First–Passage Time Moment Approximation For The Wright–Fisher Diffusion Process With Absorbing Barriers

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Abstract: *Consider the simplest Wright-Fisher diffusion process which is a model for depicting fluctuations of gene frequency of the A-type within a population having both A-and a-types subject to selection influences. This paper describes an accurate method of approximating the moments of the first – passage time for the simplest Wright-Fisher diffusion process with absorbing barriers. This was done by approximating the differential equations by an equivalent difference equations.*

Keywords: First Passage Time, Wright-Fisher Diffusion Process, Difference Equations.

1. INTRODUCTION

First – passage time play an important rule in the area of applied probability theory especially in stochastic modeling. Several examples of such problems are the extinction time of a branching process, or the cycle lengths of a certain vehicle actuated traffic signals.

Many important results related to the first – passage time have been studied from different points of view of different authors. For example, McNeil (1970) has derived the distribution of the integral functional $Wx = \int_0^{Tx} g\{X(t)\}dt$, where T_x is the first – passage time to the origin in a general birth – death process with X(0) = x and g(.) is an arbitrary function. Also, Iglehart (1965) , McNeil and Schach (1973) have been shown a number of classical birth and death processes upon taking diffusion limits to asympotically approach the Ornstein – Uhlenbeck (O.U.) .

Many properties such as a first – passage time to a barrier, absorbing or reflecting, located some distance from an initial starting point of the O.U. process and the related diffusion process and the related diffusion process such as the case of the first passage time of a Wiener process to a linear barrier is a closed form expression for the density available is discussed in Cox and Miller (1965). Also, others such as, Karlin and Taylor (1981), Thomas (1975), Ferebee (1982), Alawneh and Al-Eideh (2002), etc.have been discussed the first passage time from different points of view.

In particular, Thomas (1975) describes some mean first – passage time approximation for the Ornstein – Uhlendeck process. Also, Alawneh and Al-Eideh (2002) have discussed the problem of finding the moments of the first passage time distribution for the Ornstien-Uhlenbeck process with a single absorbing barrier using the method of approximating the differential equations by difference equations.

In this paper, we consider the simplest Wright and Fisher diffusion process and study the first – passage time for such a process to an absorbing barriers. More specifically, the moment approximations are derived using the method of difference equations.

2. FIRST – PASSAGE TIME MOMENT PPROXIMATIONS

Consider the simplest Wright and Fisher diffusion Process $\{X(t): t \geq 0\}$ with infinitesimal mean $bx(1-x)$ and variance $x(1-x)$ starting at some $x_0 >$ 0, where b is the selection coefficient and satisfies the Ito stochastic differential equation

$$dX(t) = bX(t)\big(1 - X(t)\big)dt + \sqrt{X(t)\big(1 - X(t)\big)}dW(t)$$

Where $\{W(t): t \geq 0\}$is a standard Wiener process with zero mean and variance t. Assume that the existence and uniqueness conditions are satisfied (Cf. Gihman and Skorohod (1972)).

Note $\{X(t): t \geq 0\}$is a Markov procees with state – space S = [0, 1] . Notice that 0 and 1 are absorbing states. Denote the first – passage time of a process $X(t)$ to a moving linear barrier $Y(t) = \beta$ by the random variables

$$T_Y = \inf\{t \geq 0 : X(t) \geq \beta\}$$

with probability density function

$$g(t; x_0) = -\frac{d}{dt}\int_{-\infty}^{\beta} p(x_0; x; t)dx$$

Here $p(x_0; x; t)$ is the probability density function of $X(t)$ conditional on $X(0) = x_0$.

Let $M_n(x_0, \beta; t)$; n = 1,2,3,…… , be the n-th moment of the first – passage time T_Y , i.e.

$M_n(x_0, \beta; t) = E(T_\beta^n)$; n = 1,2,3,…,

It follows from the forward Kolmogorov equation that the n-th moment of T_Y must satisfy the ordinary differential equation

$$\begin{aligned} &x(1-x)M_n''(x_0, \beta; t) \\ &+ bx(1-x)M_n'(x_0, \beta; t) \\ &= -nM_{n-1}(x_0, \beta; t) \end{aligned} \qquad (1)$$

Or equivalently

$$\begin{aligned} &M_n''(x_0, \beta; t) + bM_n'(x_0, \beta; t) \\ &= -\frac{n}{x(1-x)}M_{n-1}(x_0, \beta; t) \end{aligned} \qquad (2)$$

Where $M_n'(x_0, \beta; t)$ and $M_n''(x_0, \beta; t)$ are the first derivatives of $M_n(x_0, \beta; t)$ with respect to x $(x_0 \leq x \leq \beta)$, with appropriate boundary conditions for n=1,2,3,…….Note that $M_0(x_{0,} \beta; t) = 1$.

Now, rewrite the equation in (2), we obtain

$$\begin{aligned} &M_n''(x_0, \beta; t) = \\ &-\frac{n}{x(1-x)}M_{n-1}(x_0, \beta; t) \\ &-bM_n'(x_0, \beta; t) \end{aligned} \qquad (3)$$

Let Δ be the difference operator. Then we defined the first order difference of $M_n(x_0, \beta; t)$ as follows:

$$\begin{aligned} \Delta M_n(x_0, \beta; t) &= M_{n+1}(x_0, \beta; t) \\ &- M_n(x_0, \beta; t) \end{aligned} \qquad (4)$$

(cf. Kelley and peterson (1991)).

Note that equation (3) can be approximated by

$$M_n''(x_0,\beta;t) = -\frac{n}{x(1-x)}M_{n-1}(x_0,\beta;t) - b\Delta M_n(X_0,\beta;t) \qquad (5)$$

By applying equation (4) to equation (5) we get :

$$M_n''(x_0\beta;t) = -\frac{n}{x(1-x)}M_{n-1}(x_0,\beta;t) + bM_n(x_0,\beta;t) - bM_{n+1}(x_0,\beta;t) \qquad (6)$$

Now, we will use the matrix theory to solve the differential equation defined in equation (6). If we let

$$\vec{M}(x_0,\beta;t) = \left[M_1(x_0,\beta;t), M_2(x_0,\beta;t),......\right]'$$

Then we get

$$\frac{d^2\vec{M}(x_0,\beta;t)}{dx^2} = A\vec{M}(x_0,\beta;t) \qquad (7)$$

Where

$$A = \begin{bmatrix} b & -b & 0 & 0 & ...... & & \\ -\frac{2}{x(1-x)} & & b & -b & 0 & .... & \\ 0 & - & \frac{3}{x(1-x)} & & b & -b & .... \\ . & & . & . & . & & \\ . & & . & . & . & & \\ . & & . & . & . & & \end{bmatrix}$$

Now let

$$\frac{d\vec{M}(x_0,\beta;t)}{dx} = \vec{R}(x_0,\beta;t) \qquad (8)$$

This imply

$$\frac{d^2\vec{M}(x_0,\beta;t)}{dx^2} = \frac{d\vec{R}(x_0,\beta;t)}{dx} \qquad (9)$$

Apply to equation (7), we get

$$\frac{d}{dx}\begin{bmatrix}\vec{R}(x_0,\beta;t)\\ \vec{M}(x_0,\beta;t)\end{bmatrix} = \begin{bmatrix}0 & \mathrm{A}\\ \mathrm{I} & 0\end{bmatrix}\cdot\begin{bmatrix}\vec{\mathrm{R}}(\mathrm{x}_0,\beta;t)\\ \vec{M}(x,\beta;t_0)\end{bmatrix} \qquad (10)$$

Where **I** is the identity matrix and 0 is the zero matrix. Thus, the solution of the system of equation in (10) is then given by

$$\begin{bmatrix}\vec{R}(x_0,\beta;t)\\ \vec{M}(x_0,\beta;t)\end{bmatrix} = \mathrm{e}^{\begin{bmatrix}0 & \mathrm{A}^*\\ \mathrm{D} & 0\end{bmatrix}}\cdot\begin{bmatrix}\vec{R}(\mathrm{x}_0,\beta;t)\\ \vec{M}(x,\beta;t_0)\end{bmatrix} \qquad (11)$$

Where $\mathrm{D}^* = \left[d_{ij}^*\right]; i, j \geq 1$ is the diagonal matrix with entries

$$d_{ij} = \begin{cases}(\beta - x_0) & ; \mathrm{j} = \mathrm{i}\\ 0 & ; \mathrm{Otherwise}\end{cases} \qquad (12)$$

And $A^* = \left[a_{ij}^*\right]; i, j \geq 1$ is the matrix with entries

$$a_{ij}^* = \begin{cases} -\frac{i}{1-x}\ln\left(\frac{\beta}{x_0}\right) & ; j=i-1\\ b(\beta - x_0) & ; j=i\\ -b(\beta - x_0) & ; j=i+1\\ 0 & ; Otherwise\end{cases} \qquad (13)$$

Note that the matrix e^B where $B = \begin{bmatrix}0 & A^*\\ D & 0\end{bmatrix}$ is defined by

$$e^B = I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + ...$$

This series is convergent since it is a cauchy operator of equation (2.6) (cf. Zeifman (1991)).

3. CONCLUSION

In conclusion the advantage of this technique is to use the difference equation to approximate the ordinary differential equation since it is the discretization of the ODE. Also, the system of solution in equation (11) gives an explicit solution to the first – passage time moments for such a process which increase the applicability of the diffusion process in stochastic modeling or in all area of applied probability theory.

4. REFERENCES

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