STATISTICAL PROPERTIES OF ESTIMATES OF MUTUAL SPECTRAL DENSITIES WITH POLYNOMIAL WINDOW OF DATA VIEWING

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Abstract. The estimation of mutual spectral density with polynomial window of data viewing of stationary stochastic process $X^r(t),\ t \in Z$ with discrete time is constructed. The properties of polynomial window of data viewing are investigated. Moments of this estimation are found; their asymptotic behavior and limit distribution of the estimation are investigated. In this report the rate of mathematical expectancy convergence of modified periodogram with polynomial window of data viewing to the mutual spectral density of stationary stochastic process is investigated, if spectral density satisfies the Holder condition continuous with index α, 0 < α < 1.

Let $X^r(t) = \{X_a(t)\}, a = \overline{1,r},\ t \in Z$, be a strictly stationary process with mean zero, Holder condition continuous with index α, 0 < α ≤ 2 second order mutual spectra $f_{ab}(\lambda),\quad \lambda \in \Pi = [-\pi, \pi],\quad a, b = \overline{1,r}$.

Let $X_a(0), X_a(1),\ldots, X_a(T-1)$ – T sequential observations of the process $X_a(t)$, $t \in Z$, which were made in equal time periods, $a = \overline{1,r}$. We will suppose that the number T of observations of the process $X_a(t)$, $t \in Z$, satisfy $T = r(N-1) + 1$, where $r \in \{1, 2, \ldots\},\ N \in \{1, 2, \ldots\}$.

Let's consider the following statistics as an estimate of the spectral density:

$$I_{N,r}^{ab}(\lambda) = \left(2\pi \sum_{t=0}^{r(N-1)} Q_{N,r}^2(t)\right)^{-1} \left(\sum_{t_1=0}^{r(N-1)} Q_{N,r}(t)\, X_a(t)\, e^{-i\lambda t} \sum_{t_2=0}^{r(N-1)} Q_{N,r}(t)\, X_b(t)\, e^{i\lambda t}\right), \tag{1}$$

which we will call a periodogram, where $\lambda \in \Pi$, function $Q_{N,r}(t)$ is defined as the solution of the equation:

$$\sum_{t=0}^{r(N-1)} Q_{N,r}(t) e^{itx} = \left(\sum_{t=0}^{N-1} e^{itx}\right)^r .$$

Lemma 1. *If the function $Q_{N,r}(t)$, $t \in Z$, is limited by the constant L and have the variation limited by the constant M, then*

$$\left|\sum_{t=0}^{N-1} Q_{N,r}^2(t) e^{-i\lambda t}\right| \le 2LM \left|\sin\frac{\lambda}{2}\right|^{-1} \tag{2}$$

for any λ, $\lambda \in \Pi$ u $\lambda \ne 0 \pmod{2\pi}$, $N, r \in \{1, 2, \ldots\}$.

Lemma 2. *The function*

$$\Phi_{N,r}(x) = H_{N,r}^{-1} \Delta_N^{2r}(x), \tag{3}$$

$r \in \{1, 2, \ldots\}, N \in \{1, 2, \ldots\},\ x \in R$, *where* $\mathrm{H}_{N,r}$ *and* $\Delta_N(x)$ *are defined by equation*

$$\mathrm{H}_{N,r} = 2\pi \sum_{t=0}^{r(N-1)} Q_{N,r}^2(t), \tag{4}$$

$$\Delta_N(x) = \sin\frac{Nx}{2}\left(\sin\frac{x}{2}\right)^{-1}, \tag{5}$$

has the following properties:

1. $\int_{\Pi} \Phi_{N,r}(x)dx = 1$, (6)

2. $\lim_{N\to\infty} \int_{\Pi \setminus [-a,a]} \Phi_{N,r}(x)dx = 0$, (7)

3. $\lim_{N\to\infty} \int_{-a}^{a} \Phi_{N,r}(x)dx = 1$, (8)

where $0<a<\pi$.

It is known [1], that if the spectral density $f_{ab}(y)$ is continuous in a point $y=\lambda$ and is limited on Π, then the statistics $I_{N,r}^{ab}(\lambda)$, set by equation (1), is asymptotic unbiased, but not consistency estimation for spectral density.

We suppose, that the spectral density satisfies to the following inequality:

$$|f_{ab}(y+\lambda)-f_{ab}(\lambda)|\le C|y|^{\gamma},$$

where C - some constant, y, $\lambda\in\Pi$, a $0<\gamma\le 1$. According to [1], we have:

$$\left|MI_{N,r}^{ab}(\lambda)-f_{ab}(\lambda)\right|\le C\int_{-\pi}^{\pi}|y|^{\gamma}\Phi_{N,r}(y)dy,$$

the function $\Phi_{N,r}(y)$ is defined by equation (3).

Let's consider asymptotic properties of the first two moments of the periodogram $I_{N,r}^{ab}(\lambda)$, $\lambda\in\Pi$.

Theorem 1. *The mean of the statistic (1) satisfy*

$$MI_{N,r}^{ab}(\lambda)=\int_{-\pi}^{\pi}\Phi_{N,r}(y)f_{ab}(y+\lambda)\,dy, \tag{9}$$

$\lambda\in\Pi$, $a,b=\overline{1,r}$, *where the function* $\Phi_{N,r}(y)$ *is defined in Lemma 2.*

Theorem 2. *If the spectral density* $f_{ab}(y)$ *is continuous in the point* $y=\lambda$ *and is limited on* Π , *then for the periodogram (1) the following equation is true*

$$\lim_{N\to\infty}MI_{N,r}^{ab}(\lambda)=f_{ab}(\lambda), \tag{10}$$

$\lambda\in\Pi$, $a,b=\overline{1,r}$.

Proof. Using (9) and considering (6), we have

$$MI_{N,r}^{ab}(\lambda)-f_{ab}(\lambda)=\int_{\Pi}\Phi_{N,r}(y)\big(f_{ab}(y+\lambda)-f_{ab}(\lambda)\big)dy.$$

Hence, for any *a*, $0<a\le\pi$,

$$\left|MI_{N,r}^{ab}(\lambda)-f_{ab}(\lambda)\right|\le\int_{-a}^{a}\left|\Phi_{N,r}(y)\right|\left|\big(f_{ab}(y+\lambda)-f_{ab}(\lambda)\big)\right|dy+$$
$$+\int_{a<|y|\le\pi}\left|\Phi_{N,r}(y)\right|\left|\big(f_{ab}(y+\lambda)-f_{ab}(\lambda)\big)\right|dy=I_1+I_2.$$

As the function $f_{ab}(y)$ is continuous in the point $y=\lambda$, so for any $\varepsilon>0$ exists *a* > 0, that for $|y|\le a$, $|f_{ab}(y+\lambda)-f_{ab}(\lambda)|\le\varepsilon$. Hence, considering properties of Fejers kernel (6), we have

$$I_1\le\varepsilon\int_{-a}^{a}\Phi_{N,r}(y)dy\le\varepsilon\int_{\Pi}\Phi_{N,r}(y)dy=\varepsilon.$$

Hence, we can make the left part of the last inequality much smaller with the help of choosing *ε.*
Let's consider I_2 . As the spectral density $f_{ab}(y)$ is limited on Π , so

$$I_2\le 2\max_{y\in\Pi}f_{ab}(y)\int_{a<|y|\le\pi}\left|\Phi_{N,r}(y)\right|dy.$$

Using the property of Fejers kernel (7), we have $\lim_{N\to\infty}I_2=0$. The theorem is proved.

Lemma3. *If the spectral density* $f_{ab}(x)$ *is continuous in the point* $x=\lambda_1$ *and is limited on* Π , *data window* $Q_{N,r}(t)$, $t\in R$, *is limited by the constant L and has the variation limited by the constant M, then the following equations are true*

$$\lim_{N\to\infty}\int_{\Pi}\Phi_{N,r}(x-\lambda_1,x+\lambda_2)f_{ab}(x)dx=\begin{cases}f_{ab}(\lambda_1), & if\ \lambda_1+\lambda_2=0(\mathrm{mod}\ 2\pi)\\ 0, & if\ \lambda_1+\lambda_2\ne 0(\mathrm{mod}\ 2\pi)\end{cases} \tag{11}$$

$$\lim_{N\to\infty}\int_{\Pi}\Phi_{N,r}(x-\lambda_1,x-\lambda_2)f_{ab}(x)dx=\begin{cases}f_{ab}(\lambda_1), & if\ \lambda_1-\lambda_2=0(\text{mod }2\pi)\\ 0, & if\ \lambda_1-\lambda_2\neq 0(\text{mod }2\pi)\end{cases} \quad (12)$$

where $\lambda_1,\lambda_2\in\Pi$, $a,b=\overline{1,r}$.

Theorem 3. *For any points λ_1, $\lambda_2\in\Pi$ the covariation of the periodogram (1) satisfy*

$$\operatorname{cov}\left\{I_{N,r}^{a_1b_1}(\lambda_1),I_{N,r}^{a_2b_2}(\lambda_2)\right\}= \quad (13)$$

$$=\frac{(2\pi)^3\sum_{t=0}^{r(N-1)}Q_{N,r}^4(t)}{(H_{N,r})^2}\iiint_{\Pi^3}\Phi_{N,r}(y_1,y_2,y_3)f_{a_1b_1a_2b_2}(y_1+\lambda_1,y_2-\lambda_1,y_3+\lambda_2)dy_1\,dy_2\,dy_3+$$

$$+\int_{\Pi}\Phi_{N,r}(x-\lambda_1,x+\lambda_2)f_{a_1a_2}(x)dx\int_{\Pi}\Phi_{N,r}(x-\lambda_1,x+\lambda_2)f_{b_1b_2}(x)dx+$$

$$+\int_{\Pi}\Phi_{N,r}(x-\lambda_1,x-\lambda_2)f_{a_1b_2}(x)dx\int_{\Pi}\Phi_{N,r}(x-\lambda_1,x-\lambda_2)f_{b_1a_2}(x)dx\ ,$$

where function $f_{a_1b_1a_2b_2}(y_1,y_2,y_3)$ *is the semi-invariant spectral density of the forth order,*

$$\Phi_{N,r}(y_1,y_2,y_3)=\frac{1}{(2\pi)^3\sum_{t=0}^{r(N-1)}Q_{N,r}^4(t)}\Delta_N^r(y_1)\Delta_N^r(y_2)\Delta_N^r(y_3)\Delta_N^r(y_1+y_2+y_3), \quad (14)$$

$$\Phi_{N,r}(y_1,y_2)=H_{N,r}^{-1}\Delta_N^r(y_1)\Delta_N^r(y_2), \quad (15)$$

the function $\mathrm{H}_{N,r}$ *is defined by* (4), *and* $\Delta_N(x)$ *by the equality* (5), y_1,y_2,y_3, λ_1, $\lambda_2\in\Pi$, $a_1,b_1,a_2,b_2=\overline{1,r}$.

Theorem 4. *If the spectral density $f_{ab}(x)$ is continuous in the point $x=\lambda$ and is limited on Π, the semi-invariant spectral density of the forth order is limited on Π^3, the equation is true*

$$\sup_N\iiint_{\Pi^3}\left|\Phi_{N,r}(y_1,y_2,y_3)\right|dy_1dy_2dy_3\le C\ , \quad (16)$$

where some positive constant C is in depended on N, then

$$\lim_{N\to\infty}\operatorname{cov}\left\{I_{N,r}^{a_1b_1}(\lambda_1),I_{N,r}^{a_2b_2}(\lambda_2)\right\}=\begin{cases}0, & if\ \lambda_1\pm\lambda_2\neq 0(\text{mod }2\pi)\\ f_{a_1a_2}(\lambda_1)f_{b_1b_2}(\lambda_1), & if\ \lambda_1+\lambda_2=0(\text{mod }2\pi),\lambda_1,\lambda_2\neq 0(\text{mod }\pi)\\ f_{a_1b_2}(\lambda_1)f_{b_1a_2}(\lambda_1), & if\ \lambda_1-\lambda_2=0(\text{mod }2\pi),\lambda_1,\lambda_2\neq 0(\text{mod }\pi)\\ f_{a_1a_2}(0)f_{b_1b_2}(0)+f_{a_1b_2}(0)f_{b_1a_2}(0), & if\ \lambda_1=\lambda_2=0(\text{mod }\pi)\end{cases},$$

λ_1, $\lambda_2\in\Pi$, $a_1,b_1,a_2,b_2=\overline{1,r}$.

Proof. According to (13), we denote every item as A_1, A_2, A_3. Then

$$|A_1|\le\frac{(2\pi)^3\sum_{t=0}^{r(N-1)}Q_{N,r}^4(t)}{(H_{N,r})^2}\iiint_{\Pi^3}\left|\Phi_{N,r}(y_1,y_2,y_3)\right|\left|f_4(y_1+\lambda_1,y_2-\lambda_1,y_3+\lambda_2)\right|dy_1\,dy_2\,dy_3$$

We suppose, that the semi-invariant spectral density of the forth order is limited by the constant *F*, and the condition (16) is true, we have

$$|A_1|\le\frac{(2\pi)^3\sum_{t=0}^{r(N-1)}Q_{N,r}^4(t)}{(H_{N,r})^2}FC\underset{N\to\infty}{\to}0\,.$$

The proof for A_2 and A_3 follows from Lemma3. The theorem is proved.

Theorem 5. *If the conditions of the theorem 4 are true, then the following equation is true*

$$\lim_{N\to\infty}DI_{N,r}^{ab}(\lambda)=\begin{cases}f_{aa}(\lambda)f_{bb}(\lambda), & if\ \lambda\neq 0(\text{mod }\pi)\\ f_{aa}(0)f_{bb}(0)+f_{ab}(0)f_{ba}(0), & if\ \lambda=0(\text{mod }\pi)\end{cases},$$

where $\lambda\in\Pi$, $a,b=\overline{1,r}$.

The proof follows from the theorem 4, supposing, $\lambda_1 = \lambda_2 = \lambda$, $a_1 = a_2 = a$, $b_1 = b_2 = b$.

Some questions of the estimation of the spectral densities are considered in articles [2], [3].

References:

[1]. Troush N.N. Asymptotic methods of statistic analysis of time series. — Minsk: BSU, 1999.

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