portfolio optimization problem for return k=8.8%, get the optimal portfolio (x_1=0.0757, x_2= 0.0051, x_3=0, x_4=0.9192) with risk $\sigma(x)$= 0.1819% (the last corner portfolio).

The efficient portfolio x_{opt} is calculated as

$$x_{opt} = \lambda_1 x_{opt1} + \lambda_2 x_{opt2} + \lambda_3 x_{opt3},$$

where x_{opt1} (x_1= 0.0537, x_2 = 0, x_3 = 0.1776, x_4 = 0.7687), x_{opt2} (x_1=0.0757, x_2 = 0.0051, x_3 =0, x_4=0.9192), x_{opt3} (x_1=0.9523, x_2 = 0.0437, x_3 =0, x_4=0.0040) and $\lambda_1 + \lambda_2 + \lambda_3 = 1$, $0 \le \lambda_j \le 1$.

3. MODEL WITH RISK-FREE ASSET

Risk-free asset hypothetically corresponds to be short-term government securities. Conditionally it is assumed that the variation of the government securities return r_0 is equal zero. Considering the following Tobin model [9] for portfolio $x = (x_0, x_1, \ldots, x_n)$ with risk free asset x_0:

$$x_0 + \sum_{j=1}^{n} x_j = 1, \; x_j \ge 0, \; j = \overline{0, n}. \tag{1.2}$$

$$r(x_0, x) = r_0 x_0 + r(x) = r_0 x_0 + \sum_{j=1}^{n} x_j r_j \to \max. \tag{2.1}$$

$$\sigma(x_0, x) = \sqrt{x_0^2 \sigma_0^2 + x_p^2 \sigma_p^2 + 2 x_0 x_p \sigma_{0p}} = \sqrt{x_p^2 \sigma_p^2} = x_p \sigma_p = \sigma(x). \tag{3.1}$$

Obviously, the expected rates of return on all risky assets are not less asset, i.e. $r_j \ge r_0$.

If we take some definite efficient portfolio, we could figure all portfolios with risk free assets on CML (Capital Market Line):

$$E(R_C) = r_0 + \sigma_C \frac{E(r_m) - r_0}{\sigma_m},$$

where r_m is return of the market portfolio (depending on the market index and its risk is σ_m).

It is interesting to note, if someone has the possibility to choose not only between the given risk portfolio and risk-free assets but also to choose a structure of the risk portfolio then there exists the unique optimal solution (x_1=0.0570312, x_2=-0.00594004, x_3=0.265938, x_4=0.682971), not depended on investor's risk (solving by the "Mathematica" system).

4. SHARPE MODEL WITH FRACTIONAL CRITERIA

The main content of this model is replacement of the bi-criterion model (1), (2), (3) for the one-criterion model with budget constraint (1) and linear-fractional objective function [8]:

$$\sum_{j=1}^{n} r_j x_j \Bigg/ \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} x_i \sigma_{ij} x_j} \to \max .$$

In [4] describes a direct method to obtain the optimal risky portfolio by constructing a convex quadratic programming problem equivalent to Sharpe-ratio. In that form, this problem is not easy to solve. But the "Mathematica" system easily does it by using only one build-in function Maximise. The unique optimal portfolio is (x_1 = 0.0537, x_2 = 0, x_3 = 0.1776, x_4 = 0.7687) with risk 0.0883% and return 8.687% (the corner portfolio with minimal risk).

5. CONCLUSIONS

The expected return and the risk measured by the variance are the two main characteristics of an optimal portfolio. The optimal portfolio is desirable (the target portfolio). The real portfolio of assets can not be done by human intuition alone and some other characteristics [1]: closeness to the target portfolio; exposure to different economic sectors close to that of the target portfolio; a small number of names; a small number of transactions; high liquidity; low transaction costs.

The mathematical problem can be formulated in many ways but the principal problems can be summarized as follows: bicriterial convex quadratic optimization with simple budget constraints; linear optimization with simple polymatroidal budget and risk diversification constraints; convex quadratic or linear bicreterial optimization with integer (mixed integer variables) [3].

All models are easily and visually solved by using the "Mathematica" system. That allows to see the optimal variant of capital investments among valid range of solutions.

6. REFERENCES

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