Dirac fermions in strong gravitational fields

Yuri N. Obukhov*

Department of Mathematics and Institute of Origins, University College London, Gower Street, London, WC1E 6BT, UK

Alexander J. Silenko†

Research Institute of Nuclear Problems, Belarusian State University, Minsk 220080, Belarus

Oleg V. Teryaev‡

Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna 141980, Russia

(Dated: file "ostrong21.tex", August 31, 2011)

Abstract

We discuss the dynamics of the Dirac fermions in the general strong gravitational and electromagnetic fields. We derive the general Hermitian Dirac Hamiltonian and transform it to the Foldy-Wouthuysen representation for the spatially isotropic metric. The quantum operator equations of motion are obtained and the semiclassical limit is analyzed. The comparison of the quantum mechanical and classical equations shows their complete agreement. The helicity dynamics in strong fields is discussed. Squaring the covariant Dirac equation explicitly shows a similarity of the interactions of electromagnetic and gravitational fields with a charged and spinning particle.

PACS numbers: 04.20.Cv; 04.62.+v; 03.65.Sq

*Electronic address: obukhov@math.ucl.ac.uk

†Electronic address: silenko@inp.bsu.by

‡Electronic address: teryaev@theor.jinr.ru

I. INTRODUCTION

The quantum and semiclassical theory of Dirac particles in the gravitational field was studied in the numerous papers (see Refs. [1] to mention but a few). The main attention was usually paid to the dynamics of fermions in the weak gravitational fields when the components of the spacetime metric tensor do not significantly deviate from the flat Minkowski metric. The analysis of the strong gravitational fields is of interest for the investigation of the processes near black holes and massive compact astrophysical objects. They may be also applied to describe the spaces of variable geometry and dimension discussed recently [2–4]. Yet another application corresponds to the fermions (quarks) motion in rotating media which may be realized [5] in heavy-ion collisions.

In this paper, we apply the methods of the Foldy-Wouthuysen (FW) transformation previously used for the analysis of the spin dynamics in weak static and stationary gravitational fields [6–9] for the case of arbitrary strong gravitational fields. We use the isotropic spatial coordinates and systematically investigate, in particular, the spin dynamics of Dirac fermions. Specifically, we derive the general result for the angular velocity of spin precession and compare this quantum mechanical result with classical predictions.

The paper is organized as follows. In Sec. II we derive the general Hermitian Dirac Hamiltonian for a charged particle in an arbitrary curved spacetime interacting with the electromagnetic fields. In this general framework, we further specialize to the dynamics of the spin and momentum of the particle in a spatially isotropic metric. Sec. III presents the FW Hamiltonian and operator equations of motion. The equations describing the dynamics of a classical spin are obtained in Sec. IV and we compare them with the corresponding quantum equations, testing the equivalence principle. The validity of the latter as well as the similarity of the spin interactions with the gravitational and with the electromagnetic fields is also manifested in the analysis of squared covariant Dirac equation performed in Sec. VI. The evolution of the helicity in the strong fields is discussed in Sec. V. The results obtained are summarized in Sec. VII.

We denote world indices by Latin letters $i, j, k, \ldots = 0, 1, 2, 3$ and reserve first Greek letters for tetrad indices, $\alpha, \beta, \ldots = 0, 1, 2, 3$. Spatial indices are denoted by Latin letters from the beginning of the alphabet, $a, b, c, \ldots = 1, 2, 3$. The separate tetrad indices are distinguished by hats. As usual, $\wedge$ and * denote the exterior product and the Hodge operator, respectively.

II. SPINNING PARTICLE IN CURVED SPACETIME

A. General spacetime metric

A classical spinning particle is characterized by its position in spacetime, $x^i(\tau)$ which is a function of the proper time τ, and by the 4-vector of spin S^α. The 4-velocity of a particle $U^\alpha = e_i^\alpha dx^i/d\tau$ is normalized by the condition $g_{\alpha\beta}U^\alpha U^\beta = c^2$ where $g_{\alpha\beta} = \mathrm{diag}(c^2, -1, -1, -1)$ is the flat Minkowski metric. To describe spinning particles in flat and curved spacetimes (as well as in arbitrary curvilinear coordinates), we use the tetrad e_i^α. When the gravitational field is absent, one can choose the Cartesian coordinates and apply the coincidence of the holonomic orthonormal frame with the natural one ($e_i^\alpha = \delta_i^\alpha$). For an arbitrary spacetime metric, $g_{\alpha\beta}e_i^\alpha e_i^\beta = g_{ij}$. Let t be the time and x^a $(a = 1, 2, 3)$ be spatial local coordinates. The general form of the line element of an arbitrary gravitational field can be given by

$$ds^2 = V^2c^2dt^2 - \delta_{\widehat{a}\widehat{b}}W^{\widehat{a}}{}_c W^{\widehat{b}}{}_d\,(dx^c - K^c cdt)\,(dx^d - K^d cdt). \tag{2.1}$$

The functions V and K^a, as well as the components of the 3×3 matrix $W^{\widehat{a}}{}_b$ may depend arbitrarily on t, x^a. The total number $1 + 3 + 9 = 13$ of these functions is larger than the number of components of the spacetime metric. However, it is obvious that the structure of (2.1) allows for a redefinition $W^{\widehat{a}}{}_b \longrightarrow L^{\widehat{a}}{}_{\widehat{c}}W^{\widehat{c}}{}_b$ with the help of an arbitrary local rotation $L^{\widehat{a}}{}_{\widehat{c}}(t, x) \in SO(3)$. Taking this freedom into account, we end up with exactly 10 independent variables that describe the general spacetime metric.

The off-diagonal metric components g_{0a} are related to the effects of rotation, as is well known. They arise when the functions K^a are nontrivial. For example, the *exact* metric of the flat spacetime seen by an accelerating and rotating observer is obtained for the special case [10]

$$V = 1 + \frac{\boldsymbol{a}\cdot\boldsymbol{r}}{c^2}, \qquad W^{\widehat{a}}{}_b = \delta^a_b, \qquad K^a = -\frac{1}{c}\,(\boldsymbol{\omega}\times\boldsymbol{r})^a, \tag{2.2}$$

where $\boldsymbol{a}$ describes acceleration of the observer and $\boldsymbol{\omega}$ is an angular velocity of a noninertial reference system. Both are independent of the spatial coordinates, but may depend arbitrarily on time t. The slowly rotating massive body produces the gravitational field of similar form; far from the source the Lense-Thirring [11] metric is described by

$$V = V(x), \qquad W^{\widehat{a}}{}_b = \delta^a_b\, W(x), \qquad K^a = \frac{1}{c}\epsilon^{abc}\omega_b(x)x_c. \tag{2.3}$$

The functions $V(x^a), W(x^a), \boldsymbol{\omega}(x^a)$ can be recovered from the Kerr metric in the isotropic coordinates in the limit of $r \to \infty$:

$$V = \left(1-\frac{\mu}{2r}\right)\left(1+\frac{\mu}{2r}\right)^{-1} - \frac{\mu a^2 - 3\mu(\boldsymbol{a}\cdot\boldsymbol{n})^2}{2r^3} + \mathcal{O}(a^2r^{-4}), \tag{2.4}$$

$$W = \left(1+\frac{\mu}{2r}\right)^{2} + \frac{\mu a^2 - 3\mu(\boldsymbol{a}\cdot\boldsymbol{n})^2}{2r^3} + \mathcal{O}(a^2r^{-4}), \tag{2.5}$$

and K^a is given by the same equation (2.3) with

$$\boldsymbol{\omega} = \frac{2\mu c}{r^3}\,\boldsymbol{a}\left(1 - \frac{3\mu}{r} + \frac{21\mu^2}{4r^2}\right) + \mathcal{O}(a^3r^{-5}). \tag{2.6}$$

Here $r := \sqrt{\boldsymbol{x}\cdot\boldsymbol{x}}$ and $\boldsymbol{n} = \boldsymbol{r}/r$. The constant vector $\boldsymbol{a} = (0,0,a)$ is the rotation parameter of the Kerr solution and $\boldsymbol{a}\cdot\boldsymbol{n} = az/r$. Also, $\mu = GM/c^2$; the total mass M and the total angular momentum $\boldsymbol{J} = Mc\boldsymbol{a}$ define the Kerr black hole uniquely. These equations are obtained from the Arnowitt-Deser-Misner form [12] of the Kerr solution performed earlier by Hergt and Schäfer [13] after dropping the terms violating the isotropy.

In the weak field approximation, we have studied the dynamics of quantum and classical spin for the cases (2.2) and (2.3) in Ref. [9].

B. Dirac fermions

In order to discuss the Dirac spinors, we need the orthonormal frames. The preferable choice [9] is the Schwinger gauge:

$$e_i^{\widehat{0}} = V\,\delta^0_i, \qquad e_i^{\widehat{a}} = W^{\widehat{a}}{}_b\left(\delta^b_i - cK^b\,\delta^0_i\right), \qquad a = 1,2,3. \tag{2.7}$$

Tetrad (2.7) is characterized by the condition $e_a^{\widehat{0}} = 0, a = 1,2,3$. The same is automatically true for the inverse tetrad:

$$e^i_{\widehat{0}} = \frac{1}{V}\left(\delta^i_0 + \delta^i_a cK^a\right), \qquad e^i_{\widehat{a}} = \delta^i_b W^b{}_{\widehat{a}}, \qquad a = 1,2,3, \tag{2.8}$$

where the inverse 3×3 matrix, $W^a{}_{\widehat{c}}W^{\widehat{c}}{}_b = \delta^a_b$, is introduced.

The covariant Dirac equation for spin-1/2 particles in gravitational and electromagnetic fields has the form

$$(i\hbar\gamma^\alpha D_\alpha - mc)\Psi = 0, \qquad \alpha = 0,1,2,3. \tag{2.9}$$

The Dirac matrices γ^α are defined in local Lorentz (tetrad) frames. The spinor covariant derivatives are given by

$$D_\alpha = e^i_\alpha D_i, \qquad D_i = \partial_i + \frac{iq}{\hbar c}\,A_i + \frac{i}{4}\sigma^{\alpha\beta}\Gamma_{i\,\alpha\beta}. \tag{2.10}$$

Here $\Gamma_i{}^{\alpha\beta} = -\Gamma_i{}^{\beta\alpha}$ are the Lorentz connection coefficients, $\sigma^{\alpha\beta} = \frac{i}{2}\left(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha\right)$. The Dirac particle is characterised by the electric charge q, and A_i is the 4-potential of the electromagnetic field.

Eqs. (2.9),(2.10) show that the gravitational and inertial effects are encoded in coframe and connection (see the relevant discussion in Refs. [14, 15] and references therein). For the general metric (2.1) with the tetrad (2.7) we find explicitly

$$\begin{aligned}\Gamma_{i\,\widehat{a}\widehat{0}} &= \frac{c^2}{V}\,W^b{}_{\widehat{a}}\,\partial_b V\,e_i{}^{\widehat{0}} - \frac{c}{V}\,\mathcal{Q}_{(\widehat{a}\widehat{b})}\,e_i{}^{\widehat{b}}, &(2.11)\\ \Gamma_{i\,\widehat{a}\widehat{b}} &= \frac{c}{V}\,\mathcal{Q}_{[\widehat{a}\widehat{b}]}\,e_i{}^{\widehat{0}} + \left(\mathcal{C}_{\widehat{a}\widehat{b}\widehat{c}} + \mathcal{C}_{\widehat{a}\widehat{c}\widehat{b}} + \mathcal{C}_{\widehat{c}\widehat{b}\widehat{a}}\right)e_i{}^{\widehat{c}}. &(2.12)\end{aligned}$$

Here

$$\begin{aligned}\mathcal{Q}_{\widehat{a}\widehat{b}} &= g_{\widehat{a}\widehat{c}}W^d{}_{\widehat{b}}\left(\frac{1}{c}\dot{W}^{\widehat{c}}{}_d + K^e\partial_e W^{\widehat{c}}{}_d + W^{\widehat{c}}{}_e\partial_d K^e\right), &(2.13)\\ \mathcal{C}_{\widehat{a}\widehat{b}}{}^{\widehat{c}} &= W^d{}_{\widehat{a}}W^e{}_{\widehat{b}}\,\partial_{[d}W^{\widehat{c}}{}_{e]} = -\mathcal{C}_{\widehat{b}\widehat{a}}{}^{\widehat{c}}, \qquad \mathcal{C}_{\widehat{a}\widehat{b}\widehat{c}} = g_{\widehat{c}\widehat{d}}\mathcal{C}_{\widehat{a}\widehat{b}}{}^{\widehat{d}}. &(2.14)\end{aligned}$$

The dot $\dot{}$ denotes the derivative with respect to the time t. As one notices, $\mathcal{C}_{\widehat{a}\widehat{b}}{}^{\widehat{c}}$ is nothing but the anholonomity object for the spatial triad $W^{\widehat{a}}{}_b$. The indices (that all run from 1 to 3) are raised and lowered with the help of the spatial part of the flat Minkowski metric, $g_{\widehat{a}\widehat{b}} = -\,\delta_{ab} = \mathrm{diag}(-1,-1,-1)$.

Dirac equation can be derived from the action

$$I = \int d^4x\,\mathcal{L}, \qquad \mathcal{L} = \sqrt{-g}\,L \tag{2.15}$$

with the Lagrangian

$$L = \frac{i\hbar}{2}\left(\overline{\Psi}\gamma^\alpha D_\alpha\Psi - D_\alpha\overline{\Psi}\gamma^\alpha\Psi\right) - mc\,\overline{\Psi}\Psi. \tag{2.16}$$

As it is well known, the naive Hamiltonian form the Schrödinger form of the Dirac equation is not Hermitian. The most straightforward way to derive a Hermitian Hamiltonian is to redefine the wave function. Substituting the tetrad and connection into the Lagrangian density, we find explicitly

$$
\begin{aligned}
I \;=\; \int dt d^3x \Bigg[& \frac{i\hbar}{2}\sqrt{-g}e_{\widehat{0}}^{0}\left(\Psi^\dagger\partial_t\Psi - \partial_t\Psi^\dagger\Psi\right) - qA_0\sqrt{-g}e_{\widehat{0}}^{0}\Psi^\dagger\Psi - mc^2\sqrt{-g}\Psi^\dagger\Psi \\
& + \frac{i\hbar}{2}\sqrt{-g}e_{\widehat{0}}^{a}\left(\Psi^\dagger\partial_a\Psi - \partial_a\Psi^\dagger\Psi\right) - \frac{q}{c}\sqrt{-g}e_{\widehat{0}}^{a}A_a\Psi^\dagger\Psi \\
& + \frac{i\hbar c}{2}\sqrt{-g}e_{\widehat{a}}^{b}\left(\Psi^\dagger\alpha^a\partial_b\Psi - \partial_b\Psi^\dagger\alpha^a\Psi\right) - q\sqrt{-g}e_{\widehat{a}}^{b}A_b\Psi^\dagger\alpha^a\Psi \\
& - \frac{\hbar}{4}\sqrt{-g}\epsilon_{\widehat{a}\widehat{b}\widehat{c}}\Gamma_{\widehat{0}}{}^{\widehat{a}\widehat{b}}\Psi^\dagger\Sigma^a\Psi + \frac{\hbar c}{4}\sqrt{-g}\epsilon_{\widehat{a}\widehat{b}}{}^{\widehat{c}}\Gamma_{\widehat{c}}{}^{\widehat{a}\widehat{b}}\Psi^\dagger\gamma_5\Psi\Bigg]. \qquad (2.17)
\end{aligned}
$$

A direct check shows that the Schrödinger equation derived from this action has a non-Hermitian Hamiltonian. This problem disappears if we define a new wave function by

$$
\psi = \left(\sqrt{-g}e_{\widehat{0}}^{0}\right)^{\frac{1}{2}}\,\Psi. \qquad (2.18)
$$

Such a non-unitary transformation may be also interpreted in the framework of the pseudo-Hermitian quantum mechanics [16, 17] (cf. also [18]). Substituting (2.18) into (2.17), and recalling (2.7) and (2.8), we find the action

$$
\begin{aligned}
I \;=\; \int dt d^3x \Bigg[& \frac{i\hbar}{2}\left(\psi^\dagger\partial_t\psi - \partial_t\psi^\dagger\psi\right) - qA_0\psi^\dagger\psi - mc^2V\psi^\dagger\psi \\
& + K^a\left\{\frac{i\hbar}{2}\left(\psi^\dagger\partial_a\psi - \partial_a\psi^\dagger\psi\right) - \frac{q}{c}A_a\psi^\dagger\psi\right\} \\
& + \mathcal{F}^b{}_a\left\{\frac{i\hbar c}{2}\left(\psi^\dagger\alpha^a\partial_b\psi - \partial_b\psi^\dagger\alpha^a\psi\right) - q\,A_b\psi^\dagger\alpha^a\psi\right\} \\
& - \frac{\hbar c}{4}\left(\Xi_a\psi^\dagger\Sigma^a\psi + \Upsilon\,\psi^\dagger\gamma_5\psi\right)\Bigg]. \qquad (2.19)
\end{aligned}
$$

Here $V = e_0^{\widehat{0}}$, $\mathcal{F}^b{}_a = \sqrt{-g}e_{\widehat{a}}^{b} = VW^b{}_{\widehat{a}}$, and

$$
\Upsilon = -V\epsilon^{\widehat{a}\widehat{b}\widehat{c}}\Gamma_{\widehat{a}\widehat{b}\widehat{c}} = -V\epsilon^{\widehat{a}\widehat{b}\widehat{c}}\mathcal{C}_{\widehat{a}\widehat{b}\widehat{c}}, \qquad \Xi_{\widehat{a}} = \frac{V}{c}\,\epsilon_{\widehat{a}\widehat{b}\widehat{c}}\,\Gamma_{\widehat{0}}{}^{\widehat{b}\widehat{c}} = \epsilon_{\widehat{a}\widehat{b}\widehat{c}}\,\mathcal{Q}^{\widehat{b}\widehat{c}}. \qquad (2.20)
$$

For the static and stationary rotating configurations, the pseudoscalar invariant vanishes ($\epsilon^{\widehat{a}\widehat{b}\widehat{c}}\mathcal{C}_{\widehat{a}\widehat{b}\widehat{c}} = 0$), and thus the corresponding term was absent in the special cases considered earlier [9]. But in general this term contributes to the Dirac Hamiltonian.

Variation of the action (2.19) with respect to the rescaled wave function yields the Dirac equation in Schrödinger form $i\hbar\dfrac{\partial\psi}{\partial t} = \mathcal{H}\psi$. The corresponding *Hermitian* Hamiltonian reads

$$
\begin{aligned}
\mathcal{H} \;=\; & \beta mc^2V + q\Phi + \frac{c}{2}\left(\pi_b\,\mathcal{F}^b{}_a\alpha^a + \alpha^a\mathcal{F}^b{}_a\pi_b\right) \\
& + \frac{c}{2}\left(\boldsymbol{K}\cdot\boldsymbol{\pi} + \boldsymbol{\pi}\cdot\boldsymbol{K}\right) + \frac{\hbar c}{4}\left(\Upsilon\gamma_5 + \boldsymbol{\Xi}\cdot\boldsymbol{\Sigma}\right). \qquad (2.21)
\end{aligned}
$$

The kinetic momentum operator $\pi_i = i\hbar\partial_i - \frac{q}{c}A_i = p_i - \frac{q}{c}A_i$ accounts of the interaction with the electromagnetic field $A_i = (\Phi, A_a)$.

Hamiltonian (2.21) is one of our central results and covers the general case of a spin-1/2 particle in an arbitrary curved spacetime. Remarkably, the form of the Hamiltonian remains the same even when the connection is non-Riemannian. If the torsion $T^\alpha = d\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta$ is nontrivial, we just have to replace

$$\Upsilon \to \Upsilon - \check{T}_{\widehat{0}}, \qquad \Xi_a \to \Xi_a + \check{T}_{\widehat{a}}, \tag{2.22}$$

where $\check{T}_\alpha\vartheta^\alpha = {}^*(\vartheta_\alpha \wedge T^\alpha)$ is the axial covector of torsion (i.e., the totally antisymmetric piece of torsion). It is worthwhile to mention that the recent discussion [19] of the Dirac fermions in an arbitrary gravitational field is very different in that the non-Hermitian Hamiltonian is used in that work, in deep contrast to the explicitly Hermitian one (2.21).

Since the metric tensor is symmetric, it can be brought to the form diagonal *in spatial coordinates*. For example, the Kerr metric in the both spherical and Boyer-Lindquist coordinates belongs to this form.

A spatially diagonal metric tensor can often be reduced to an isotropic form by an appropriate transformation of *spatial coordinates*. In spatially isotropic coordinates, $W^{\widehat{a}}{}_b = W\delta^a{}_b$ and the final form of the line element is defined by

$$ds^2 = V^2c^2dt^2 - W^2\,\delta_{ab}\,(dx^a - K^a cdt)\,(dx^b - K^b cdt). \tag{2.23}$$

Evidently, none of the two transformations changes the temporal coordinate. Since only the spatial coordinates are transformed, the transformations do not change the physical frame (particle rest frame) used for a definition of the three-component physical spin.

For isotropic metric (2.23) with the Schwinger gauge, the exact Hermitian Dirac Hamiltonian reads [9]

$$\begin{aligned}\mathcal{H} \;=\; & \beta mc^2 V + \frac{c}{2}\left[(\boldsymbol{\alpha}\cdot\boldsymbol{p})\mathcal{F} + \mathcal{F}(\boldsymbol{\alpha}\cdot\boldsymbol{p})\right] \\ & + \frac{c}{2}\left(\boldsymbol{K}\cdot\boldsymbol{p} + \boldsymbol{p}\cdot\boldsymbol{K}\right) + \frac{\hbar c}{4}\left(\boldsymbol{\nabla}\times\boldsymbol{K}\right)\cdot\boldsymbol{\Sigma},\end{aligned} \tag{2.24}$$

where

$$\mathcal{F} = V/W. \tag{2.25}$$

It covers the general case of a spin-1/2 particle in an isotropic stationary metric.

III. THE FOLDY-WOUTHUYSEN REPRESENTATION

Let us now derive the FW Hamiltonian for stationary spatially isotropic metric (2.23). We perform the FW transformation of the exact Dirac Hamiltonian (2.24) with the help of the method developed in Ref. [20]. Since the gravitational field is supposed to be strong, we do not make any approximations for the functions V, W, K^a, and the only small parameter is the Planck constant $\hbar$. In the FW Hamiltonian, we retain all the terms of the zero and first orders in $\hbar$ and the leading terms of order of $\hbar^2$ nonvanishing in the both nonrelativistic and weak field approximations. These terms describe the gravitational contact (Darwin) interaction [6]. The computations are straightforward, and the final FW Hamiltonian is given by

$$\mathcal{H}_{FW}=\mathcal{H}_{FW}^{(1)}+\mathcal{H}_{FW}^{(2)}, \tag{3.1}$$

where

$$\begin{aligned}\mathcal{H}_{FW}^{(1)}=\beta\epsilon'-\beta\frac{\hbar mc^4}{4}&\left\{\frac{1}{2\epsilon'^2+mc^2\{\epsilon',V\}},\left[\boldsymbol{\Sigma}\cdot(\boldsymbol{\Phi}\times\boldsymbol{p})-\boldsymbol{\Sigma}\cdot(\boldsymbol{p}\times\boldsymbol{\Phi})+\hbar\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}\right]\right\}\\&+\beta\frac{\hbar c^2}{16}\left\{\frac{1}{\epsilon'},\left[\boldsymbol{\Sigma}\cdot(\boldsymbol{\mathcal{G}}\times\boldsymbol{p})-\boldsymbol{\Sigma}\cdot(\boldsymbol{p}\times\boldsymbol{\mathcal{G}})+\hbar\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{G}}\right]\right\},\end{aligned} \tag{3.2}$$

$$\mathcal{H}_{FW}^{(2)}=\frac{c}{2}\left(\boldsymbol{K}\cdot\boldsymbol{p}+\boldsymbol{p}\cdot\boldsymbol{K}\right)+\frac{\hbar c}{4}\boldsymbol{\Sigma}\cdot(\boldsymbol{\nabla}\times\boldsymbol{K})-\frac{\hbar c^3}{16}\left\{\frac{1}{2\epsilon'^2+mc^2\{\epsilon',V\}},\left\{\mathcal{F}^2,\boldsymbol{\Sigma}\cdot\boldsymbol{Q}\right\}\right\}, \tag{3.3}$$

$$\epsilon'=\sqrt{m^2c^4V^2+\frac{1}{2}c^2\{\mathcal{F}^2,\boldsymbol{p}^2\}},\qquad \boldsymbol{\mathcal{G}}=\boldsymbol{\nabla}(\mathcal{F}^2),\qquad \boldsymbol{\Phi}=\mathcal{F}^2\boldsymbol{\nabla}V, \tag{3.4}$$

$$\boldsymbol{Q}=\boldsymbol{p}\times\boldsymbol{\nabla}\left(\boldsymbol{K}\cdot\boldsymbol{p}+\boldsymbol{p}\cdot\boldsymbol{K}\right)-\boldsymbol{\nabla}\left(\boldsymbol{K}\cdot\boldsymbol{p}+\boldsymbol{p}\cdot\boldsymbol{K}\right)\times\boldsymbol{p}-\boldsymbol{p}\times(\boldsymbol{p}\times(\boldsymbol{\nabla}\times\boldsymbol{K}))-((\boldsymbol{\nabla}\times\boldsymbol{K})\times\boldsymbol{p})\times\boldsymbol{p}. \tag{3.5}$$

The equivalent forms of the latter quantity

$$\begin{aligned}\boldsymbol{Q}&=\boldsymbol{p}\times(\boldsymbol{p}\times(\boldsymbol{\nabla}\times\boldsymbol{K}))+((\boldsymbol{\nabla}\times\boldsymbol{K})\times\boldsymbol{p})\times\boldsymbol{p}+2\boldsymbol{p}\times(\boldsymbol{p}\cdot\boldsymbol{\nabla})\boldsymbol{K}-2\delta^{ab}(\nabla_a\boldsymbol{K})p_b\times\boldsymbol{p},\\ \boldsymbol{Q}&=\boldsymbol{p}\times\boldsymbol{\nabla}(\boldsymbol{p}\cdot\boldsymbol{K})-\boldsymbol{\nabla}(\boldsymbol{K}\cdot\boldsymbol{p})\times\boldsymbol{p}+\boldsymbol{p}\times(\boldsymbol{p}\cdot\boldsymbol{\nabla})\boldsymbol{K}-\delta^{ab}(\nabla_a\boldsymbol{K})p_b\times\boldsymbol{p}\end{aligned} \tag{3.6}$$

may be also useful. Eqs. (3.1)–(3.6) (which in the limiting cases agree with our previous results obtained in Refs. [6, 7, 9]) belong to the principal results of this paper.

The dynamical equation for the spin is obtained from the commutator of the FW Hamiltonian with the polarization operator $\boldsymbol{\Pi}=\beta\boldsymbol{\Sigma}$ and is given by

$$\frac{d\boldsymbol{\Pi}}{dt}=\frac{i}{\hbar}[\mathcal{H}_{FW},\boldsymbol{\Pi}]=\boldsymbol{\Omega}^{(1)}\times\boldsymbol{\Sigma}+\boldsymbol{\Omega}^{(2)}\times\boldsymbol{\Pi}, \tag{3.7}$$

where $\boldsymbol{\Omega}^{(1)}$ is the operator of angular velocity of rotation of the spin in the static gravitational field,

$$\boldsymbol{\Omega}^{(1)} = -\frac{mc^4}{2}\left\{\frac{1}{2\epsilon'^2+mc^2\{\epsilon',V\}},(\boldsymbol{\Phi}\times\boldsymbol{p}-\boldsymbol{p}\times\boldsymbol{\Phi})\right\}+\frac{c^2}{8}\left\{\frac{1}{\epsilon'},(\boldsymbol{\mathcal{G}}\times\boldsymbol{p}-\boldsymbol{p}\times\boldsymbol{\mathcal{G}})\right\}, \quad (3.8)$$

and the contribution from the nondiagonal part of the metric is equal to

$$\boldsymbol{\Omega}^{(2)} = \frac{c}{2}\boldsymbol{\nabla}\times\boldsymbol{K}-\frac{c^3}{8}\left\{\frac{1}{2\epsilon'^2+mc^2\{\epsilon',V\}},\{\mathcal{F}^2,\boldsymbol{Q}\}\right\}. \quad (3.9)$$

The two different matrices appear on the right-hand side of Eq. (3.7) due to the fact that $\boldsymbol{\Omega}^{(1)}$ contains the velocity operator rather than the momentum one. Since the velocity operator is proportional to an additional β factor and is equal to $\boldsymbol{v}=\beta c\boldsymbol{p}/\epsilon$ for free particles, the operator $\boldsymbol{\Omega}^{(1)}$, when expressed in terms of $\boldsymbol{v}$, also acquires an additional β factor [9].

The corresponding semiclassical formulas describing the motion of the average spin are then explicitly given by

$$\frac{d\boldsymbol{s}}{dt} = \boldsymbol{\Omega}\times\boldsymbol{s}=(\boldsymbol{\Omega}^{(1)}+\boldsymbol{\Omega}^{(2)})\times\boldsymbol{s}, \quad (3.10)$$

$$\boldsymbol{\Omega}^{(1)} = \frac{mc^4}{\epsilon'(\epsilon'+mc^2V)}\boldsymbol{p}\times\boldsymbol{\Phi}-\frac{c^2}{2\epsilon'}\boldsymbol{p}\times\boldsymbol{\mathcal{G}}, \quad (3.11)$$

$$\boldsymbol{\Omega}^{(2)} = \frac{c}{2}\boldsymbol{\nabla}\times\boldsymbol{K}-\frac{c}{4\epsilon'(\epsilon'+mc^2V)}\mathcal{F}^2\boldsymbol{Q}, \quad (3.12)$$

where, in the semiclassical limit,

$$\epsilon'=\sqrt{m^2c^4V^2+c^2\boldsymbol{p}^2\mathcal{F}^2}, \qquad \boldsymbol{Q}=2\boldsymbol{p}\times\boldsymbol{\nabla}(\boldsymbol{p}\cdot\boldsymbol{K})+2\boldsymbol{p}\times(\boldsymbol{p}\cdot\boldsymbol{\nabla})\boldsymbol{K}. \quad (3.13)$$

FW Hamiltonian (3.1) can now be expressed in the simpler form in terms of $\boldsymbol{\Omega}^{(1)},\boldsymbol{\Omega}^{(2)}$:

$$\begin{aligned}\mathcal{H}_{FW} = {} & \beta\epsilon'+\frac{c}{2}(\boldsymbol{K}\cdot\boldsymbol{p}+\boldsymbol{p}\cdot\boldsymbol{K})+\frac{\hbar}{2}\boldsymbol{\Pi}\cdot\boldsymbol{\Omega}^{(1)}+\frac{\hbar}{2}\boldsymbol{\Sigma}\cdot\boldsymbol{\Omega}^{(2)}\\ & -\beta\frac{\hbar^2mc^4}{4}\left\{\frac{1}{2\epsilon'^2+mc^2\{\epsilon',V\}},\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}\right\}+\beta\frac{\hbar^2c^2}{16}\left\{\frac{1}{\epsilon'},\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{G}}\right\}.\end{aligned} \quad (3.14)$$

It is instructive to compare the classical and quantum Hamiltonians of a spinning particle. In order to do this, one can start from the classical Hamiltonian of a spinless particle [21]:

$$\mathcal{H}_{class}=\left(\frac{m^2c^2-\tilde{g}^{ab}\pi_a\pi_b}{g^{00}}\right)^{1/2}+\frac{g^{0a}\pi_a}{g^{00}}+qA_0, \qquad \tilde{g}^{ab}=g^{ab}-\frac{g^{0a}g^{0b}}{g^{00}}. \quad (3.15)$$

Substituting the components of the general metric (2.1), we recast this into

$$\mathcal{H}_{class}=\sqrt{m^2c^4V^2+c^2\delta^{cd}\mathcal{F}^a{}_c\,\mathcal{F}^b{}_d\,\pi_a\,\pi_b}+c\boldsymbol{K}\cdot\boldsymbol{\pi}+q\Phi. \quad (3.16)$$

In order to take into account the spin correctly, this Hamiltonian should be completed by the interaction term $\boldsymbol{s}\cdot\boldsymbol{\Omega}$, along the lines of the general discussion of the Ref. [22]:

$$\mathcal{H}_{class} = \sqrt{m^2c^4V^2 + c^2\delta^{cd}\mathcal{F}^a{}_c\,\mathcal{F}^b{}_d\,\pi_a\,\pi_b} + c\boldsymbol{K}\cdot\boldsymbol{\pi} + q\Phi + \boldsymbol{s}\cdot\boldsymbol{\Omega}. \tag{3.17}$$

In the general case, $\boldsymbol{\Omega}$ includes the both electromagnetic and gravitational contributions.

For the case of the stationary spatially isotropic metric (2.23), we obtain the resulting classical Hamiltonian (equal just to cp_0)

$$\mathcal{H}_{class} = \sqrt{m^2c^4V^2 + c^2\mathcal{F}^2\boldsymbol{p}^2} + c\boldsymbol{K}\cdot\boldsymbol{p} + \boldsymbol{s}\cdot\boldsymbol{\Omega}, \tag{3.18}$$

where terms dependent on the electromagnetic fields are omitted. It is satisfactory to notice that the quantum Hamiltonians (3.1), and (3.14) agree completely with the classical Hamiltonian (3.18).

The similarity of the quantum and the classical Hamiltonians naturally leads to the similarity of the quantum and classical equations of motion which we are going to discuss now.

IV. CLASSICAL DYNAMICS OF SPINNING PARTICLES

The dynamics of spinning particles in strong gravitational fields is described by the generally covariant Mathisson-Papapetrou [23, 24] theory. A different approach was developed by Khriplovich and Pomeransky [22] for the noncovariant 3-dimensional spin defined in the particle rest frame. In general, the analysis of the Mathisson-Papapetrou equations is a difficult task and various approximation schemes were developed for their solution. By neglecting the second order spin effects, the Mathisson-Papapetrou system is reduced to [25]

$$\frac{DU^\alpha}{d\tau} = f^\alpha_{\rm m} = -\,\frac{1}{2m}\,S^{\mu\nu}U^\beta R_{\mu\nu\beta}{}^\alpha, \tag{4.1}$$

$$\frac{DS^\alpha}{d\tau} = 0. \tag{4.2}$$

The Mathisson force $f^\alpha_{\rm m}$ in the right-hand side of (4.1) depends on the curvature $R_{\mu\nu\beta}{}^\alpha$ of spacetime. The physical spin $\boldsymbol{s}$ is defined in the rest frame of a particle. Taking into account that the 4-velocity is

$$U^i = \gamma(e^i_{\widehat{0}} + v^a\,e^i_{\widehat{a}}), \tag{4.3}$$

we can recast the Mathisson-Papapetrou system into the 3-vector form

$$\frac{d\boldsymbol{v}}{d\tau} = \boldsymbol{\mathcal{E}} - \frac{\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{\mathcal{E}})}{c^2} + \boldsymbol{v}\times\boldsymbol{\mathcal{B}} + \frac{1}{\gamma}\boldsymbol{f}_{\rm m}, \tag{4.4}$$

$$\frac{d\boldsymbol{s}}{d\tau} = \boldsymbol{\Omega}\times\boldsymbol{s}, \qquad \boldsymbol{\Omega} = -\boldsymbol{\mathcal{B}} + \frac{\gamma}{\gamma+1}\frac{\boldsymbol{v}\times\boldsymbol{\mathcal{E}}}{c^2}, \tag{4.5}$$

by introducing the fields $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$ [22],

$$-U^i\Gamma_{i\widehat{0}}{}^{\widehat{a}} = \mathcal{E}^a, \qquad -U^i\Gamma_{i\widehat{b}}{}^{\widehat{a}} = \epsilon^a{}_{bc}\mathcal{B}^c. \tag{4.6}$$

This represents a clear analogy between the gravitational and electromagnetic fields that makes it possible to speak of the gravitoelectromagnetic type effects. One should not confuse the objects (4.6) with the usual gravitoelectromagnetic fields [26, 27] that are defined in the weak-field approximation and that satisfy the Maxwell-like dynamical equations which are derived from Einstein's gravitational equation. In contrast, the fields (4.6) arise in a purely kinematic context and are defined for arbitrarily strong field configurations. Note also that they depend on the velocity of the particle, unlike the usual gravitoelectromagnetic fields [26, 27]. Nevertheless, their application is quite helpful because the equations of motion of particles and their spins, written in terms of $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$, look very similar to the corresponding equations of motion of charged spinning particles in the electrodynamics.

Let us calculate these fields explicitly. We start from Eqs. (2.11) and (2.12) and use the notation from Ref. [9]. Multiplying the connection coefficients (2.11)-(2.12) by the 4-velocity (4.3), we find:

$$U^i\,\Gamma_{i\widehat{a}\widehat{b}} = \frac{c\gamma}{V}\,\mathcal{Q}_{[\widehat{a}\widehat{b}]} + \gamma\left(\mathcal{C}_{\widehat{a}\widehat{b}\widehat{c}} + \mathcal{C}_{\widehat{a}\widehat{c}\widehat{b}} + \mathcal{C}_{\widehat{c}\widehat{b}\widehat{a}}\right)v^{\widehat{c}}, \tag{4.7}$$

$$U^i\,\Gamma_{i\widehat{0}\widehat{b}} = \frac{c\gamma}{V}\,\mathcal{Q}_{(\widehat{b}\widehat{c})}v^{\widehat{c}} - \frac{c^2\gamma}{V}\,W^c{}_{\widehat{b}}\,\partial_c V. \tag{4.8}$$

As a result, for the general metric (2.1), the gravitoelectric and gravitomagnetic fields (4.6) read

$$\mathcal{E}_a = \frac{\gamma}{V}\left(c\mathcal{Q}_{(\widehat{a}\widehat{b})}v^b - c^2\,W^b{}_{\widehat{a}}\,\partial_b V\right), \tag{4.9}$$

$$\mathcal{B}^a = \frac{\gamma}{V}\left(-\frac{c}{2}\,\Xi^a - \frac{1}{2}\Upsilon\,v^a + \epsilon^{abc}V\mathcal{C}_{bc}{}^d v_d\right). \tag{4.10}$$

The physical spin precesses (4.5) with the angular velocity $\boldsymbol{\Omega}$ that can also be written explicitly in terms of the connection [9, 22]

$$\Omega_{\widehat{a}} = \epsilon_{abc}\,U^i\left(\frac{1}{2}\Gamma_i{}^{\widehat{c}\widehat{b}} + \frac{\gamma}{\gamma+1}\,\Gamma_{i\widehat{0}}{}^{\widehat{b}}v^{\widehat{c}}/c^2\right). \tag{4.11}$$

As compared with the quantities $\boldsymbol{\Omega}^{(1)}$ and $\boldsymbol{\Omega}^{(2)}$ which describe the precession of the quantum spin using the coordinate time, $\Omega_{\widehat{a}}$ contains an extra factor $dt/d\tau = U^0 = \gamma/V$ in view of a different parameterization using the proper time.

Let us calculate the classical precession angular velocity in the Schwinger gauge explicitly. Substituting (4.7) and (4.8) into (4.11), we obtain the *exact classical formula* for the angular velocity of the spin precession in an arbitrary gravitational field:

$$\begin{aligned} \Omega^{\widehat{a}} &= \frac{\gamma}{V}\left(\frac{1}{2}\Upsilon\, v^{\widehat{a}} - \epsilon^{abc} V \mathcal{C}_{\widehat{b}\widehat{c}}{}^{d} v_{\widehat{d}} + \frac{\gamma}{\gamma+1}\epsilon^{abc} W^{d}{}_{\widehat{b}}\,\partial_d V v_{\widehat{c}} \right.\\ &\quad \left. + \frac{c}{2}\,\Xi^{\widehat{a}} - \frac{\gamma}{\gamma+1}\epsilon^{abc}\mathcal{Q}_{(\widehat{b}\widehat{d})}\frac{v^{\widehat{d}} v_{\widehat{c}}}{c}\right). \end{aligned} \tag{4.12}$$

The terms in the first line are linear in the 4-velocity of the particle, whereas the terms in the second line contain the even number of the velocity factors. For the special class of stationary geometries (2.23), this general formula reduces to the simpler expression:

$$\begin{aligned} \boldsymbol{\Omega} &= \frac{\gamma}{V}\left[-V\,\boldsymbol{v}\times\boldsymbol{\nabla}\left(\frac{1}{W}\right) - \frac{\gamma}{W(\gamma+1)}\,\boldsymbol{v}\times\boldsymbol{\nabla}V \right.\\ &\quad \left. + \frac{c}{2}\,\boldsymbol{\nabla}\times\boldsymbol{K} - \frac{\gamma}{2c(\gamma+1)}\left(\boldsymbol{v}\times\boldsymbol{\nabla}(\boldsymbol{v}\cdot\boldsymbol{K}) + \boldsymbol{v}\times(\boldsymbol{v}\cdot\nabla)\boldsymbol{K}\right)\right]. \end{aligned} \tag{4.13}$$

Equivalently,

$$\begin{aligned} \boldsymbol{\Omega} &= \frac{\gamma}{V}\left[\frac{1}{\mathcal{F}V(\gamma+1)}\,\boldsymbol{v}\times\boldsymbol{\Phi} - \frac{1}{2\mathcal{F}}\,\boldsymbol{v}\times\boldsymbol{\mathcal{G}} \right.\\ &\quad \left. + \frac{c}{2}\,\boldsymbol{\nabla}\times\boldsymbol{K} - \frac{\gamma}{2c(\gamma+1)}\left(\boldsymbol{v}\times\boldsymbol{\nabla}(\boldsymbol{v}\cdot\boldsymbol{K}) + \boldsymbol{v}\times(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{K}\right)\right]. \end{aligned} \tag{4.14}$$

Since

$$\epsilon' = mc^2 V\sqrt{1 + \frac{\boldsymbol{p}^2}{W^2 m^2 c^2}} = mc^2 V\gamma, \qquad \boldsymbol{p} = mW\gamma\boldsymbol{v}, \tag{4.15}$$

we conclude that the classical equation of the spin motion (4.5) agrees with the quantum equation (3.7) and with the semiclassical one (3.10). Thus, the classical and the quantum theories of the spin motion in gravity are in complete agreement. This is now verified for the arbitrary strong field configurations.

V. EVOLUTION OF HELICITY IN AN ISOTROPIC METRIC

Let us analyse the semiclassical evolution of the helicity of a particle propagating in a strong gravitation field. The helicity describes the spin orientation with respect to the direction of particle's motion. As usual, one means the orientation of the 3-component spin defined in the particle rest frame (physical spin). The particle motion is defined by the trajectory that shows how the contravariant spatial world coordinates of the particle change in time. Evidently, the particle motion can be correctly characterized by the evolution of the *contravariant* world velocity or the unit vector in its direction. As a result, one can unambiguously define the helicity as a projection of the 3-component spin (pseudo)vector *in the particle rest frame* onto the direction of the unit vector along the contravariant velocity *in the world frame*. Thus, the helicity should be defined as

$$\zeta = (\boldsymbol{s}/s) \cdot (\boldsymbol{U}/U) = (\boldsymbol{s}/s) \cdot \boldsymbol{\mathcal{V}}/\mathcal{V},$$

where $\boldsymbol{U} = \{U^1, U^2, U^3\}$ and $\boldsymbol{\mathcal{V}} = \boldsymbol{U}/U^0$. The investigation of the helicity is simplified when particle's trajectory is infinite. In this case, one can apply the fact that the vector $\boldsymbol{N} = \boldsymbol{U}/U$ coincides with the vectors $\boldsymbol{v}/v$ and $\boldsymbol{p}/p$ at the initial and final parts of such a trajectory of the particle because of the very large distance to the field source. Here $\boldsymbol{v} = \{v^{\hat{1}}, v^{\hat{2}}, v^{\hat{3}}\}$ is the velocity in the anholonomic *coframe* (4.3) and $\boldsymbol{p} = \{-p_1, -p_2, -p_3\}$ is the *covariant* momentum entering the classical and quantum Hamiltonians. For the isotropic metric under consideration, $p_a = e_a^{\hat{0}} p_{\hat{0}} + e_a^{\hat{b}} p_{\hat{b}} = W p_{\hat{a}}$ and $p_a/p = W v_{\hat{a}}/(|W|v) = v_{\hat{a}}/v$. As a result, the change of the helicity on the whole trajectory can be given by $\Delta\zeta' = \Delta\zeta$, where $\zeta' = (\boldsymbol{s}/s) \cdot \boldsymbol{n}$, $\boldsymbol{n} = \boldsymbol{v}/v = \boldsymbol{p}/p$. Evidently, $\zeta' = \cos\chi$, where χ is an angle between the $\boldsymbol{s}$ and $\boldsymbol{n}$ vectors.

In the general case, the evolution of the covariant momentum operator is defined by

$$\frac{d\boldsymbol{p}}{dt} = \frac{i}{\hbar}[\mathcal{H}_{FW}, \boldsymbol{p}]. \tag{5.1}$$

In the discussion of the semiclassical dynamics of the particle, we can neglect the small spin-dependent force which enters into the equation of motion with an additional factor $\hbar$. Thus, in the semiclassical approximation

$$\frac{dp^{-1}}{dt} = -\frac{1}{p^3}\boldsymbol{p} \cdot \frac{d\boldsymbol{p}}{dt},$$

and the dynamics of the unit vector $\boldsymbol{n}$ is given by

$$\frac{d\boldsymbol{n}}{dt} = -\frac{1}{p}\boldsymbol{n} \times \left(\boldsymbol{n} \times \frac{d\boldsymbol{p}}{dt}\right). \tag{5.2}$$

Therefore, the vector $\boldsymbol{n}$ rotates with the angular velocity

$$\tilde{\boldsymbol{\omega}} = \frac{1}{p}\boldsymbol{n} \times \frac{d\boldsymbol{p}}{dt}, \tag{5.3}$$

and one can add a quantity $\kappa\boldsymbol{n}$ with an arbitrary factor κ. Using the Eqs. (3.4), (3.14), (4.15), (5.1), (5.3), we derive

$$\tilde{\boldsymbol{\omega}} = -\frac{c^2}{\mathcal{F}V\gamma^2 v}\boldsymbol{n}\times\boldsymbol{\Phi} - \frac{v}{2\mathcal{F}}\boldsymbol{n}\times\boldsymbol{\mathcal{G}} - c\boldsymbol{n}\times\boldsymbol{\nabla}(\boldsymbol{n}\cdot\boldsymbol{K}). \tag{5.4}$$

The nabla operator does not act on $\boldsymbol{n}$.

We can add the quantity $\frac{c}{2}\boldsymbol{n}(\boldsymbol{n}\cdot(\boldsymbol{n}\times\boldsymbol{K}))$ to $\tilde{\boldsymbol{\omega}}$ and use the identity

$$2\boldsymbol{n}\times\boldsymbol{\nabla}(\boldsymbol{n}\cdot\boldsymbol{K}) = \boldsymbol{n}(\boldsymbol{n}\cdot(\boldsymbol{n}\times\boldsymbol{K})) - \boldsymbol{\nabla}\times\boldsymbol{K} + \boldsymbol{n}\times\boldsymbol{\nabla}(\boldsymbol{n}\cdot\boldsymbol{K}) + \boldsymbol{n}\times(\boldsymbol{n}\cdot\boldsymbol{\nabla})\boldsymbol{K}.$$

As a result, Eq. (5.4) is recast into

$$\tilde{\boldsymbol{\omega}} = -\frac{c^2}{\mathcal{F}V\gamma^2 v}\boldsymbol{n}\times\boldsymbol{\Phi} - \frac{v}{2\mathcal{F}}\boldsymbol{n}\times\boldsymbol{\mathcal{G}} + \frac{c}{2}\boldsymbol{\nabla}\times\boldsymbol{K} - \frac{c}{2}\left[\boldsymbol{n}\times\boldsymbol{\nabla}(\boldsymbol{n}\cdot\boldsymbol{K}) + \boldsymbol{n}\times(\boldsymbol{n}\cdot\boldsymbol{\nabla})\boldsymbol{K}\right]. \tag{5.5}$$

The angular velocity of the spin rotation defined by Eq. (3.12) can be expressed in the form similar to Eq. (4.14):

$$\begin{aligned}\boldsymbol{\Omega} \;=\; & \frac{v}{\mathcal{F}V(\gamma+1)}\boldsymbol{n}\times\boldsymbol{\Phi} - \frac{v}{2\mathcal{F}}\boldsymbol{n}\times\boldsymbol{\mathcal{G}} \\ & + \frac{c}{2}\boldsymbol{\nabla}\times\boldsymbol{K} - \frac{c(\gamma-1)}{2\gamma}\left[\boldsymbol{n}\times\boldsymbol{\nabla}(\boldsymbol{n}\cdot\boldsymbol{K}) + \boldsymbol{n}\times(\boldsymbol{n}\cdot\boldsymbol{\nabla})\boldsymbol{K}\right].\end{aligned} \tag{5.6}$$

Therefore, the vector of spin rotates with respect to the momentum direction, and the angular velocity of such rotation is

$$\boldsymbol{o} = \boldsymbol{\Omega} - \tilde{\boldsymbol{\omega}} = \frac{c^2}{\mathcal{F}V\gamma v}\boldsymbol{n}\times\boldsymbol{\Phi} + \frac{c}{2\gamma}\left[\boldsymbol{n}\times\boldsymbol{\nabla}(\boldsymbol{n}\cdot\boldsymbol{K}) + \boldsymbol{n}\times(\boldsymbol{n}\cdot\boldsymbol{\nabla})\boldsymbol{K}\right]. \tag{5.7}$$

The same result can be obtained for an arbitrary metric by the purely classical method considered in the previous section. Indeed, let us use classical equations (4.4),(4.5) and neglect the relatively small Mathisson force $\boldsymbol{f}_{\mathrm{m}}$. Since

$$\frac{d\boldsymbol{n}}{d\tau} = -\frac{1}{v}\boldsymbol{n}\times\left(\boldsymbol{n}\times\frac{d\boldsymbol{v}}{d\tau}\right), \tag{5.8}$$

Eq. (4.4) leads to the following formula:

$$\frac{d\boldsymbol{n}}{d\tau} = \boldsymbol{\omega}'\times\boldsymbol{n}, \tag{5.9}$$

where

$$\boldsymbol{\omega}' = -\boldsymbol{\mathcal{B}} + \frac{1}{v^2}\boldsymbol{v}\times\boldsymbol{\mathcal{E}}. \tag{5.10}$$

Consequently, we find that the spin vector rotates with respect to the momentum direction and the angular velocity of this rotation is

$$\boldsymbol{o}=\frac{V}{\gamma}(\boldsymbol{\Omega}-\boldsymbol{\omega}')=-V\,\frac{\boldsymbol{v}\times\boldsymbol{\mathcal{E}}}{\gamma^2v^2}. \tag{5.11}$$

The gravitomagnetic field $\boldsymbol{\mathcal{B}}$ does not influence particle's helicity [31]. This conclusion is valid for an arbitrary gravitational field when the effect of the Mathisson force on the particle motion is neglected. $\boldsymbol{\mathcal{E}}$ is proportional to γ/V. The use of the world time shows that any gravitational field does not affect the helicity of a particle with a negligible mass.

The covariant Dirac equation can be used and the FW transformation can be performed for a massless particle. However, the interpretation of the results is a serious problem. The 3-component spin is defined in the particle rest frame which cannot be introduced for such a particle.

For stationary isotropic metric (2.23), the gravitoelectric and the gravitomagnetic fields (4.9) and (4.10) reduce to

$$\mathcal{E}_a \;=\; -\frac{\gamma c^2}{V}\left(\frac{1}{W}\partial_a V+\frac{K^b\partial_b W}{W}\,\frac{v_a}{c}+\partial_{(a}K_{b)}\,\frac{v^b}{c}\right), \tag{5.12}$$

$$\mathcal{B}^a \;=\; \frac{\gamma}{V}\left(-\frac{c}{2}\,\epsilon^{abc}\partial_b K_c+\epsilon^{abc}V\,\frac{\partial_b W}{W^2}\,v_c\right). \tag{5.13}$$

This demonstrates the perfect agreement between the quantum and classical analyses, cf. Eqs. (5.7) and (5.11).

As a final remark, we should mention that the results presented in this section cannot, generally speaking, be applied for the particle moving on a finite trajectory because the replacement of ζ by ζ' is not always possible in this case.

A. General noninertial frame

A general noninertial frame is an important application of the results above. This frame is characterized by the acceleration $\boldsymbol{a}$ and the rotation $\boldsymbol{\omega}$ of an observer. The *exact* metric of the flat spacetime seen by the accelerated and rotating observer has the form (2.23) and is given by Eq. (2.2).

We can apply our general results (3.7)–(3.9), (3.14) for the metric (2.2) of the flat spacetime. For the FW Hamiltonian we then find

$$
\begin{aligned}
\mathcal{H}_{FW} &= \mathcal{H}_0 + \frac{\hbar}{2}\boldsymbol{\Pi}\cdot\boldsymbol{\Omega}^{(1)} + \frac{\hbar}{2}\boldsymbol{\Sigma}\cdot\boldsymbol{\Omega}^{(2)},\\
\mathcal{H}_0 &= \frac{\beta}{2}\left\{\left(1+\frac{\boldsymbol{a}\cdot\boldsymbol{r}}{c^2}\right), \sqrt{m^2c^4+c^2\boldsymbol{p}^2}\right\} - \boldsymbol{\omega}\cdot\boldsymbol{l},\\
\boldsymbol{\Omega}^{(1)} &= \frac{\boldsymbol{a}\times\boldsymbol{p}}{mc^2(\gamma+1)}, \qquad \boldsymbol{\Omega}^{(2)} = -\boldsymbol{\omega}, \qquad \gamma = \frac{\sqrt{m^2c^4+c^2\boldsymbol{p}^2}}{mc^2}.
\end{aligned}
\tag{5.14}
$$

Here $\boldsymbol{l} = \boldsymbol{r}\times\boldsymbol{p}$ is the angular momentum operator. There are no any terms of order of $\hbar^2$ nonvanishing in both the nonrelativistic and the weak field approximations. We also emphasize the absence of the spin-dependent Mathisson force. Let us stress that Eq. (5.14) is derived for the strong kinematical effects when the ratios $|\boldsymbol{a}\cdot\boldsymbol{r}|/c^2$ and $|\boldsymbol{\omega}\times\boldsymbol{r}|/c$ are not small. Nevertheless, Hamiltonian (5.14) coincides in the case of $\boldsymbol{\omega} = 0$ with the result [6] obtained for the special case of the accelerated frame in the weak field approximation $|\boldsymbol{a}\cdot\boldsymbol{r}|/c^2 \ll 1$. When $\boldsymbol{a} = 0$, we reproduce the exact FW Hamiltonian deduced in Ref. [7] for Dirac particles in the rotating frame. Eq. (5.14) and the corresponding Hamiltonian obtained in Ref. [7] generalize the approximate expression derived by Hehl and Ni [10].

The equivalent form of $\mathcal{H}_0$ reads

$$
\mathcal{H}_0 = \beta\left(mc^2 + m\boldsymbol{a}\cdot\boldsymbol{r}\right)\gamma - \frac{i\hbar\beta\,\boldsymbol{a}\cdot\boldsymbol{p}}{2mc^2\,\gamma} - \boldsymbol{\omega}\cdot\boldsymbol{l}. \tag{5.15}
$$

The metric under consideration is non-Minkowskian at any parts of finite and infinite trajectories of particles. Therefore, we need to use the contravariant 4-velocity U^i or the world velocity $\boldsymbol{\mathcal{V}} = c\boldsymbol{U}/U^0$. Evidently, $\boldsymbol{\mathcal{V}}/\mathcal{V} = \boldsymbol{U}/U$. The world velocity operator is given by

$$
\boldsymbol{\mathcal{V}} = \frac{d\boldsymbol{r}}{dt} = \frac{i}{\hbar}[\mathcal{H}_{FW}, \boldsymbol{r}] = \frac{\beta}{2}\left\{\left(1+\frac{\boldsymbol{a}\cdot\boldsymbol{r}}{c^2}\right), \frac{c\boldsymbol{p}}{\sqrt{m^2c^2+\boldsymbol{p}^2}}\right\} - \boldsymbol{\omega}\times\boldsymbol{r}. \tag{5.16}
$$

As a result, the semiclassical approximation for the world acceleration operator

$$
\boldsymbol{w} = \frac{d\boldsymbol{\mathcal{V}}}{dt} = \frac{i}{\hbar}[\mathcal{H}_{FW}, \boldsymbol{\mathcal{V}}]
$$

reads

$$
\boldsymbol{w} = -\boldsymbol{a}\left(1+\frac{\boldsymbol{a}\cdot\boldsymbol{r}}{c^2}\right) - \boldsymbol{\omega}\times(2\boldsymbol{\mathcal{V}}+\boldsymbol{\omega}\times\boldsymbol{r}) + \frac{(\boldsymbol{a}\cdot(2\boldsymbol{\mathcal{V}}+\boldsymbol{\omega}\times\boldsymbol{r}))}{c^2+\boldsymbol{a}\cdot\boldsymbol{r}}\left(\boldsymbol{\mathcal{V}}+\boldsymbol{\omega}\times\boldsymbol{r}\right). \tag{5.17}
$$

The exact semiclassical Eq. (5.17) agrees with the particular results obtained in Refs. [6] and [7] (which are approximate for the acceleration $\boldsymbol{a}$).

The expression for the angular velocity of rotation of the vector $\boldsymbol{N} = \boldsymbol{\mathcal{V}}/\mathcal{V}$ is similar to Eq. (5.3):

$$\boldsymbol{\omega}_{\mathcal{V}} = \frac{1}{\mathcal{V}}\,\boldsymbol{N} \times \frac{d\boldsymbol{\mathcal{V}}}{dt}. \tag{5.18}$$

Explicitly,

$$\boldsymbol{\omega}_{\mathcal{V}} = -2\boldsymbol{\omega} + \frac{\boldsymbol{N}}{\mathcal{V}} \times \left[-\boldsymbol{a}\left(1 + \frac{\boldsymbol{a}\cdot\boldsymbol{r}}{c^2}\right) - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) + \frac{(\boldsymbol{a}\cdot(2\boldsymbol{\mathcal{V}} + \boldsymbol{\omega}\times\boldsymbol{r}))}{c^2 + \boldsymbol{a}\cdot\boldsymbol{r}}(\boldsymbol{\omega}\times\boldsymbol{r})\right]. \tag{5.19}$$

The exact semiclassical expression for the angular velocity of spin rotation can be written in terms of $\boldsymbol{\mathcal{V}}$:

$$\boldsymbol{\Omega} = \frac{\boldsymbol{a}\times(\boldsymbol{\mathcal{V}} + \boldsymbol{\omega}\times\boldsymbol{r})}{c^2\left[1 + \frac{\boldsymbol{a}\cdot\boldsymbol{r}}{c^2} + \sqrt{\left(1 + \frac{\boldsymbol{a}\cdot\boldsymbol{r}}{c^2}\right)^2 - \left(\frac{\boldsymbol{\mathcal{V}}+\boldsymbol{\omega}\times\boldsymbol{r}}{c}\right)^2}\right]} - \boldsymbol{\omega}. \tag{5.20}$$

To consider the helicity evolution, it is sufficient to examine three specific cases. If $|\boldsymbol{\omega}\times\boldsymbol{r}| \ll c,\ |\boldsymbol{a}\cdot\boldsymbol{r}| \ll c^2$, one may retain only the terms linear in $\boldsymbol{\omega}, \boldsymbol{a}$:

$$\boldsymbol{o}_{\mathcal{V}} = \boldsymbol{\Omega} - \boldsymbol{\omega}_{\mathcal{V}} = \boldsymbol{\omega} + \frac{\boldsymbol{\mathcal{V}}\times\boldsymbol{a}}{\gamma\mathcal{V}^2}. \tag{5.21}$$

When $\boldsymbol{a} = 0$ (a purely rotating frame), we find

$$\boldsymbol{o}_{\mathcal{V}} = \boldsymbol{\omega} + \frac{\boldsymbol{\mathcal{V}}\times(\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}))}{\mathcal{V}^2}. \tag{5.22}$$

The presented solution for the rotating frame is exact.

When $\boldsymbol{\omega} = 0$ (an uniformly accelerated frame), we obtain

$$\boldsymbol{o}_{\mathcal{V}} = \frac{\boldsymbol{\mathcal{V}}\times\boldsymbol{a}}{\mathcal{V}^2}\left(1 + \frac{\boldsymbol{a}\cdot\boldsymbol{r}}{c^2}\right) - \frac{\boldsymbol{\mathcal{V}}\times\boldsymbol{a}}{c^2\left[1 + \frac{\boldsymbol{a}\cdot\boldsymbol{r}}{c^2} + \sqrt{\left(1 + \frac{\boldsymbol{a}\cdot\boldsymbol{r}}{c^2}\right)^2 - \frac{\mathcal{V}^2}{c^2}}\right]}. \tag{5.23}$$

As follows from the Eq. (5.16), $\mathcal{V}/c = 1 + \boldsymbol{a}\cdot\boldsymbol{r}/c^2$ for the ultrarelativistic particles with negligible masses. For such particles, we find $\boldsymbol{o}_{\mathcal{V}} = 0$. Therefore, their helicity remains unchanged in the uniformly accelerated frame.

We can conclude that the motion in the general noninertial (arbitrarily rotating and accelerating) frame leads to the change of the helicity even for the ultrarelativistic particle with a negligible mass. A similar effect takes place in a gravitational field when a particle trajectory is finite.

Another possible application of our general results is a rotating massive thin shell which metric obtained by Brill and Cohen [28] has the form (2.3),(2.23). We will analyse this elsewhere.

VI. SQUARED DIRAC EQUATION AND THE EQUIVALENCE PRINCIPLE

The electromagnetic and gravitational contributions to the covariant derivative (2.10) manifest an obvious similarity of the electromagnetic and gravitational effects. In this section we further clarify this similarity by analysing the squared Dirac equation.

The commutator of the covariant derivative (2.10) reads

$$D_iD_j - D_jD_i = \frac{iq}{\hbar c}\,F_{ij} + \frac{i}{4}\sigma_{\alpha\beta}R_{ij}{}^{\alpha\beta}. \tag{6.1}$$

Here $F_{ij} = \partial_i A_j - \partial_j A_i$ is the electromagnetic field tensor, and $R_{ij}{}^{\alpha\beta}$ is the Riemann curvature tensor.

Acting with the conjugate Dirac operator $(i\hbar\gamma^\alpha D_\alpha + mc)$ on (2.9), we find the squared Dirac equation

$$\left(-\hbar^2 g^{ij}D_iD_j - \hbar^2\gamma^{[i}\gamma^{j]}D_iD_j - m^2c^2\right)\psi = 0. \tag{6.2}$$

Substituting Eq. (6.1), we find

$$\left(-\hbar^2 g^{ij}D_iD_j - \frac{i\hbar q}{2c}\,\gamma^{[i}\gamma^{j]}F_{ij} + \frac{\hbar^2}{8}\gamma^{[i}\gamma^{j]}\gamma_{[\alpha}\gamma_{\beta]}R_{ij}{}^{\alpha\beta} - m^2c^2\right)\psi = 0. \tag{6.3}$$

Expanding the product of gamma matrices, we derive the equation

$$\left(-\hbar^2 g^{ij}D_iD_j - \frac{\hbar q}{2c}\,\sigma^{\alpha\beta}F_{\alpha\beta} + \frac{\hbar^2}{4}\,R - m^2c^2\right)\psi = 0, \tag{6.4}$$

where $F_{\alpha\beta} = e^i_\alpha e^j_\beta F_{ij}$ are the tensor-like electromagnetic field coefficients.

The special form of Eq. (6.4) for the Dirac particle in a gravitational field has been obtained in Ref. [29].

Explicit computation yields

$$-\,\hbar^2 g^{ij}D_iD_j = \pi^i\pi_i - \frac{\hbar}{4}\sigma^{\alpha\beta}\left\{\pi^i, \Gamma_{i\,\alpha\beta}\right\} + \frac{\hbar^2}{16}\sigma^{\alpha\beta}\sigma^{\mu\nu}\Gamma^i{}_{\alpha\beta}\Gamma_{i\,\mu\nu}. \tag{6.5}$$

Since

$$\sigma^{\alpha\beta}\sigma^{\mu\nu}\Gamma^i{}_{\alpha\beta}\Gamma_{i\,\mu\nu} = \frac{1}{2}\{\sigma^{\alpha\beta},\sigma^{\mu\nu}\}\Gamma^i{}_{\alpha\beta}\Gamma_{i\,\mu\nu} = 2\Gamma^i{}_{\alpha\beta}\Gamma_i{}^{\alpha\beta} + i\varepsilon^{\alpha\beta\mu\nu}\Gamma^i{}_{\alpha\beta}\Gamma_{i\,\mu\nu}\gamma_5,$$

we get finally

$$\left[\pi^i\pi_i - \frac{\hbar}{2}\sigma^{\alpha\beta}\left(\frac{q}{c}F_{\alpha\beta} + m\Phi_{\alpha\beta}\right) + \frac{\hbar^2}{4}\,R + \frac{\hbar^2}{16}\left(2\Gamma^i{}_{\alpha\beta}\Gamma_i{}^{\alpha\beta} + i\varepsilon^{\alpha\beta\mu\nu}\Gamma^i{}_{\alpha\beta}\Gamma_{i\,\mu\nu}\gamma_5\right) - m^2c^2\right]\psi = 0, \tag{6.6}$$

where

$$\Phi_{\alpha\beta} = \frac{1}{2m}\left\{\pi^i, \Gamma_{i\,\alpha\beta}\right\}, \qquad \gamma_5 = -i\gamma^{\hat{0}}\gamma^{\hat{1}}\gamma^{\hat{2}}\gamma^{\hat{3}}. \tag{6.7}$$

In the semiclassical approximation, $\pi^i = mU^i$, and $\Phi_{\alpha\beta}$ coincides with the spin (and momentum) transport matrix in a gravitational field (see Ref. [9]) and with the tensor-like coefficients $\gamma_{\alpha\beta\lambda}u^\lambda$ [22]. It is analogous to the electromagnetic field tensor and leads to the Dirac gyro-gravitomagnetic ratio $g_{grav} = 2$ in perfect agreement with the equivalence principle which is also manifested in the interaction of spin with gravity [30]. This means [31] the absence of both the anomalous gravitomagnetic moment and the gravitoelectric dipole moment which are gravitational analogs of the anomalous magnetic moment and the electric dipole moment, respectively.

Eq. (6.6) explicitly shows a similarity of the Dirac particle interactions with electromagnetic and gravitational fields. This similarity is caused by the similarity of the motion of spinning particles in any external classical fields shown in Ref. [9].

VII. CONCLUSIONS

In this paper we have studied the quantum and classical dynamics of spin 1/2 fermions in the strong fields. We derived the Hermitian Dirac Hamiltonian (2.21) that describes the fermion in arbitrary electromagnetic and gravitational fields in the Schwinger gauge. Applying the FW transformation, we constructed the respective Hamiltonian (3.1) for an arbitrary isotropic metric. This allowed us to obtain the operator equation of spin precession (3.7) and its semiclassical limit (3.10). As a specific application, we proved that the equations of motion in the uniformly accelerated rotating frame obtained earlier for the small acceleration are in fact valid for arbitrary large acceleration. We also derived the classical equation (4.12) for the spin precession in the general gravitational field (in the Schwinger gauge) and observed its consistency with the semiclassical limit of quantum equation, in complete agreement with the equivalence principle. We confirmed this conclusion by deriving and analyzing the squared Dirac equation. The general evolution equations for the helicity were obtained in quantum and classical pictures. Their application to the dynamics in the general noninertial frame revealed the change of the helicity even for the ultrarelativistic particle with a negligible mass.

Acknowledgements

We are indebted to V.P. Neznamov, P. Fiziev and D.V. Shirkov for stimulating discussions. This work was supported in part by JINR, the Belarusian Republican Foundation for Fundamental Research (Grant No. Φ10D-001) and the Russian Foundation for Basic Research (Grants No. 11-02-01538, 11-01-12103).

---

[1] C.G. de Oliveira and J. Tiomno, Nuovo Cim. **24**, 672 (1962); F.W. Hehl, J. Lemke, and E.W. Mielke, *Two lectures on fermions and gravity*, in: *Geometry and Theoretical Physics*, Proc. of the Bad Honnef School 12–16 Feb. 1990, J. Debrus and A.C. Hirshfeld, eds. (Springer: Heidelberg, 1991) pp. 56–140; E. Fischbach, B.S. Freeman, and W.-K. Cheng, *Phys. Rev.* **D23** (1981) 2157; J.C. Huang, *Ann. d. Phys.* **3** (1994) 53; K. Konno and M. Kasai, *Prog. Theor. Phys.* **100** (1998) 1145; K. Varjú and L.H. Ryder, *Phys. Lett.* **A250** (1998) 263; arXiv:gr-qc/0603035; L. Ryder, *J. Phys. A: Math. Gen.* **A31** (1998) 2465; K. Varjú and L.H. Ryder, *Phys. Rev.* **D62** (2000) 024016; C. Kiefer and C. Weber, *Ann. Physik (Berlin)* **14** (2005) 253; arXiv:gr-qc/0408010; P. Gosselin, A. Bérard, and H. Mohrbach, *Phys. Lett.* **A368** (2007) 356; arXiv:hep-th/0604012.

[2] P. P. Fiziev and D. V. Shirkov, Teor. Matem. Fiz **167**, 323 (2011) [Theor. Math. Phys. **167** 680 (2011)]; arXiv:1009.5309 [hep-th].

[3] J. R. Mureika and D. Stojkovic, Phys. Rev. Lett. **106**, 101101 (2011); arXiv:1102.3434 [gr-qc].

[4] T. P. Sotiriou, M. Visser and S. Weinfurtner, arXiv:1104.1223 [gr-qc].

[5] O. Rogachevsky, A. Sorin, and O. Teryaev, Phys. Rev. C **82** (2010) 054910; arXiv:1006.1331 [hep-ph].

[6] A.J. Silenko and O.V. Teryaev, Phys. Rev. D **71**, 064016 (2005); arXiv:gr-qc/0407015.

[7] A.J. Silenko and O.V. Teryaev, Phys. Rev. D **76**, 061101(R) (2007); arXiv:gr-qc/0612103.

[8] A.J. Silenko, Acta Phys. Polon. B Proc. Suppl. **1** 87 (2008); arXiv:0802.4443 [gr-qc].

[9] Yu.N. Obukhov, A.J. Silenko, and O.V. Teryaev, Phys. Rev. D **80**, 064044 (2009); arXiv:0907.4367 [gr-qc].

[10] F. W. Hehl and W. T. Ni, Phys. Rev. D **42**, 2045 (1990).

[11] H. Thirring, Phys. Z. **19**, 33 (1918) [Gen. Rel. Grav. **16**, 712 (1984)]; Phys. Z. **22**, 29 (1921)

[Gen. Rel. Grav. **16**, 725 (1984)]; J. Lense and H. Thirring, Phys. Z. **19**, 156 (1918) [Gen. Rel. Grav. **16**, 727 (1984)].

[12] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962).

[13] S. Hergt and G. Schäfer, Phys. Rev. D **77**, 104001 (2008); arXiv:0712.1515 [gr-qc].

[14] Yu.N. Obukhov, *Phys. Rev. Lett.* **86**, 192 (2001); arXiv:gr-qc/0012102.

[15] Yu.N. Obukhov, *Fortschr. Phys.* **50**, 711 (2002); arXiv:gr-qc/0112080.

[16] M. V. Gorbatenko and V. P. Neznamov, Phys. Rev. D **82**, 104056 (2010); arXiv:1007.4631 [gr-qc].

[17] M. V. Gorbatenko and V. P. Neznamov, Phys. Rev. D **83**, 105002 (2011); arXiv:1102.4067 [gr-qc].

[18] M. Leclerc, Class. Quant. Grav. **23**, 4013 (2006); arXiv:gr-qc/0511060.

[19] P. Gosselin and H. Mohrbach, arXiv:1009.2607 [hep-th].

[20] A. J. Silenko, Phys. Rev. A **77**, 012116 (2008); arXiv:0710.4218 [math-ph].

[21] G. Cognola, L. Vanzo, and S. Zerbini, Gen. Rel. Grav. **18**, 971 (1986).

[22] A. A. Pomeransky and I. B. Khriplovich, J. Exp. Theor. Phys. **86**, 839 (1998) [Zh. Eksp. Teor. Fiz. **113**, 1537 (1998)]; arXiv:gr-qc/9710098.

[23] M. Mathisson, Acta Phys. Polon. **6**, 163 (1937).

[24] A. Papapetrou, Proc. Roy. Soc. Lond. A **209**, 248 (1951).

[25] C. Chicone, B. Mashhoon, and B. Punsly, *Phys. Lett.* **A343**, 1 (2005); arXiv:gr-qc/0504146.

[26] B. Mashhoon, *Gravitoelectromagnetism: A brief review*, in: *"The Measurement of Gravitomagnetism: A Challenging Enterprise"*, ed. L. Iorio (Nova Science, New York, 2007), pp. 29-39; arXiv:gr-qc/0311030.

[27] M.L. Ruggiero and A. Tartaglia, *Nuovo Cim.* **117B** (2002) 743-768; arXiv:gr-qc/0207065.

[28] D.R. Brill and J.M. Cohen, Phys. Rev. **143**, 1011 (1966); J.M. Cohen, Phys. Rev. **148**, 1264 (1966); J.M. Cohen and D.R. Brill, Nuovo Cim. B **56**, 209 (1968); J.M. Cohen, Astrophys. Space. Sci. **6**, 263 (1970).

[29] S. M. Christensen and M. J. Duff, Nucl. Phys. B **154**, 301 (1979).

[30] I.Yu. Kobzarev, L.B. Okun, Zh. Eksp. Teor. Fiz. **43**, 1904 (1962) [Sov. Phys. JETP **16**, 1343 (1963)].

[31] O. V. Teryaev, Czech. J. Phys. **53** (2003) 47; arXiv:hep-ph/0306301.