ABOUT STATISTICAL ESTIMATOR OF PULSE TRANSFER FUNCTION FOR LINEAR SYSTEMS

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Abstract: In this paper, the statistical estimation of pulse transfer function of linear system is entered. The behaviour of an error estimate is studied. Conditions at which takes place asymptotic normality of finite dimensional distributions of the normalized error of an estimation of pulse transfer function are established.

Let physically feasible time homogeneous linear system with pulse transfer function $H(\tau), \tau \in R = (-\infty, \infty)$ is set. It means, that real valued function H satisfies with a condition $H(\tau) = 0, \tau < 0,$ and reaction of system to an allowable entrance signal $x(t), t \in R$ looks like

$$y(t) = \int_0^\infty H(\tau)x(t-\tau)d\tau = \int_{-\infty}^t H(t-s)x(s)ds.$$

One of research problems of the theory of linear systems is how to have an estimate or identification of function H if the obvious form of it is unknown, on supervision over reaction of system on some signal. In this paper, the statistical estimation of pulse transfer function of linear system is entered. The behaviour of an error estimate is studied. Conditions at which takes place asymptotic normality of finite dimensional distributions of the normalized error of an estimation of pulse transfer function are established.

Let $g_\Delta = (g_\Delta(x), x \in R), \Delta \in (0, \infty)$ is the real valued non-negative function family, satisfying following conditions:

a) $g_\Delta(x) = g_\Delta(-x), x \in R$; b) $g_\Delta \in L_1(R)$; c) $g_\Delta \in L_\infty(R)$ and $M = \sup_\Delta \|g_\Delta\|_\infty < \infty \ j = 1,2\} < \infty$,

where $L_\infty(R)$ is a space of the limited functions with norm $\|g_\Delta\|_\infty = \sup_{x \in R} |g_\Delta(x)|$; d) exists $c \in (0, \infty)$, for all $x \in R$, $\lim_{\Delta \to \infty} g_\Delta(x) = \frac{c}{2\pi}$, moreover, for anyone $a \in (0, \infty)$, $\lim_{\Delta \to \infty} \sup_{-a \le x \le a} \left| g_\Delta(x) - \frac{c}{2\pi} \right| = 0$; e) for someone $\beta \ge 0$, $B_\Delta \in L_{1+\beta}(R),$ where $B_\Delta(t) = \int_{-\infty}^{\infty} e^{itx} g_\Delta(x)dx, \ t \in R$. (1)

From (1b) and (1c), we get $g_\Delta \in L_2(R),$ so $B_\Delta \in L_2(R)$. It is supposed that condition (1) is existential during the article.

Let $X_\Delta = (X_\Delta(t), t \in R)$ is measurable central Gauss processes of stationary with real value, their spectral density is g_Δ, correlation function is B_Δ. Processes $X_\Delta, \Delta > 0$ will serve as the processes revolting linear system. From condition (1b), that process X_Δ is mean-square continuous.

Now let's consider random process $Y_\Delta(t) = \int_0^\infty H(s)X_\Delta(t-s)ds, t \in R,$ being the response of system to input signal X_Δ.

Let $\tau \geq 0$, as an estimate for $H(\tau)$, we consider a random variable

$$\hat{H}_{T,\Delta}(\tau) = (cT)^{-1} \int_0^T X_\Delta(t) Y_\Delta(t+\tau) dt. \tag{2}$$

Its mathematical expectation looks like

$$E\hat{H}_{T,\Delta}(\tau) = c^{-1} \int_0^\infty B_\Delta(\tau - s) H(s) ds. \tag{3}$$

clearly , that $E\hat{H}_{T,\Delta}(\tau) \neq H(\tau)$, i.e. the estimation is displaced.

Let $Z_{T,\Delta}(\tau) = \sqrt{T}[\hat{H}_{T,\Delta}(\tau) - E\hat{H}_{T,\Delta}(\tau)],$ (4)

Lemma 1. Let for someone $\beta \geq 0$, $B_\Delta \in L_{1+\beta}(R)$, and $H \in L_{\frac{2+2\beta}{1+2\beta}} \cap L_2(R)$. For all $\tau_1, \tau_2 \geq 0$, we get the following equation:

$$EZ_{T,\Delta}(\tau_1) Z_{T,\Delta}(\tau_2) = C_{T,\Delta}(\tau_1, \tau_2), \tag{5}$$

where

$$C_{T,\Delta}(\tau_1,\tau_2) = \frac{2\pi}{c^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[e^{i(\tau_2 - \tau_1)u_1} \left| H^*(u_1) \right|^2 + e^{i(\tau_1 u_1 + \tau_2 u_2)} H^*(u_1) H^*(u_2) \right] \Phi_T(u_2 - u_1) g_\Delta(u_1) g_\Delta(u_2) du_1 du_2; \tag{6}$$

$\Phi(x) = \dfrac{1}{2\pi T} \left(\dfrac{\sin(Tx/2)}{x/2} \right)^2, \quad x \in R.$ where c is a constant from a condition (1d).

The proof is curtailed.

Let's put

$$C(\tau_1, \tau_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[e^{i(\tau - \tau_1)u} \left| H^*(u) \right|^2 + e^{i(\tau_1 + \tau_2)u} \left(H^*(u) \right)^2 \right] du, \quad k, j = 1,2. \tau_1, \tau_2 \geq 0. \tag{7}$$

Lemma 2. Let for someone $\beta \geq 0$, $B_\Delta \in L_{1+\beta}(R)$, and 1) $H \in L_{\frac{2+2\beta}{1+2\beta}} \cap L_2(R)$, 2) exists such $p > 1$, that $H^* \in L_{2p}(R)$, 3) function H^* is continuous almost everywhere on R. Then for all $\tau_1, \tau_2 \geq 0$, we get:

$$\lim_{\substack{T \to \infty \\ \Delta \to \infty}} Z_{T,\Delta}(\tau_1) Z_{T,\Delta}(\tau_2) = C(\tau_1, \tau_2). \tag{8}$$

The proof is similar to a lemma 4 in work [2] and consequently curtailed.

Let $(Z(\tau), \tau \geq 0)$ be a central Gauss process with the correlation function $C(\tau_1, \tau_2)$. Mark $Z_{T,\Delta} \Rightarrow Z$ means that, for $T \to \infty, \Delta \to \infty$, the finite dimensional distributions of process $(Z_{T,\Delta}(\tau), \tau \geq 0)$ convergence to corresponding finite dimensional distribution of process Z.

Lemma 3. For any natural number $n \geq 2$ and anyone $\tau_1, \tau_2, \cdots, \tau_n \in [0, \infty)$, then it is obtained as

$$E[\prod_{k=1}^{n} Z_{T,\Delta}(\tau_k)\} = \sum_{D_1,\cdots,D_n} (Tc^2)^{-n/2} \int_0^T \leftrightarrow \int_0^T [\prod_{k=1}^{n} \operatorname{cov} D_k] dt_1 \cdots dt_n \ , \tag{9}$$

where c is a constant from a condition (1d).

$$D = \begin{pmatrix} X_\Delta(t_1) & Y_\Delta(t_1+\tau_1) \\ X_\Delta(t_2) & Y_\Delta(t_2+\tau_2) \\ \vdots & \vdots \\ X_\Delta(t_n) & Y_\Delta(t_n+\tau_n) \end{pmatrix}$$

The sum of (9) is calculated on all disorder partition of $(D_1,\cdots,D_n)$. $D_k, k=1,2,\cdots,n$, is a crossed set of two elements in different line, and $D_k = \{\xi,\eta\}$, if $\operatorname{cov} D_k = E\xi\eta$.

The proof: Using Fubini-Tonelli theorem, the expression (9) can be gained by equality (3) and Леонов-ширяева formula [4].

Theorem 1. Let for $\beta \ge 0$, $B_\Delta \in L_{1+\beta}(R)$, and 1) $H \in L_{\frac{2+2\beta}{1+2\beta}} \cap L_2(R)$; 2) $H^* \in L_1(R) \cap L_\infty(R)$; 3) function H^* is continuous almost everywhere on *R*. Then for all $\tau_1,\tau_2,\cdots,\tau_n \in [0,\infty), n \ge 1$

$$\lim_{\substack{T\to\infty \\ \Delta\to\infty}} E\left[\prod_{k=1}^{n} Z_{T,\Delta}(\tau_k)\right] = E\left[\prod_{k=1}^{n} Z(\tau_k)\right] \tag{10}$$

$$Z_{T,\Delta} \Rightarrow Z \tag{11}$$

By lemma 1,lemma 2 and lemma 3, we can prove the theorem using the way in the [2].

The proof: As processes $Z_{T,\Delta}$ have zero average, so the equality (10) is executed at $n=1$. At $n=2$ this equality is established in a lemma 2. Therefore further $n \ge 3$.

Following the accepted terminological word, say, that elements D_{k_1}, D_{k_2} form the simple block if their association consists of two any lines of the table D. Accordingly, splitting $D_1,\cdots,D_n$ of the table D we shall name simple if its elements can be broken on the pairs forming simple blocks. Clearly, that simple splitting can take place only for even n. The splitting $D_1,\cdots,D_n$ which is not being simple, we shall name complex.

Let's break the sum of the right part of equality (9) into the sum $\sum' = \sum'_{T,\Delta}(\tau_1,\cdots,\tau_n)$ on simple splitting and the sum $\sum'' = \sum''_{T,\Delta}(\tau_1,\cdots,\tau_n)$ on complex splitting. So it is necessary to prove: that for any $n \ge 3, \tau_1,\cdots,\tau_n \in [0,\infty)$

$$\text{a)} \lim_{\substack{T\to\infty \\ \Delta\to\infty}} \sum' = E[\prod_{k=1}^{n} Z(\tau_k)]; \qquad \text{b)} \lim_{\substack{T\to\infty \\ \Delta\to\infty}} \sum'' = 0 . \tag{12}$$

At first let's prove (12a). From above known, number $n \ge 3$ and even. If the pair D_{k_1}, D_{k_2} elements of splitting form the simple block, there will be such $j, p \in \{1, \cdots, n\}$, such that

$$D_{k_1} = \{X_\Delta(t_j), X_\Delta(t_p)\}; \quad D_{k_2} = \{Y_\Delta(t_j+\tau_j), Y_\Delta(t_p+\tau_p)\}, \qquad \text{or} \qquad D_{k_1} = \{X_\Delta(t_j), Y_\Delta(t_p+\tau_p)\}; \quad D_{k_2} = \{X_\Delta(t_p), Y_\Delta(t_j+\tau_j)\}.$$

Let $D^{(1)}_{k_1}, D^{(1)}_{k_2}$ express first case above, and $D^{(2)}_{k_1}, D^{(2)}_{k_2}$ express second type. Clearly, that

$$\sum{}' = \sum_{\{k_1,k_2\}\cdots\{k_{n-1},k_n\}} (Tc^2)^{-n/2} \int_0^T \overset{n}{\longleftrightarrow} \int_0^T \prod_{\substack{j=1 \\ j-uneven}}^{n-1} \left[\operatorname{cov} D^{(1)}_{k_j} \operatorname{cov} D^{(1)}_{k_{j+1}} + \operatorname{cov} D^{(2)}_{k_j} \operatorname{cov} D^{(2)}_{k_{j+1}}\right] dt_1 \cdots dt_n ,$$

where the sum undertakes on all disorder splitting set $\{1, \cdots, n\}$ into not crossed two-element subsets $\{k_1, k_2\}, \cdots, \{k_{n-1}, k_n\}$. As (see lemma 2)

$$\operatorname{cov} D^{(1)}_{k_j} \operatorname{cov} D^{(1)}_{k_{j+1}} + \operatorname{cov} D^{(2)}_{k_j} \operatorname{cov} D^{(2)}_{k_{j+1}}$$

$$= B_{X_\Delta X_\Delta}(t_{k_{j+1}} - t_{k_j}) B_{Y_\Delta Y_\Delta}(t_{k_{j+1}} - t_{k_j} + \tau_{k_{j+1}} - \tau_{k_j}) + B_{X_\Delta Y_\Delta}(t_{k_{j+1}} - t_{k_j} + \tau_{k_{j+1}}) + B_{X_\Delta Y_\Delta}(t_{k_j} - t_{k_{j+1}} + \tau_{k_j})$$

$$= L(t_{k_j}, t_{k_{j+1}}) ,$$

that
$$\sum{}' = \sum_{\{k_1,k_2\}\cdots\{k_{n-1},k_n\}} \prod_{\substack{j=1 \\ j-unevn}}^{n-1} \frac{1}{Tc^2} \int_0^T \int_0^T L(t_{k_j}, t_{k_{j+1}}) dt_{k_j} dt_{k_{j+1}}$$

and
$$\frac{1}{Tc^2} \int_0^T \int_0^T L(t_{k_j}, t_{k_{j+1}}) dt_{k_j} dt_{k_{j+1}} = EZ_{T,\Delta}(\tau_{k_j}) Z_{T,\Delta}(\tau_{k_{j+1}}) = C_{T,\Delta}(\tau_{k_{lj}}, \tau_{k_{j+1}}).$$

From here, valid a lemma 3 and Leonov-Shirjaev formula (see also [4]) can yield equality (12a).

Let's pass to the proof of the equality (12b). We know that each complex splitting can be presented as association of final number of indecomposable blocks. The minimal indecomposable block is the simple block. Hence at representation of complex splitting always there is the indecomposable block containing not less of three lines of the table D. We shall notice, that having rearranged in appropriate way lines which form the indecomposable block, it is possible to achieve that elements of this block connect the first line to the second, the second with the third, etc., and last line incorporates to the first. Accordingly the sum $\sum{}''$ can be presented as

$$\sum{}'' = \sum{}'' \prod_{p=1}^{d} I^{(m_p)}_{T,\Delta} \tag{13}$$

where d represents the numbers of the indecomposable block of complex splitting, $m_p \geq 2$, and always will be such p, that $m_p \geq 3$, and $I^{(m_p)}_{T,\Delta}$ integrals of a kind $I^{(m)}_{T,\Delta} = (Tc^2)^{-m/2} \int_0^T \overset{m}{\longleftrightarrow} \int_0^T \prod_{j=1}^{m} [\operatorname{cov} D_j] dt_1 \cdots dt_m$.

At $m_p = 2$, $\sup_{T,\Delta} | I_{T,\Delta}^{(2)} | \le \frac{2\pi M^2}{c^2} \left\| H^* \right\|_2^2 < \infty$. (14)

At $m \ge 3$

$$I_{T,\Delta}^{(m)} = \int_{-\infty}^{\infty} \xleftarrow{m}\!\!\!\!\rightarrow \int_{-\infty}^{\infty} U_T(x_1 - x_2) U_T(x_2 - x_3) \cdots U_T(x_{m-1} - x_m) U_T(x_m - x_1)$$
$$\cdot \prod_{k=1}^{m} e^{i\alpha_k x_k} g_1(x_1) \cdots g_m(x_m) dx_1 \cdots dx_m \tag{15}$$

where $U_T(x) = \sqrt{\frac{2}{\pi T}} \frac{\sin \frac{Tx}{2}}{x}, x \in R$; α_k it is leveled 0 or $\pm\tau_k$ or $\tau_{k+1} - \tau_k$, $g_k(x)$ is $g_\Delta(x)$ or $\left| H^*(x) \right|^2 g_\Delta(x)$, or $H^*(x) g_\Delta(x)$ depending on concrete structure of the indecomposable block $\{D_1, \cdots, D_m\}$. And always there are even two various such as these functions. As, on a condition (1b) and a condition 2) in the theorem, functions $\sup_\Delta g_\Delta$, H^* are limited and $H^* \in L_1(R) \cap L_2(R)$, follows that (see [1]) $\lim_{T\to\infty} \sup_\Delta | I_{T,\Delta}^{(m)} | = 0$. From here and equality (13), (14) can yield the equality (12b) . The theorem is proved.

Theorem 2. Let for $\beta \ge 0$, $B_\Delta \in L_{1+\beta}(R)$, and 1) $H \in L_{\frac{2+2\beta}{1+2\beta}} \cap L_2(R)$, 2) exists $p > 2$, that $H^* \in L_{2p}(R)$, 3) function H^* is continuous almost everywhere on R, 4) $T \to \infty, \Delta \to \infty$, so that for any $m \ge 3$

$$\frac{\left\| g_\Delta \right\|_2^{a(m)} \left\| g_\Delta \right\|_p^{b(m)}}{T^{\frac{(m-2)(p-2)}{2p}}} \to 0 \tag{16}$$

where

$$a(m) = \begin{cases} 1, & if\ \ m \ge 4; \\ 0, & if\ \ m = 3, \end{cases} \quad ; \qquad b(m) = \begin{cases} ext(\frac{m}{2}) - 1, & if\ \ m \ge 4; \\ 1, & if\ \ m = 3. \end{cases}$$

Then equality (10) and (11) are established.

The proof is similar to the proof of the theorem 1.

References

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