Interpolation of rectangular grids using deformation of curves

Pobegailo A. P.
Assistant Professor, Faculty of Applied Mathematics and Informatics, Belarus State University, 220030, Independency avenu. 4, Minsk, Belarus,
pobegailo@bsu.by

Abstract: *The paper presents an approach to interpolation of rectangular grids which is based on deformation of curves which form borders of the patch. The deformation is fulfilled by means of polynomials which satisfy special boundary conditions. Using this approach the problem of determining conditions which ensure continuity of an interpolating surface at knot points is avoided. Instead curves for deformation have to be chosen.*

Keywords**:** Bernstein polynomials, curve deformation, Bézier patches, surface interpolation.

1. MOTIVATION

In image processing and geometric modeling it is often necessary to reconstruct curves or surfaces using grids of knot points. These problems are usually solved by construction of spline curves or surfaces interpolating or approximating these grids [1]. Classical approaches to construct spline curves or surfaces on grids needs determination of conditions at knot points which ensure necessary continuity of spline curves and surfaces. These conditions usually describe values of partial derivatives which segments of spline curves or surfaces must have at knot points. But it is not easy to determine values of the partial derivatives which ensure proper shape and other geometric characteristics of spline curves or surfaces. Usually to solve this problem some empirical assumptions or optimization criteria are used. To avoid the difficulties Bézier curves and surfaces are still widely used to solve the problem because in this case continuity of spline curves and surfaces is controlled by frame points. That is why the points are also called control points.

The paper presents a different approach to interpolation of rectangular grids by spline curves and surfaces. The approach is based on smooth deformation of curves which form border of the patch. The deformation is fulfilled by means of polynomials which satisfy special boundary conditions. Using this approach the problem of determining conditions which ensure continuity of an interpolating surface at knot points is avoided. Instead curves for deformation have to be chosen.

The proposed technique was developed by the author and inspired by early works on interpolation by means of deformation of circular arcs [2]. The proposed polynomials were firstly defined by the author at the paper [3]. But there they were defined by boundary conditions. In this paper the analytical expressions for these polynomials are introduced by means of Bernstein polynomials. Other applications of these polynomials are presented in the paper [4].

The paper is structured as follows. Firstly polynomials for deformation of curves are introduced. Then it is shown how these polynomials can be used for construction of spline curves. Next an approach to construction spline surfaces interpolating rectangular grids is presented.

2. APPROXIMATION OF A JUMP FUNCTION

In order to define approximating polynomials consider a jump function

$$\delta(u) = \begin{cases} 0, & 0 \le u < 1/2 \\ 1/2, & u = 1/2 \\ 1, & 1/2 < u \le 1 \end{cases},$$

It can be seen that the jump function $\delta(u)$ is infinitely smooth at the boundaries but has a discontinuity at the middle of the interval $[0, 1]$. In order to avoid the discontinuity the jump function $\delta(u)$ can be approximated by means of Bernstein polynomials. To explain construction of these polynomials introduce the following sequences of knots:

$$(\underbrace{0,0,\ldots,0}_{k+1},\underbrace{1,1,\ldots,1}_{k+1}),$$

for any $k \in N$. Then using Bernstein polynomials

$$b_{n,m}(u) = C_n^m (1-u)^{n-m} u^m,$$

where $0 \le u \le 1$, define the following polynomials:

$$w_k(u) = 0 \cdot b_{2k+1,\,0}(u) + \cdots + 0 \cdot b_{2k+1,\,k} +$$

$$+1 \cdot b_{2k+1,\,k+1}(u) + \cdots + 1 \cdot b_{2k+1,\,2k+1}$$

Deleting all zero terms yields the following definition of the polynomials:

$$w_k(u) = \sum_{i=k+1}^{2k+1} b_{2k+1,\,i}(u),$$

where $0 \le u \le 1$.

It follows from this definition that the introduced polynomials $w_k(u)$ satisfy the following conditions:

$$w_k(0) = 0,\ w_k(1) = 1, \tag{1}$$

$$w_k^{(l)}(0) = w_k^{(l)}(1) = 0 \tag{2}$$

for any $l \in \{1, \ldots, k\}$.

Note some other properties of the polynomials $w_k(u)$ which will be used further. The polynomials $w_k(u)$ satisfy the condition

$$w_k(u) + w_k(1-u) = 1 \tag{3}$$

for any $u \in [0,1]$. This condition follows from the following property of Bernstein polynomials:

$$\sum_{m=0}^{n} b_{n,m}(u) = 1$$

for any $u \in [0,1]$. It follows from Equation (3) that the $w_k(u)$ polynomials are symmetric relative to the point $1/2$ that is

$$w_k(1/2+v) + w_k(1/2-v) = 1 .$$

Now show that a shape the polynomials $w_k(u)$ while $k \to \infty$ infinitely close approaches to the shape of the jump function $\delta(u)$. Analytically this means that the following equation:

$$\lim_{k \to \infty} \int_0^{1/2} w_k(u) du = 0 \qquad (4)$$

is fulfilled. This is true because the polynomials $w_k(u)$ can be represented using the power basis polynomials u^{k+1}, u^{k+2}, …, u^{2k+1}. Therefore integration of these basis polynomials results the polynomials u^{k+2}, u^{k+3}, …, u^{2k+2}. To prove Equation (4) it is sufficient to substitute these polynomials into the limit instead of the integral.

It follows from these properties that the introduced polynomials $w_k(u)$ approximate the jump function $\delta(u)$.

Figure 1 illustrates profiles of the introduced polynomials.

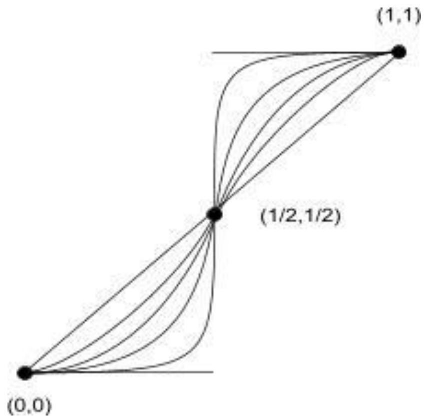

Fig. 1 – Profiles of the approximating polynomials

3. POLYNOMIAL DEFORMATION OF CURVES

Show how the introduced polynomials $w_k(u)$ can be used for deformation of parametric curves in a linear space. For this purpose consider two parametric curves $\boldsymbol{p}_1(u)$ and $\boldsymbol{p}_2(u)$, $u \in [0,1]$, which satisfy the following conditions:

$$\boldsymbol{p}_1(0) = \boldsymbol{p}_2(0) , \ \boldsymbol{p}_1(1) = \boldsymbol{p}_2(1) .$$

That is the parametric curves have the same boundary points. The purpose is to construct a parametric curve $\boldsymbol{p}(u)$, $u \in [0,1]$, which satisfies the following boundary conditions:

$$\boldsymbol{p}(0) = \boldsymbol{p}_1(0) , \ \boldsymbol{p}(1) = \boldsymbol{p}_2(1) , \qquad (5)$$

$$\boldsymbol{p}^{(l)}(0) = \boldsymbol{p}_1^{(l)}(0) , \ \boldsymbol{p}^{(l)}(1) = \boldsymbol{p}_2^{(l)}(1) . \qquad (6)$$

for all $l \in \{1,2,\dots,k\}$ where $k \in N$. The parametric curve $\boldsymbol{p}(u)$ is called a deformation of the parametric curve $\boldsymbol{p}_1(u)$ into the parametric curve $\boldsymbol{p}_2(u)$ with the k degree of continuity.

Define the parametric curve $\boldsymbol{p}(u)$ as follows:

$$\boldsymbol{p}(u) = (1 - w_{k-1}(u))\boldsymbol{p}_1(u) + w_{k-1}(u)\boldsymbol{p}_2(u) , \qquad (7)$$

$u \in [0,1]$, where $w_k(u)$ are introduced polynomials which approximate a jump function. Show that the defined parametric curve satisfies the boundary Conditions (5) and (6).

It follows Equations (1) that

$$\boldsymbol{p}(0) = (1 - w_{k-1}(0))\boldsymbol{p}_1(0) + w_{k-1}(0)\boldsymbol{p}_2(0) = \boldsymbol{p}_1(0) , \qquad (8)$$

$$\boldsymbol{p}(1) = (1 - w_{k-1}(1))\boldsymbol{p}_1(1) + w_{k-1}(1)\boldsymbol{p}_2(1) = \boldsymbol{p}_2(1) . \qquad (9)$$

Therefore boundary Conditions (4) are fulfilled.

Now determine parametric derivatives of the parametric curve $\boldsymbol{p}(u)$. It is obtained that

$$\boldsymbol{p}^{(l)}(u) = \sum_{i=1}^{l} C_l^i ((1 - w_{k-1}(u))^{(l-i)} (\boldsymbol{p}_1(u))^{(i)} +$$

$$+ (w_{k-1}(u))^{(l-i)} (\boldsymbol{p}_2(u))^{(i)}) .$$

Using Equation (3) the last equation can be transformed as follows:

$$\boldsymbol{p}^{(l)}(u) = \sum_{i=1}^{l} C_l^i (w_{k-1}(1-u))^{(l-i)} (\boldsymbol{p}_1(u))^{(i)} +$$

$$+ (w_{k-1}(u))^{(l-i)} (\boldsymbol{p}_2(u))^{(i)}) .$$

It follows from this equation that boundary Conditions (6) are also fulfilled taking into account Equations (2).

Thus the parametric curve $\boldsymbol{p}(u)$ defined Equation (7) is a deformation of the parametric curve $\boldsymbol{p}_1(u)$ into the parametric curve $\boldsymbol{p}_2(u)$ with the k degree of continuity.

Show how deformation of parametric curves can be applied to construction of spline curves. For this purpose consider four knot points $\boldsymbol{p}_{i-1}$, $\boldsymbol{p}_i$, $\boldsymbol{p}_{i+1}$, $\boldsymbol{p}_{i+2}$ which have to be interpolated. Chose two parametric curves $\boldsymbol{p}_1(u)$ and $\boldsymbol{p}_2(u)$ which interpolates points $\boldsymbol{p}_{i-1}$, $\boldsymbol{p}_i$, $\boldsymbol{p}_{i+1}$ and $\boldsymbol{p}_i$, $\boldsymbol{p}_{i+1}$, $\boldsymbol{p}_{i+2}$ respectively. In general a choice of the parametric curves $\boldsymbol{p}_1(u)$ and $\boldsymbol{p}_2(u)$ depends on the problem to be solved or can be conditioned by some optimization criteria. Take only those segments of these curves which connect the knot points $\boldsymbol{p}_i$ and $\boldsymbol{p}_{i+1}$. Then a segment of interpolating curve $\boldsymbol{p}(u)$ which connects the same knot points can de determined using Equation (6).

Figure 2 illustrates the proposed approach to construction of spline curve segments interpolating the knot points.

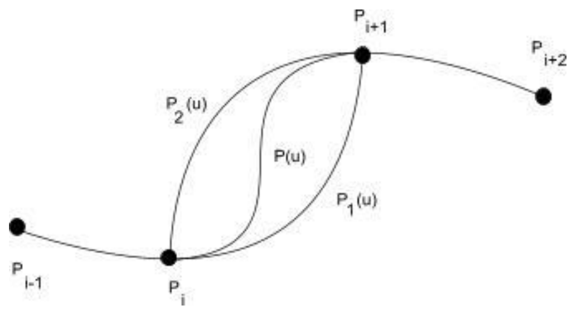

Fig. 2 – Construction of interpolating curve segments

It is obvious that to ensure C^k continuity of a parametric spline curve $\boldsymbol{p}(u)$ passing through the knot points $\boldsymbol{p}_i$ where $i \in \{0, 1, \ldots, m\}$ it is sufficient to deform consecutive parametric curves which smoothly or at least C^k continuously joined at the knot points.

It should be noted that the constructed spline curve $\boldsymbol{p}(u)$ has local control. That is changing a knot point of the spline curve forces changing shapes of only two segments which are incident to the knot point.

4. INTERPOLATION OF RECTANGULAR GRIDS

Consider a rectangular grid of points $\boldsymbol{p}_{ij}$ in a linear space where $i \in \{0, 1, \ldots, m\}$, $j \in \{0, 1, \ldots, n\}$. The problem is to construct a C^k continuous surface which interpolates points of this grid. Using approximating polynomials $w_k(u)$ Solution of this problem can be performed as follows.

Firstly mesh of parametric curves which will be used to determine borders of patches must be constructed. Figure 3 illustrates construction of this mesh. The simplest way to construct this mesh is to use parabolas in Bézier form for deformation curves.

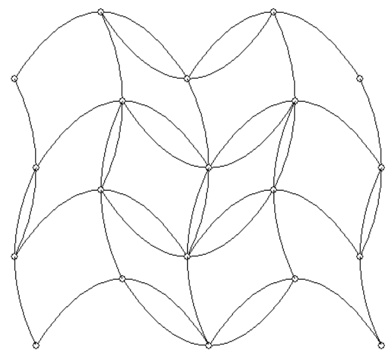

Fig. 3 – Mesh of parametric curves for deformation

Then it is necessary to determine borders of patches. The borders can be constructed using Equation (7). Thus a rectangular mesh of spline curves is constructed and segments of these spline curves are borders of patches. Figure 4 illustrates construction of the mesh.

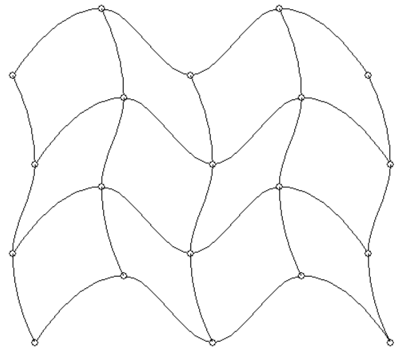

Fig. 4 – Rectangular mesh of spline curves

Now patches of surface which interpolates the rectangular grid can be constructed. A patch of the surface will be determined by means of deformation of its borders. In order to obtain an analytical expression of a patch consider for corner points $\boldsymbol{p}_{ij}$, $\boldsymbol{p}_{i+1,\,j}$, $\boldsymbol{p}_{i,\,j+1}$ and $\boldsymbol{p}_{i+1,\,j+1}$ of the patch. Let $\boldsymbol{p}_{i,\,j+1}(u)$, $\boldsymbol{p}_{i+1,\,j+1}(u)$, $\boldsymbol{q}_{i+1,\,j}(v)$, and $\boldsymbol{q}_{i+1,\,j+1}(v)$ are parametric curves which satisfy the following boundary conditions:

$$\boldsymbol{p}_{i,\,j+1}(0) = \boldsymbol{q}_{i+1,\,j}(0) = \boldsymbol{p}_{ij}, \tag{10}$$

$$\boldsymbol{p}_{i,\,j+1}(1) = \boldsymbol{q}_{i+1,\,j+1}(0) = \boldsymbol{p}_{i+1,\,j}, \tag{11}$$

$$\boldsymbol{p}_{i+1,\,j+1}(0) = \boldsymbol{q}_{i+1,\,j}(1) = \boldsymbol{p}_{i,\,j+1}, \tag{12}$$

$$\boldsymbol{p}_{i+1,\,j+1}(1) = \boldsymbol{q}_{i+1,\,j+1}(1) = \boldsymbol{p}_{i+1,\,j+1}. \tag{13}$$

Then determine a patch by means of the following expression:

$$\begin{aligned} \boldsymbol{p}(u, v) &= (1 - w_{k-1}(v))\,\boldsymbol{p}_{i,\,j+1}(u) + \\ &+ w_{k-1}(v)\,\boldsymbol{p}_{i+1,\,j+1}(u) + (1 - w_{k-1}(u))\boldsymbol{q}_{i+1,\,j}(v) + , \\ &+ w_{k-1}(u)\boldsymbol{q}_{i+1,\,j+1}(v) - \boldsymbol{r}(u, v), \end{aligned} \tag{14}$$

$u, v \in [0, 1]$, where

$$\begin{aligned} \boldsymbol{r}(u, v) &= (1 - w_{k-1}(u))(1 - w_{k-1}(v))\,\boldsymbol{p}_{ij} + \\ &+ w_{k-1}(u)(1 - w_{k-1}(v))\,\boldsymbol{p}_{i+1,\,j} + \\ &+ (1 - w_{k-1}(u))\,w_{k-1}(v)\,\boldsymbol{p}_{i,\,j+1} + \\ &+ w_{k-1}(u)\,w_{k-1}(v)\,\boldsymbol{p}_{i+1,\,j+1}. \end{aligned}$$

Show that the parametric curves $\boldsymbol{p}_{i,\,j+1}(u)$, $\boldsymbol{p}_{i+1,\,j+1}(u)$, $\boldsymbol{q}_{i+1,\,j}(v)$, and $\boldsymbol{q}_{i+1,\,j+1}(v)$ are borders of the parametric surface $\boldsymbol{p}(u, v)$. Taking into account Equations (8), (9) and (10) – (13) it is obtained that

$$\boldsymbol{p}(u, 0) = \boldsymbol{p}_{i,\,j+1}(u) +$$

$$+(1-w_{k-1}(u))\boldsymbol{q}_{i+1,\,j}(0)+$$

$$+w_{k-1}(u)\boldsymbol{q}_{i+1,\,j+1}(0)-\boldsymbol{r}(u,0)=$$

$$=\boldsymbol{p}_{i,\,j+1}(u)+(1-w_{k-1}(u))\,\boldsymbol{p}_{ij}+$$

$$+w_{k-1}(u))\,\boldsymbol{p}_{i+1,\,j}-$$

$$-(1-w_{k-1}(u))\,\boldsymbol{p}_{ij}-w_{k-1}(u)\,\boldsymbol{p}_{i+1,\,j}=$$

$$=\boldsymbol{p}_{i,\,j+1}(u)$$

Analogously it can be shown that

$$\boldsymbol{p}(u,1)=\boldsymbol{p}_{i+1,\,j+1}(u)\,,$$

$\boldsymbol{p}(0,v)=\boldsymbol{q}_{i+1,\,j}(v)\,,$	
$\boldsymbol{p}(1,v)=\boldsymbol{q}_{i+1,\,j+1}(v)\,.$	

Now prove that the interpolating surface composed of patches defined by Equation (14) is C^k continuous. Since the patches have common borders it is sufficient to prove that the surface is C^k continuous along these borders. For this purpose consider three arbitrary adjacent patches $\boldsymbol{p}_{ij}(u,v)$, $\boldsymbol{p}_{i+1,\,j}(u,v)$ and $\boldsymbol{p}_{i,\,j+1}(u,v)$ for $u,v\in[0,1]$. Determine partial derivatives along the parametric curves which are borders of these patches. It is obtained using Equations (2) and (6) that

$$\left.\frac{\partial^l \boldsymbol{p}_{ij}(u,v)}{\partial u^l}\right|_{u=1}=$$

$$=(1-w_{k-1}(v))\frac{\partial^l \boldsymbol{p}_{i+1,\,j}(u)}{\partial u^l}(1)+$$

$$+w_{k-1}(v)\frac{\partial^l \boldsymbol{p}_{i+1,\,j+1}(u)}{\partial u^l}(1)=$$

$$=(1-w_{k-1}(v))\frac{\partial^l \boldsymbol{p}_{i+2,\,j}(u)}{\partial u^l}(0)+$$

$$+w_{k-1}(v)\frac{\partial^l \boldsymbol{p}_{i+2,\,j+1}(u)}{\partial u^l}(0)=$$

$$=\left.\frac{\partial \boldsymbol{p}_{i+1,\,j}(u,v)}{\partial u^l}\right|_{u=0}$$

for all $l\in\{1,2,\ldots,k\}$. Analogously can be proven that

$$\left.\frac{\partial^l \boldsymbol{p}_{ij}(u,v)}{\partial v^l}\right|_{v=1}=\left.\frac{\partial^l \boldsymbol{p}_{i,\,j+1}(u,v)}{\partial v^l}\right|_{v=0}.$$

for all $l\in\{1,2,\ldots,k\}$.

It should be noted that the constructed spline surface $\boldsymbol{p}(u,v)$ has local control. That is changing a knot point of the spline surface forces changing shapes of only four patches which are incident to the knot point.

Figure 5 shows interpolation surface which is constructed by the proposed approach.

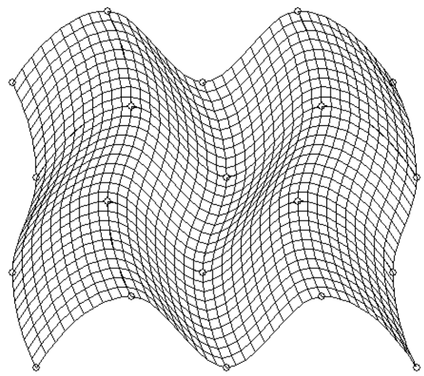

Fig. 5 – Interpolating spline surface

5. CONCLUSION

The paper presents a method to construct surfaces interpolating rectangular grids. The method is based on deformation of border curves using polynomials satisfying special boundary conditions. Analytical expressions for these polynomials by means of Bernstein polynomials are presented.

The distinguished feature of the presented approach to curve and surface interpolation is that the problem of determining boundary conditions at knot points is substituted by the problem of determining curves for deformation. The new problem can be simpler in some cases and have more visual representation.

Interpolating spline curves and surfaces constructed with the presented approach have local control. That is changing a knot point of a spline curve or surface forces changing only those shapes of segments or patches which are incident to the knot point.

6. REFERENCES

[1] G. Farin. *Curves and Surfaces for Computer-Aided Geometric Design. A Practical Guide*. Fourth Edition. Acadimic Press. San Diego, 1997. p. 447.

[2] Ю. С. Завьялов, В. А. Леус, В. А. Скороспелов. *Сплайны в инженерной геометрии*. Машиностроение. Москва, 1985. с. 224.

[3] A. P. Pobegailo. Local interpolation with weight functions for variable-smoothness curve design, *Computer-Aided Design* 23 (8) (1991). p. 579-582

[4] A. P. Pobegailo. C^n interpolation on smooth manifolds with one-parameter transformations, *Computer-Aided Design* 28 (12) (1996). p. 973-979