FORWARD INTEREST RATES AND VOLATILITY OF ZERO COUPON YIELD

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Abstract. The analysis of the forward rate curve for enough wide class of one-factor affine models of the term structure that includes not only Vasiček's Gaussian model and CIR model «with a square root» but also models with any levels of the lower boundary of the short term (riskless) interest rate (see [13, 14, 16]) is resulted. The special attention is given to the problem connected with the tendency for the term structure of long term forward rates to slope downwards.

1. Introduction

One of classical problems of financial economics is the analysis of behaviour of the yield on default free bonds depending on their maturities. At the certain assumptions it is possible to use mathematical model of available yield curve to extrapolate it to obtain the future values of yield rate. For example by that way the forward rates are obtained on the basis of knowledge of time structure of discount bonds (see details in [12]).

The forward rate term structure it was investigated in a number of references from which we shall mention only a few. A forward rate curve for affine models of term structure has been received in [3]. A minimax way of modelling forward curve on the basis of supervision of yield has been offered in [15]. The comparative analysis of yield curves and forward rate curves has been presented in [16].

At the same time the segment of term structure for long-term forward rates was not exposed to detailed research as for short-term interest rates. In [4] it has been discovered that the new information about the yield term structure can be received from the analysis of the long-term end of the forward rate curve. They have noted that the empirical data show that the forward rate curve for long term maturity is usually sloped downwards. Discussion of two-factor Gaussian affine model of term structure at the certain assumptions has allowed to draw an inference that the slope of the forward rate curve on the long-term end always should be negative and this effect is connected to properties volatility of long-term zero coupon yield. Moreover that is possible to predict volatility of long-term yield by observation a slope of the forward rate curves. This problem was empirically investigated in [5], where it is provided cautious support to results of [5].

In the present paper the analysis of the forward rate curve is made for wider class of affine models of the term structure including not only Vasiček's Gaussian model [19] and model «with a square root» CIR [7], but also the models generated by various levels of the lower boundary of the short term (riskless) interest rates [13, 14, 16].

2. Properties of forward rate curves for affine models of term structure

The affine models of term structure occur when the short term (riskless) interest rate $r(t)$ follows the stochastic process described by the stochastic differential equation

$$dr(t) = k\,(\theta - r(t))\,dt + \sqrt{2kD\frac{r(t)-x}{\theta - x}}\,dW(t), \quad r(0) > x. \tag{1}$$

The parameters of equation (1) have the following concrete practical sense: θ – stationary expectation of the short term interest rate $r(t)$; D – its stationary variance; x – the parameter that has sense of lower reflecting boundary of process $r(t)$: $r(t) \geq x$ for every t (according to Feller (1951) this boundary is unattainable if $(\theta - x)^2 > D$); k – the parameter that determines a velocity of transfer of process (1) into stationary mode; there is another interpretation of the parameter k: it determines an autocorrelation coefficient of process (1) in a form

$$\rho(\tau) = E[(r(t) - \theta)(r(t + s) - \theta)]/D = \exp\{-k|s|\}.$$

Let in the market a non arbitrage conditions are satisfied at the short term interest rates described by the equation (1). Then a market price of risk $\lambda(r)$ is defined by expression [see Ilieva (2001)]:

$$\lambda(r) = -\lambda\sqrt{(r-x)/(\theta - x)}\,, \tag{2}$$

were the parameter λ determines a value of risk premium, $\lambda \geq 0$.

The parameter k, x, θ and D of equation (1) are constant therefore the process $r(t)$ is homogenous in the time and at the current time t, when $r(t) = r$. The price $P(r, t, T)$ of zero coupon bond, on which at date maturity T the one money unit is paid, is determined by formula

$$P(r, t, T) = \exp\{A(T - t) - rB(T - t)\}. \tag{3}$$

Further for brevity a term to maturity of the zero coupon bond we shall designate $\tau = T - t$. Models of the interest rates that allow to express the price of bond $P(r, t, T)$ as (3) form a class of affine term structures of interest rates. Functions of term structures $A(\tau)$ and $B(\tau)$ satisfy to the equations

$$\frac{dB}{d\tau} = 1 - \left(k + \lambda\frac{\sqrt{2kD}}{\theta - x}\right)B(\tau) - \frac{kD}{\theta - x}[B(\tau)]^2, \quad B(0) = 0, \tag{4}$$

$$\frac{dA}{d\tau} = -\left(k\theta + \lambda x\frac{\sqrt{2kD}}{\theta - x}\right)B(\tau) - \frac{kDx}{\theta - x}[B(\tau)]^2, \quad A(0) = 0. \tag{5}$$

Solutions of these equations are expressed as:

$$B(\tau) = \left(\frac{\varepsilon}{e^{\varepsilon\tau} - 1} + V\right)^{-1}, \tag{6}$$

$$A(\tau) = x\,[B(\tau) - \tau] - \frac{(\theta - x)^2}{D}[v\tau - \ln(1 + vB(\tau))], \tag{7}$$

where it is designated for brevity

$$\varepsilon = \sqrt{\left(k + \lambda\frac{\sqrt{2kD}}{\theta - x}\right)^2 + \frac{4kD}{\theta - x}} \ge k > 0, \quad v = \frac{1}{2}\left(\varepsilon - k - \lambda\frac{\sqrt{2kD}}{\theta - x}\right) \ge 0, \quad V = \frac{1}{2}\left(\varepsilon + k + \lambda\frac{\sqrt{2kD}}{\theta - x}\right) \ge k > 0. \tag{8}$$

Note that $v + V = \varepsilon$, $vV = kD/(\theta - x)$.

Properties of functions of affine term structure $A(\tau)$ and $B(\tau)$ that are determined by formulae (6) – (7) are in detail investigated in [13].

The forward rate $f(t, T, T')$ determines the bond yield between dates T and T' such that $t < T < T'$ on the base of information about yield that is available at time t:

$$f(t, T, T') = \frac{1}{T' - T}\ln\left[\frac{P(t, r, T)}{P(t, r, T')}\right] = \frac{r[B(\tau') - B(\tau)] - A(\tau') + A(\tau)}{\tau' - \tau}, \tag{9}$$

where $\tau' = T' - t$. As $T' \to T$, i.e. $\tau' \to \tau$, the forward rate (9) turn into the so-called instantaneous forward rate

$$f(t, T) = f(\tau) = -\frac{\partial \ln P(t, r, T)}{\partial T} = r\frac{dB(\tau)}{d\tau} - \frac{dA(\tau)}{d\tau}, \tag{10}$$

which is used more often as it is connected by simple relations with bond yield to maturity $y(t, T) = y(\tau)$

$$f(t, T) = f(\tau) = \frac{\partial[(T - t)y(t, T)]}{\partial T} = y(\tau) + \tau\frac{dy(\tau)}{d\tau}. \tag{11}$$

Therefore more often a word-combination «the forward rate» means the instantaneous forward rate.

For the term structure models of the affine yield class in [3] it has offered to consider a forward rate curve of $f(\tau)$ as the complex function that depends on term to maturity τ only through the function of affine structure $B(\tau)$, i.e. $f(\tau) \equiv F(B(\tau))$. First, it is convenient because an interval of possible values of function $B(\tau)$ is finite according to (6). In this connection the properties of functions $F(B)$ can be illustrated visually by plots on the whole of interval $(0, \infty)$ possible values of terms to maturity τ. Second, as it was mentioned in [6] it is possible to consider the function $B(\tau)$ as a measure of a duration because by analogy with the standard duration of the bond price with respect to the interest rates (in this case with respect to short term rates) this function is determined by the formula $B(\tau) = -[\partial P/\partial r]/P$.

It is obtained from expression (6), (7) and (10) that

$$f(\tau) \equiv F(B(\tau)) = r + \left[k(\theta - r) - \lambda\sqrt{2kD}\frac{r - x}{\theta - x}\right]B(\tau) - kD\frac{r - x}{\theta - x}B(\tau)^2 \equiv$$

$$\equiv r + [k(\theta - x) - (V - v)(r - x)]\,B(\tau) - vV(r - x)\,B(\tau)^2. \tag{12}$$

The general properties of the forward rate curves $f(\tau)$ are written in [16]. Here we shall consider in more detail behavior of a forward curve for the long terms to maturity τ.

From expression (10) it follows that

$$\frac{df(\tau)}{d\tau} = r\frac{d^2B(\tau)}{d\tau^2} - \frac{dA^2(\tau)}{d\tau^2}$$

From the relations (4) – (7) it is possible to find that

$$\frac{dB(\tau)}{d\tau} = (1+vB(\tau))(1-VB(\tau)), \quad \frac{d^2B(\tau)}{d\tau^2} = -(V - v + 2vVB(\tau))\frac{dB(\tau)}{d\tau},$$

$$\frac{d^2A(\tau)}{d\tau^2} = -k(\theta - x)(1+vB(\tau))(1-VB(\tau)) + x\frac{d^2B(\tau)}{d\tau^2}.$$

Therefore

$$\frac{df(\tau)}{d\tau} = (r-x)\frac{d^2B(\tau)}{d\tau^2} + k(\theta - x)(1+vB(\tau))(1-VB(\tau)) =$$

$$= [k(\theta - x) - [v + V - 2v(1 - VB(\tau))](r - x)]\,(1+vB(\tau))(1-VB(\tau)). \tag{13}$$

From (6) it follows that as $\tau \to \infty$ the function $B(\tau) \to V^{-1}$. Hence for $f(\tau)$ and $df(\tau)/d\tau$ the limit relations are valid as $\tau \to \infty$

$$f(\tau) \to f(\infty) \equiv \frac{k}{V}\theta + \left(1 - \frac{k}{V}\right)x, \qquad \frac{df(\tau)}{d\tau} \to 0. \tag{14}$$

It means that if the terms to maturity increase then the forward rates tend to a constant, which takes values from an interval $[x, \theta]$ since by definition $0 < k/V \le 1$.

It is interesting to explain under what conditions the forward rate curve has a negative slope at long term to maturity. As follows from representation (13) the derivative $df(\tau)/d\tau$ is expressed in the form of product of three factors, two of which (the second and the third) are negative by definition. Therefore the forward rate curve has a negative slope if the following inequality is valid

$$k(\theta - x) - [v + V - 2v(1 - VB(\tau))](r - x) < 0,$$

that it is more convenient to write

$$\frac{r-x}{\theta - x} > \frac{k}{v + V - 2v(1 - VB(\tau))}. \tag{15}$$

Because the function $B(\tau)$ increases monotonously from the value $B(0) = 0$ at $\tau = 0$ up to the value $B(\infty) = V^{-1}$ at $\tau = \infty$, hence the right part of the inequality (15) is monotonously decreasing function τ from value $k/(V - v)$ at $\tau = 0$ up to $k/(V + v)$ at $\tau = \infty$.

Thus the forward rate curve has a negative slope for anyone τ, if

$$\frac{r-x}{\theta - x} > \frac{k}{V - v}, \tag{16}$$

and can have the negative slope for the some enough long term to maturity $\tau > \tau_0(r)$ if

$$\frac{k}{V - v} > \frac{r-x}{\theta - x} > \frac{k}{V + v}. \tag{17}$$

Here it is designated by $\tau_0(r)$ such value τ, at which the inequality (15) turns into equality.

Using formulas (8) it is possible to write down inequalities (15) – (17) in the explicit form through parameters of model (1).

Let's remind that by definition $r = r(t)$ – value of process of the short term rate at time t. Hence, this value can be considered as random variable. It is known also [13] that process $r(t)$, that is determined by the equation (1), under the conditions accepted above has the shifted gamma distribution with the following parameters: parameter of shift x, parameter of scale $D/(\theta - x)$ and parameter of the form $(\theta - x)^2/D$.

The probability density function of this random variable r is

$$g(r) = \frac{[(\theta - x)D^{-1}]^{\frac{(\theta-x)^2}{D}}(r-x)^{\frac{(\theta-x)^2}{D}-1}}{\Gamma[(\theta - x)^2 D^{-1}]}\exp\left(-\frac{(\theta - x)(r-x)}{D}\right), \qquad r > x. \tag{18}$$

Let's remind that according to Feller’s condition for unattainability of the lower boundary of the short term rate process $r(t)$ accepted above the inequality $(\theta - x)^2 > D$ takes place, therefore the parameter of the form $(\theta - x)^2/D > 1$.

It is more convenient to deal with affine transformation $z = (r - x)/(\theta - x)$ of the random variable r. The random variable z has the ordinary gamma distribution with parameter of the form $(\theta - x)^2/D$ and parameter of scale $D/(\theta - x)^2$. Let's designate the gamma distribution function with parameter of the form α and parameter of scale β through $G(r \mid \alpha, \beta)$. Then the probability PSD of the fact that for enough long terms to maturity the forward interest rates will slope downwards according to (17) is equal to probability of performance of inequalities (16) or (17), i.e.

$$\text{PSD} = 1 - G\left(\frac{k}{v+V}\middle|\frac{(\theta-x)^2}{D}, \frac{D}{(\theta-x)^2}\right). \tag{19}$$

Thus, PSD depends not only from parameters k, x, θ and D of process (1) but also on a parameter of a market price of risk λ through values v and V (see the formula (8)). The values of PSD as function of arguments x and λ were computed for values of parameters $k = 0{,}2339$, $\theta = 0{,}0808$ and $D = 0{,}00126$ corresponding to the empirical estimations received in [8] at analysis of the annualized one-month U.S. Treasury bill yield from June 1964 to December 1989 (306 observations). The values of PSD as function arguments x and D were computed too for values of parameters $k = 0{,}8922$, $\theta = 0{,}0905$ and $\lambda = 0{,}0789$ according to the empirical estimations received in [1] at analysis the 7-day Eurodollar deposit spot rate, daily from 1 Jun 1973 to 25 Feb 1995 (5505 observations). From these calculation it follows that probability PSD is more sensitive to parameter λ market price of risk, than to stationary variance D of the short term interest rate, i.e. to volatility of yield process that is directly proportional $\sqrt{D}$.

In conclusion of this section note that from representation (6) follows that for the long terms to maturities $VB(\tau) \approx 1 - \varepsilon\, e^{-\varepsilon\tau}/V$. Therefore for the long term to maturity τ the formula (13) for a derivative of forward rate curve can be presented as

$$\frac{df(\tau \mid r)}{d\tau} = [k(\theta - x) - \varepsilon(r - x)]\left(\frac{\varepsilon}{V}\right)^2 e^{-\varepsilon\tau} + o(e^{-\varepsilon\tau}). \tag{20}$$

This implies that on the long term end of term structure the derivative $df(\tau \mid r)/d\tau$ on absolute value exponentially decreases.

3. Forward rates and yield volatility

The term structure of forward rates $f(\tau) \equiv f(\tau|r)$ it is defined by equation (12). In this expression as already it has above been told only one variable can be considered as random. It is r – a value of riskless interest rate at present time t. According to (3) by the same reason the yield to maturity is also random variable and its volatility $\sigma_y(\tau|r)$ differs from volatility of short term rates only by multiplier $B(\tau)/\tau$, i.e.

$$\sigma_y(\tau|r) = \frac{B(\tau)}{\tau}\sqrt{2kD\frac{r-x}{\theta-x}}. \tag{21}$$

The random variable r has the probability density function that is determined by expression (13). Therefore it is possible to calculate the moments $f(\tau|r)$ and $\sigma_y(\tau|r)$ that results in to formulae

$$E[f(\tau|r)] = \theta + [k - (V - v) - vVB(\tau)](\theta - x)B(\tau) = \theta - [\lambda\sqrt{2} + \sqrt{kD}\, B(\tau)]\sqrt{kD}\, B(\tau). \tag{22}$$

$$\text{Var}[f(\tau|r)] = [(1 - VB(\tau))(1 + vB(\tau))]^2 D. \tag{23}$$

$$E[\sigma_y(\tau|r)] = \sqrt{2kD}\, B(\tau)Q/\tau. \tag{24}$$

$$\text{Var}[\sigma_y(\tau|r)] = 2kD[B(\tau)/\tau]^2(1 - Q^2). \tag{25}$$

Here for brevity the designation is used

$$Q \equiv \Gamma\left(\frac{(\theta-x)^2}{D} + \frac{1}{2}\right) \Bigg/ \left[\frac{\theta - x}{\sqrt{D}}\,\Gamma\left(\frac{(\theta-x)^2}{D}\right)\right], \tag{26}$$

where by symbol $\Gamma(.)$ the gamma function is designated.

Covariance of the forward rate $f(\tau|r)$ and volatility $\sigma_y(\tau|r)$ is expressed by the formula

$$\text{Cov}[f(\tau|r), \sigma_y(\tau|r)] = [(1 - VB(\tau))(1 + vB(\tau))]\sqrt{2kD}\,\frac{B(\tau)}{2\tau}\,\frac{DQ}{\theta - x}. \tag{27}$$

From this by expressions (23) and (25) it is easy to calculate the correlation coefficient of forward rate $f(\tau|r)$ and volatility $\sigma_y(\tau|r)$ in the form

$$\rho = \frac{1}{2}DQ/(\theta - x)^2 \Big/ \sqrt{1 - Q^2} > 0. \tag{28}$$

Let's analyze the received relation. From properties of the gamma function follows that the function $Q(u) = \Gamma(u + 1/2)/[u^{1/2}\Gamma(u)]$ in formula (26) is determined for positive values of argument and is monotonously increase. And also $Q(u) \to 1$ as $u \to \infty$. Because it is supposed that the Feller's condition for unattainability of the lower boundary by process of riskless rates $r(t)$ is carried out, it is equivalent to an inequality $u = (\theta - x)^2/D > 1$. The calculations show that for $u \in (1, \infty)$ there is $0{,}8 < Q(u) < 1$. Therefore from the formula (28) follows that correlation of the forward rate $f(\tau|r)$ and volatility $\sigma_y(\tau|r)$ is always positive.

In addition it is interesting to notice, that in spite of the fact that expectations and variances of the forward rate $f(\tau|r)$ and volatility yield process $\sigma_y(\tau|r)$ depend on the term to maturity τ, correlation between these functions does not depend on τ and is identical for all terms to maturities and monotonously decreases with growth of value $u = (\theta - x)^2/D$. For example, as u increases from 1 up to 20 the correlation coefficient ρ decreases from 0,9565 up to 0,2229.

For affine Gaussian model of term structure, i.e. Vasiček's model, the lower boundary of riskless rates $x = -\infty$. Therefore for this model $u \to \infty$ and then $Q(u) \to 1$. Estimate the correlation coefficient for this case. At great values u for asymptotic representations of gamma function it is possible to use the Stirling's formula

$$\Gamma(u) = \sqrt{2\pi}\, u^{u-1/2} e^{-u} \left(1 + \frac{1}{2u} + o\left(\frac{1}{u}\right)\right).$$

Use of this decomposition in the formula (28) results in following expression for great values of argument $u = (\theta - x)^2/D$:

$$\rho = \frac{\sqrt{D}}{\theta - x}\left(1 - \frac{3}{32}\frac{D}{(\theta - x)^2} + o\left(\frac{D}{(\theta - x)^2}\right)\right).$$

Thus, as $x \to -\infty$ the correlation coefficient $\rho \to 0$. It is a natural result (in Vasiček's model the volatility is constant) as in Gaussian models of term structure correlation of the forward rate $f(\tau|r)$ and the yield process volatility $\sigma_y(\tau|r)$ is absent.

For model of the term structure generated by riskless rate process $r(t)$ «with a square root», i.e. the CIR models, the lower boundary of riskless rates $x = 0$. In Table 1 the empirical data are given about the CIR model parameter estimates according to different authors and corresponding values of the correlation coefficient ρ are given too. In this table by symbol σ the volatility parameter of the short term interest rate is designated for standard representation of CIR model for process $r(t)$

$$dr(t) = k\,(\theta - r(t))\,dt + \sigma\sqrt{r(t)}\,dW(t).$$

In Brown and Schaefer (2000) it is offered to use a slope of a curve of forward rates for a prediction volatility of long term yield rates. As a result of the analysis of forward rate spread there it is found that for real parameters of model of the interest rate the forward rate spread it is linearly connected with (local) variance of yield on a time unit, i.e. a square volatility process of yield. We shall consider this problem for examined model.

Table 1. *Correlation between the forward rate and volatility for some empirical data fitted by model CIR*

References	k	θ	σ	D	u	ρ
CKLS (1992)	0,2339	0,0808	0,0854	0,00126	5,1778	0,4345
Sun (1992)	1,1570	0,0520	0,1223	0,00034	8,0496	0,3498
Gibbons & Ramaswamy (1993), I	12,4300	0,0154	0,4900	0,00015	1,5945	0,7673
Gibbons & Ramaswamy (1993), II	14,4477	0,0264	0,5459	0,00027	2,5600	0,6118
Pearson & Sun (1994)	0,8762	0,0311	0,1707	0,00052	1,8704	0,7111
Ait-Sahalia (1996)	0,8922	0,0905	0,1809	0,00166	4,9320	0,4450
Duffie & Singleton (1997), I	0,5440	0,3740	0,0230	0,00018	769,21	0,0008
Duffie & Singleton (1997), II	0,0030	0,2580	0,0190	0,01552	4,2881	0,4764
Bali (1999)	0,0317	0,0642	0,0265	0,00071	5,8147	0,4105

The slope of the forward rate curve is characterized by derivative (13) of forward curve relative to term to maturity. If in expression (13) the value $r = r(t)$ is a random variable then two first moments of a derivative forward curve will be calculated by formulae

$$E\left[\frac{df(\tau|r)}{d\tau}\right] = (1 + vB(\tau))(1 - VB(\tau))\,[k - (V - v) - vVB(\tau)](\theta - x) =$$

$$= -\sqrt{2kD}\ [\lambda + B\sqrt{2kD}\ ]\,(1+vB(\tau))(1-VB(\tau)). \tag{29}$$

$$\mathrm{Var}\left[\frac{df(\tau\,|\,r)}{d\tau}\right] = (1+vB(\tau))^2(1-VB(\tau))^2\,[V - v + vVB(\tau)]^2 D. \tag{30}$$

Two first moments of $\sigma_y^2(\tau|r) = \left(\frac{B(\tau)}{\tau}\right)^2 2kD\,\frac{r-x}{\theta-x}$**,** i.e. the (local) variance of yield on a time unit, are expressed in the form

$$E[\sigma_y^2(\tau|r)] = \left(\frac{B(\tau)}{\tau}\right)^2 2kD. \quad \mathrm{Var}[\sigma_y^2(\tau|r)] = \left(\frac{B(\tau)}{\tau}\right)^4 (2kD)^2\,\frac{D}{(\theta-x)^2}. \tag{31}$$

The covariance of the forward curve derivative and the variance of yield is determined by the formula

$$\mathrm{Cov}\left[\frac{df}{d\tau},\sigma_y^2\right] = -\,(1+vB(\tau))(1-VB(\tau))\,[V - v + vVB(\tau)]\left(\frac{B(\tau)}{\tau}\right)^2 \frac{2kD^2}{\theta-x}.$$

From here it follows that the correlation coefficient of the forward curve derivative and the variance of yield for all terms to maturity τ is equal to the minus unit:

$$\rho = \frac{\mathrm{Cov}[df/d\tau,\sigma_y^2]}{\sqrt{\mathrm{Var}[df/d\tau]\,\mathrm{Var}[\sigma_y^2]}} = -1. \tag{32}$$

It means that these characteristics are linearly connected between themselves. Actually this fact is easy to set. If from formulae (13) and (21) to express value $(r-x)$ and to equate the received results then we shall obtain the following relation that is valid with probability 1 for any τ:

$$\sigma_y^2(\tau|r) = \left(\frac{B(\tau)}{\tau}\right)^2 \frac{[k(1+vB(\tau))(1-VB(\tau))(\theta-x) - df(\tau\,|\,r)/d\tau]2kD}{(1+vB(\tau))(1-VB(\tau))[V - v + 2vVB(\tau)](\theta-x)}. \tag{33}$$

Thus on the basis of the above analysis it is possible to draw the following conclusions:

- the expectation of the derivative of forward rate curve relative to terms to maturities (29) is negative. It means, that in mean the slope of forward rate curve is negative for any terms to maturities τ;
- the derivative of the forward rate curve with probability unit is connected with (local) variance of yield by affine relation (33) for any terms to maturity τ;
- the correlation coefficient of the forward curve derivative and variance of yield (32) for all terms to maturity τ is equal to the minus unit at any values of the parameters having real economic sense.

Generally speaking, the formula (33) can be considered as a basis for determination of the local variance of yield $\sigma_y^2(\tau|r)$ by the forward curve derivative $df(\tau\,|\,r)/d\tau$ if this derivative can be estimated. Let's consider such opportunity. For convenience of reasoning we shall present the formula (33) as:

$$\sigma_y^2(\tau|r) = \xi(\tau)\left(k(\theta-x) - \zeta(\tau)\,\frac{df(\tau\,|\,r)}{d\tau}\right), \tag{34}$$

where

$$\xi(\tau) \equiv \left(\frac{B(\tau)}{\tau}\right)^2 \frac{2kD}{[V - v + 2vVB(\tau)](\theta-x)}, \quad \zeta(\tau) \equiv \frac{1}{(1+vB(\tau))(1-VB(\tau))}.$$

Note that the difference in the right part (34) is always positive as from representation (13) follows that

$$k(\theta-x) - \zeta(\tau)\,\frac{df(\tau\,|\,r)}{d\tau} = \frac{[V - v + 2vVB(\tau)](r-x)}{(1+vB(\tau))(1-VB(\tau))} > 0.$$

At small τ function $B(\tau)$ has representation $B(\tau) = \tau + O(\tau^2)$. Therefore for small τ the formula (33) can be written down as

$$\sigma_y^2(\tau|r) \approx \frac{2kD}{k(\theta-x) + \lambda\sqrt{2kD}}\left(k(\theta-x) - \frac{df(\tau\,|\,r)}{d\tau}\right). \tag{35}$$

At long τ there is representation $VB(\tau) \approx 1 - \varepsilon\, e^{-\varepsilon\tau}/V$ and the formula (34) can be approximately written down as:

$$\sigma_y^2(\tau|r) \approx \frac{2v}{\varepsilon V \tau^2}\left(k(\theta - x) - \frac{V^2 e^{\varepsilon\tau}}{\varepsilon^2}\,\frac{df(\tau\,|\,r)}{d\tau}\right). \tag{36}$$

Thus as it follows from representation (20) at long τ the derivative of forward rate curve exponentially decreases as τ increases, and the factor before a derivative in the formula (36) exponentially increases. It means that even small errors in estimation of the derivative of forward rate curve will be increased "exponentially" by factor facing to it and the calculation of $\sigma_y^2(\tau|r)$ by formula (34) can have the big errors.

Table 2. *Coefficients φ and ψ of formula (37) for some empirical data fitting by the CIR model*

Reference	ε	v	V	φ	ψ
CKLS (1992)	0,23390	1,261E-06	0,23390	4,609E-05	0,018899
Sun (1992)	1,15700	3,363E-07	1,15700	5,024E-07	0,060200
Gibbons & Ramaswamy (1993), I	1,03620	1,487E-07	1,03620	2,769E-07	0,015952
Gibbons & Ramaswamy (1993), II	1,20451	2,721E-07	1,20451	3,751E-07	0,031785
Pearson & Sun (1994)	0,87639	5,17E-07	0,87633	1,347E-06	0,027250
Ait-Sahalia (1996)	0,89219	1,660E-06	0,89219	4,172E-06	0,080742
Duffie & Singleton (1997), I	0,54400	1,818E-07	0,54400	1,228E-06	0,203456
Duffie & Singleton (1997), II	0,00303	1,544E-05	0,00302	3,36133	0,000774
Bali (1999)	0,03170	7,088E-07	0,03170	1,411E-03	0,002035

For the real empirical data that presented in table 1 the values of parameters are such that $V \approx \varepsilon$, and v it is very small. In that case the formula (36) can be rewritten as

$$\sigma_y^2(\tau|r) \approx \frac{\varphi}{\tau^2}\left(\psi - e^{\varepsilon\tau}\,\frac{df(\tau\,|\,r)}{d\tau}\right), \tag{37}$$

where $\varphi = 2v/\varepsilon^2$ and $\psi = k(\theta - x)$. In Table 2 values of these coefficients are presented. We shall remind that empirical data of Tables 1 concern to approximation of the real data in a case when the short term rate is generated by the CIR model with $x = 0$.

In conclusion note that the problem of a prediction of volatility of the long-term yield by an the forward rate curve slope was investigated in Brown and Schaefer (2000) for two factor affine Gaussian models of term structure. The results received in this paper differ from conclusions Brown and Schaefer (2000).

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