Investigation of the law of the iterated logarithm for extreme queue length in multiphase queues

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Abstract: *Interest in the field of multiphase queueing systems has been stimulated by the theoretical values of the results as well as by their possible applications in information and computing systems, communication networks, and automated technological processes. The object of this research in queueing theory is the law of the iterated logarithm (LIL) under the conditions of heavy traffic in multiphase queueing systems (MQS). In this paper, the LIL is proved for extreme values of important probabilistic characteristics of the MQS investigated as well as maxima of the summary queue length of customers and maxima of the queue length of customers.*

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1. INTRODUCTION

Interest in the field of multiphase queueing systems has been stimulated by the theoretical values of the results as well as by their possible applications in information and computing systems, communication networks, and automated technological processes (see, for example, [5]). The MQS is a queueing system when a customer does not visit same queueing node twice (see, for example, [2]). Therefore, such a system is a special case of the open Jackson network (see, for example, [4]).

Thus, in this paper, theorems on the LIL for the extreme values of the main probability characteristics of MQS in heavy traffic (maxima of the summary queue length of customers, maxima of the queue length of customers) are proved. The main tools for the analysis of MQS in heavy traffic are the functional LIL for a Wiener process and a renewal process (proof can be found in [1]).

2. MAIN RESULTS

We investigate here a k-phase MQS (i.e., when a customer is served in the j-th phase of the MQS, he goes to the $j+1$-st phase of the MQS, and after the customer is served in the k-phase of MQS, then he leaves the MQS). Let us denote t_n as the time of arrival of the n-th customer; $S_n^{(j)}$ as the service time of the n-th customer in the j-th phase of the MQS; $z_n = t_{n+1} - t_n$. Let us introduce mutually independent renewal processes $x_j(t) = \{\max_k \sum_{i=1}^{k} S_i^{(j)} \le t\}$ (such a total number of customers can be served in the j-th phase of the MQS until time t (if devices are working without time wasted)), $e(t) = \{\max_k \sum_{i=1}^{k} z_i \le t\}$ (total number of customers which arrive at MQS until a time moment t). Next, denote by $\tau_j(t)$ the total number of customers after service departure from the j-th phase of the MQS until time t; $Q_j(t)$ as the queue length of customers in the j-th phase of MQS at the time moment t; $\nu_j(t) = \sum_{i-1}^{j} Q_i(t)$ stands for the summary queue length of customers until the j-th phase of the MQS at the time moment t, $j = 1, 2, \ldots, k$ and $t > 0$.

Suppose that the queue length of customers in each phase of the MQS is unlimited, the service principle of customers is “first come, first served” (FCFS). All random variables are defined on the common probability space (Ω, F, P).

Let interarrival times (z_n) at the MQS and service times (S_n^j) in every phase of the MQS for $j = 1, 2, \ldots, k$ be mutually independent identically distributed random variables.

Let us define $\beta_j = (ES_1^{(j)})^{-1}$, $\beta_0 = (Ez_1)^{-1}$,

$$\alpha_j = \beta_0 - \beta_j, \ \alpha_0 = 0,$$

$$\hat{\sigma}_j^2 = DS_1^{(j)}(ES_1^{(j)})^{-3} > 0, \ \hat{\sigma}_0^2 = Dz_1(Ez_1)^{-3} > 0,$$

$$\tilde{\sigma}_j^2 = \hat{\sigma}_0^2 + \hat{\sigma}_j^2, \ \sigma_j^2 = \hat{\sigma}_j^2 + \hat{\sigma}_{j-1}^2,$$

$$\hat{x}_j(t) = e(t) - x_j(t), \quad j = 1, 2, \ldots, k,$$

$$a(t) = \sqrt{2t \ln\ln t},$$

$$\hat{x} = \sup_{1 \le i \le k}\{\sup_{0 \le s \le t}(\hat{x}_{i-1}(s) - \hat{x}_i(s))\} \ge 0, \ \hat{x}_0(\cdot) \equiv 0.$$

Assume the following condition to be fulfilled $\beta_0 > \beta_1 > \ldots > \beta_k > 0$ Then

$$\alpha_k > \alpha_{k-1} > \ldots > \alpha_1 > 0. \qquad (1)$$

One of the main results of the work is a theorem on the LIL for the summary length of customers.

Theorem 2.1. *If condition (1) is fulfilled, then*

$$P\left(\overline{\lim_{t\to\infty}}\frac{\sup\limits_{0\le s\le t} v_j(s)-\alpha_j\cdot t}{\tilde{\sigma}_j\cdot a(t)}=1\right)=$$

$$P\left(\underline{\lim_{t\to\infty}}\frac{\sup\limits_{0\le s\le t} v_j(s)-\alpha_j\cdot t}{\tilde{\sigma}_j\cdot a(t)}=-1\right)=1,$$

for $j=1,2,\dots,k.$

In relations

$$Q_j(t)=\tau_{j-1}(t)-\tau_j(t), \tag{2}$$

$$Q_j(t)=f_t(\tau_{j-1}(\cdot)-x_j(\cdot)), \tag{3}$$

$$Q_j(t)=f_t(€_j(\cdot)-\sum_{i=1}^{j-1}Q_i(\cdot)) \tag{4}$$

are obtained for $j=1,2,\dots,k$ and $f_t\left(x(\cdot)\right)=x(t)-$ $-\inf\limits_{0\le s\le t} x(s)$ (see [3]). In view of (3) and (4) we have $v_j(t)=€_j(t)-\inf\limits_{0\le s\le t}(€_j(s)-v_{j-1}(s))$ and $v_0(\cdot)\equiv 0.$

Next, using (2) and (3), we obtain $\tau_j(t)=\tau_{j-1}(t)-Q_j(t)=x_j(t)+\inf\limits_{0\le s\le t}(\tau_{j-1}(s)-x_j(s)),$ for $j=1,2,\dots,k$ and $\tau_0(t)=e(t)$.

Thus,

$$x_j(t)-\tau_j(t)=\sup_{0\le s\le t}(x_j(s)-\tau_{j-1}(s))$$
$$=\sup_{0\le s\le t}(x_j(s)-x_{j-1}(s)+\sup_{0\le v\le s}(x_{j-1}(s)-\tau_{j-2}(s)))$$
$$\le\sup_{0\le s\le t}(x_j(s)-x_{j-1}(s))+\sup_{0\le s\le t}\sup_{0\le v\le s}(x_{j-1}(v)-\tau_{j-2}(v))$$
$$\le\sup_{0\le s\le t}(x_j(s)-x_{j-1}(s))+\sup_{0\le s\le t}(x_{j-1}(s)-\tau_{j-2}(s)) \tag{5}$$
$$\le\dots\le\sum_{i=1}^{j}\left\{\sup_{0\le s\le t}\left(x_i(s)-x_{i-1}(s)\right)\right\}$$
$$\le\sum_{i=1}^{k}\{\sup_{0\le s\le t}(€_{i-1}(s)-€_i(s))\}$$
$$\le k\cdot\sup_{0\le i\le k}\left\{\sup_{0\le s\le t}\left(€_{i-1}(s)-€_i(s)\right)\right\}=k\cdot x, \text{ for}$$
$j=1,2,\dots,k.$

From (2) and (5) we get

$$v_j(t)=\sum_{i=1}^{j}Q_i(t)=\sum_{i=1}^{j}\{\tau_{i-1}(t)-\tau_i(t)\}$$
$$=e(t)-\tau_j(t)=e(t)-x_j(t)+x_j(t)-\tau_j(t) \tag{6}$$
$$\le €_j(t)+\sum_{i=1}^{k}\{\sup_{0\le s\le t}(€_{i-1}(s)-€_i(s))\}\le €_j(t)+k\cdot €,$$

for $j=1,2,\dots,k.$

So (see (6))

$$\sup_{0\le s\le t} v_j(s)\le\sup_{0\le s\le t}(€_j(s)+k\cdot €)= \sup_{0\le s\le t}€_j(s)+k\cdot €,\quad j=1,2,\dots,k. \tag{7}$$

Since for any j $(j=1,2,\dots,k)$

$$v_j(t)=e(t)-\tau_j(t)\ge e(t)-x_j(t)=€_j(t)$$

or

$$\sup_{0\le s\le t} v_j(s)\ge\sup_{0\le s\le t}€_j(s)\ge\sup_{0\le s\le t}€_j(s)-k\cdot €$$

we have from (7) the following estimate

$$|\sup_{0\le s\le t} v_j(s)-\sup_{0\le s\le t}€_j(s)|\le k\cdot €,\ \text{ for }\ j=1,2,\dots,k. \tag{8}$$

Denote the families of random functions as $0\le t\le 1$

$$V€_j^n=(\sup_{0\le s\le nt} v_j(s)-\alpha_j\cdot nt)/a(n),$$
$$€_j^n=(\sup_{0\le s\le nt}€_j(s)-\alpha_j\cdot nt)/a(n),$$
$$x_j^n=(€_j(nt)-\alpha_j\cdot nt)/a(n),\ \text{ for }\ j=1,2,\dots,k.$$

Therefore, making use of (8), we can get

$$d(V€_j^n,x_j^n)\le\rho(V€_j^n,x_j^n)\le\rho(V€_j^n,€_j^n)+\rho(€_j^n,x_j^n)$$
$$\le k\cdot\{\sup_{1\le i\le k}(\sup_{0\le s\le t}(€_{i-1}(s)-€_i(s))\}+\rho(€_j^n,x_j^n) \tag{9}$$
$$\le k\cdot\{\sum_{j=1}^{k}(\sup_{0\le s\le nt}(€_{j-1}(s)-€_j(s))/a(n)\}+\rho(€_j^n,x_j^n).$$

Note that $\sup\limits_{0\le s\le nt}(€_{j-1}(s)-€_j(s))/a(n)\ge 0,\ \ 0\le t\le 1.$

Also we note that

$$\lim_{t\to\infty}\frac{€_{j-1}(t)-€_j(t)}{t}=\alpha_{j-1}-\alpha_j<0$$

almost everywhere for $j=1,2,\dots k$ (see [1]). Thus, similarly as in [1] we can prove that the first term in (9) tends to zero.

Now we prove that the second term in (9) also tends to zero. Using $\alpha_{j-1}-\alpha_j<0$ for $j=1,2,\dots,k$ and again, just like in [6], we prove that the second term in inequality (9) tends to zero.

Hence we prove

$$P\left(\overline{\lim_{t\to\infty}}\frac{\sup\limits_{0\le s\le t} v_j(s)-\alpha_j\cdot t}{\tilde{\sigma}_j\cdot a(t)}=1\right)=$$

$$P\left(\varliminf_{t\to\infty}\frac{\sup\limits_{0\le s\le t} v_j(s)-\alpha_j\cdot t}{\tilde{\sigma}_j\cdot a(t)}=-1\right)=1,$$

for $j=1,2,...,k$ The proof is complete.

Now we present the theorem on the LIL for the maxima of the queue length of customers.

Theorem 2.2. If condition (1) is fulfilled, then

$$P\left(\varlimsup_{t\to\infty}\frac{\sup\limits_{0\le s\le t} Q_j(s)-(\alpha_j-\alpha_{j-1})\cdot t}{\sigma_j\cdot a(t)}=1\right)=$$

$$P\left(\varliminf_{t\to\infty}\frac{\sup\limits_{0\le s\le t} Q_j(s)-(\alpha_j-\alpha_{j-1})\cdot t}{\sigma_j\cdot a(t)}=-1\right)=1,$$

for $j=1,2,...,k$.

The theorem is proved similarly as Theorem 2.1. The proof is complete.

3. References.

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