REAL SOLUTIONS IN A SECOND DEGREE EQUATION WITH FUZZY COEFFICIENTS. ECONOMICAL APPLICATIONS

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Abstract. Using mathematical techniques in the analysis of economic functions, generally polynomial, we obtain certain equations that may have as solutions imaginary or complex numbers. Unfortunately, the significance of this kind of solutions is the loss of some function properties, such as convexity, properties that are essential for the conservation of his economic meaning and his applications to the business world.

This first work is based on the analysis of a certain "approximation's matrix", which allow us the deduction of the minimal supports that must have the fuzzy coefficients of a second degree equation with the purpose that all his solutions are real numbers.

1. Second degree equation with real coefficients

Let a polynomial function, for example the cubic $f(x)=a\cdot x^3+b\cdot x^2+c\cdot x+d$, were $a,b,c,d\in R$, and suppose we wish, for economic reasons, that this function must have always a maximum and a minimum. Consequently his derivative function, that we can put in the simplified form $f'(x)=x^2-m\cdot x+n$, shall be equal to zero, $x^2-m\cdot x+n=0$, and furthermore his two roots or solutions x and x' must be two real numbers.

Our aim in this work is to begin with the second degree equation:

$$x^2-m\cdot x+n=0 \qquad [1]$$

where the coefficients m and n are two real positive crisp numbers, that's to say $m,n\in R^+$. We know that the coefficients m and n are respectively the addition and the product of the two roots x and x':

$$m=x+x' \quad \text{and} \quad n=x\cdot x' \qquad [2]$$

being these roots of [1] expressed by:

$$x=\frac{m+\sqrt{m^2-4\cdot n}}{2} \quad \text{and} \quad x'=\frac{m-\sqrt{m^2-4\cdot n}}{2} \qquad [3]$$

As we have said it's necessary, for economic reasons, that x and x' shall be real and positive, $x,x'\in R^+$. Therefore, if we don't wish complex roots, must be positive the discriminant $m^2-4\cdot n$. In other words:

$$m^2-4\cdot n>0 \qquad [4]$$

2. Fuzzy real coefficients in a second degree equation

Since the great majority of the functions applied to the Economic Sciences are deduced by statistics observing the data of previous periods (months, years, etc.) and comparing these information with the data of another similar enterprises, we obtain polynomial functions with fuzzy coefficients that can be represented efficiently by means of Triangular fuzzy numbers. And, logically, the equation $x^2-m\cdot x+n=0$ must be generalized in another of fuzzy coefficients.

We represent these coefficients by the two Triangular fuzzy numbers:

$$\tilde{m}=(m_1,m_2,m_3) \quad \text{and} \quad \tilde{n}=(n_1,n_2,n_3) \qquad [5]$$

where the centers m_2 and n_2 are precisely two coefficients crisp m and n of the second degree equation:

$$m_2=m \quad \text{and} \quad n_2=n \qquad [6]$$

If we choose m_i and n_j inside the support of the two fuzzy coefficients, $m_1\leq m_i\leq m_3$ and $n_1\leq n_j\leq n_3$, then we shall have a double infinity of second degree equations:

$$x2-mi\cdot x+nj=0 \quad \text{or} \quad x2=mi\cdot x-nj \qquad [7]$$

Since we consider very important the approximation of the m_i and n_j to the centers m_2 and n_2 of the fuzzy coefficients, we can summarize all these double infinity equation's family by:

$$\tilde{x}^2=\tilde{m}(\cdot)\tilde{x}(-)\tilde{n} \qquad [8]$$

Depending of the discriminant value:

$$m_i^2 - 4 \cdot n_j \geq 0 \quad \text{or} \quad m_i^2 - 4 \cdot n_j < 0 \qquad [9]$$

the roots of [8] will be real or complex numbers.

Our purpose is to gradually reduce the supports, $[m_1, m_3]$ and $[n_1, n_3]$, of the Triangular fuzzy coefficients, as we can see in the *Fig.1*. Therefore we shall increase the value of the alpha-cuts, $0 \leq \alpha \leq 1$. Since we also consider the centers m_2 and n_2 of the supports, we obtain news and more thin Triangular fuzzy numbers:

$$\tilde{m}(\alpha) = (m_1(\alpha), m_2, m_3(\alpha)) \quad \text{and} \quad \tilde{n}(\alpha) = (n_1(\alpha), n_2, n_3(\alpha)) \qquad [10]$$

where the extreme of the supports are given by:

$$m_1(\alpha) = m_1 + (m_2 - m_1) \cdot \alpha \quad , \quad m_3(\alpha) = m_3 - (m_3 - m_2) \cdot \alpha \qquad [11]$$

$$n_1(\alpha) = n_1 + (n_2 - n_1) \cdot \alpha \quad , \quad n_3(\alpha) = n_3 - (n_3 - n_2) \cdot \alpha \qquad [12]$$

It's easy to see that the two limits of the α-cuts, α=0 and α=1, verifies:

$$\tilde{m}(0) = \tilde{m} \quad , \quad \tilde{n}(0) = \tilde{n} \quad , \quad \tilde{m}(1) = m \quad \text{and} \quad \tilde{n}(1) = n \qquad [13]$$

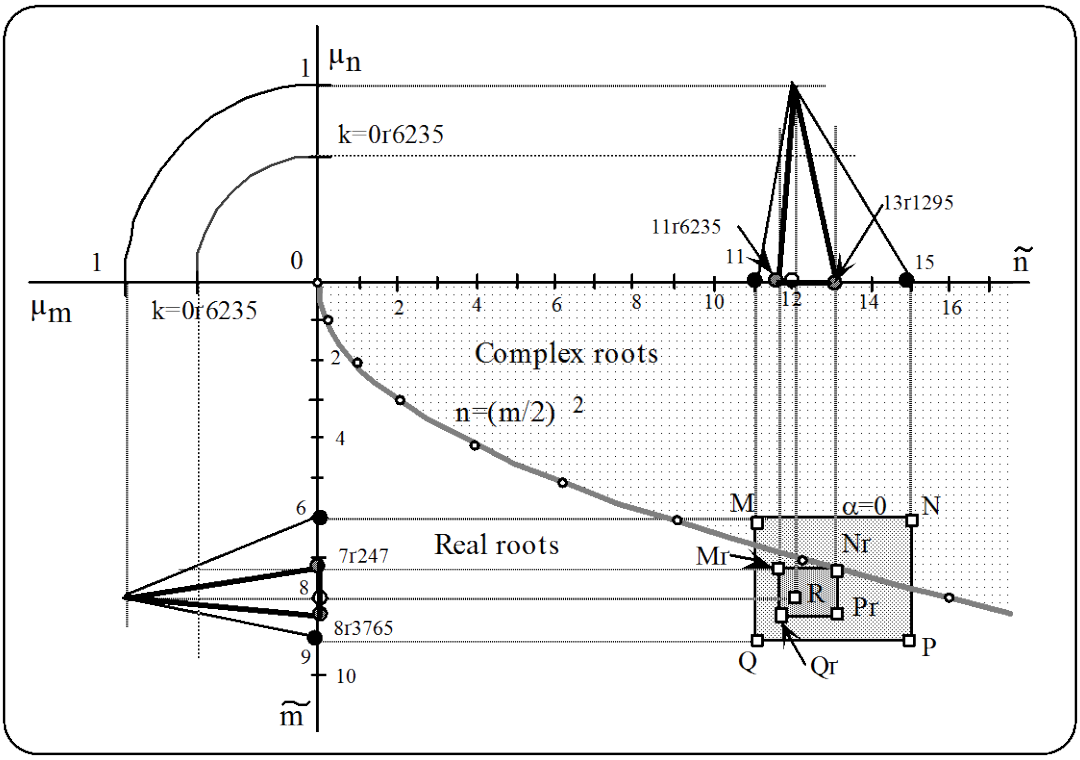

Fig. 1

Afterwards we wish to calculate the minimum value k of the alpha-cuts with the aim that all the solutions of the fuzzy equation are real numbers, that's to say there will be no complex root.

In this paper we suppose that the equation [8] has the signify of the double infinity of equations $x^2 - m_i \cdot x + n_j = 0$, with the coefficients verifying that $m_i \in [m_1, m_3]$ and $n_j \in [n_1, n_3]$.

3. Approximation matrix

To begin with we will use the equations [10], [11] and [12]. Then with the two triplets $(m_1(\alpha)\ m_2, m_3(\alpha))$ and $(n_1(\alpha), n_2, n_3(\alpha))$ we shall form the "product set" of 9 elements:

$$P = \{(m_1(\alpha), n_1(\alpha)), (m_1(\alpha), n_2), \ldots, (m_3(\alpha), n_3(\alpha))\} \qquad [14]$$

Five of these nine pairs of elements, corresponding α=0 and α=1, are indicated in *Fig.1* by means of the little rectangles M, N, P, Q and R. We observe that the last point R is the pair crisp (m, n). Consequently the nine second degree equations will be:

$$x^2 - m_i(\alpha)\cdot x + n_j(\alpha) = 0 \quad (i,j=1,2,3) \qquad [15]$$

These nine equations have for solutions:

$$x_{ij}(\alpha) = \frac{m_i(\alpha) + \sqrt{[m_i(\alpha)]^2 - 4\cdot n_j(\alpha)}}{2} \quad \text{and} \quad x'_{ij}(\alpha) = \frac{m_i(\alpha) - \sqrt{[m_i(\alpha)]^2 - 4\cdot n_j(\alpha)}}{2} \qquad [16]$$

With the finality to observe the approximation of the roots $x_{ij}(\alpha)$ and $x'_{ij}(\alpha)$ to the crisp solutions x and x' of the crisp equation x^2-m·x+n=0, we use a matrices whose elements are the roots [16] of the fuzzy equations. This kind of matrices, that we call "Approximation's matrices" will depend of α.

According to the positive or negative sign of [16] we obtain two Approximation's matrices. The first, X(α),approximates to the solution crisp x, and the second, X'(α), to the another solution crisp x':

$$X(\alpha) = \begin{pmatrix} x_{11}(\alpha) & x_{12}(\alpha) & x_{13}(\alpha) \\ x_{21}(\alpha) & x_{22}(\alpha) & x_{23}(\alpha) \\ x_{31}(\alpha) & x_{32}(\alpha) & x_{33}(\alpha) \end{pmatrix} \quad \text{and} \quad X'(\alpha) = \begin{pmatrix} x'_{11}(\alpha) & x'_{12}(\alpha) & x'_{13}(\alpha) \\ x'_{21}(\alpha) & x'_{22}(\alpha) & x'_{23}(\alpha) \\ x'_{31}(\alpha) & x'_{32}(\alpha) & x'_{33}(\alpha) \end{pmatrix} \qquad [17]$$

Since the fuzzy solutions of [16] verifies that $x_{ij}(\alpha)+x'_{ij}(\alpha)=m_i(\alpha)$, the X'($\alpha$) can be deduced from X($\alpha$), and we can simplify further operations using only the first approximation matrix, X(α).

4. Obtaining the minimum value of the alpha-cut

The discriminant of the second degree equation [8] with the news coefficients $\tilde{m}(\alpha)$ and $\tilde{n}(\alpha)$, which is equivalent to [15], can be expressed by:

$$\tilde{D}(\alpha) = [\tilde{m}(\alpha)]^2 - 4\cdot\tilde{n}(\alpha) \qquad [18]$$

where $\tilde{m}(\alpha) = (m_1(\alpha), m_2, m_3(\alpha))$ and $\tilde{n}(\alpha) = (n_1(\alpha), n_2, n_3(\alpha))$.

The minimum value of the fuzzy discriminant $\tilde{D}(\alpha)$, which we denote by D(α), is assumed when $\tilde{m}(\alpha)$ has his smaller value and when $\tilde{n}(\alpha)$ has his greater value:

$$D(\alpha) = [m_1(\alpha)]^2 - 4\cdot n_3(\alpha) \qquad [19]$$

Furthermore, since we wish to find the frontier value k of the alpha-cut α that is bordering the complex roots from the real roots, this discriminant must be zero. Then, $[m_1(k)]^2=4\cdot n_3(k)$, and putting it in function of the extreme and center of the two triangular fuzzy coefficients (m_1, m_2, m_3) and (n_1, n_2, n_3) we obtain the important relationship:

$$\boxed{[m_1 + (m_2 - m_1)\cdot k]^2 = 4\cdot[n_3 + (n_2 - n_3)\cdot k]} \qquad [20]$$

5. An easy application to the economic field

Let it be the Benefits function $f(x)=x^3-12\cdot x^2+36\cdot x+158$, where x is the number of units produced that we suppose is $x\leq 7$, and f(x) is the number of monetary units. Since his derivative is $f'(x)=3x^2-24x+36=3\cdot(x^2-8x+12)$ and we wish know his optimal points, f'(x)=0, we obtain the second degree equation $x^2-8x+12=0$. His roots are x=6 and x'=2, and since f(x)=50 and f(x')=190 the benefits are a minimum at P(6,50) and a maximum at P'(2,190).

If we wish to extend this analysis to the fuzzy field we can enlarge the crisp coefficients m=8 and n=12 of the equation $x^2-8x+12=0$ by means, for example, of the two Triangular fuzzy numbers:

$$\tilde{m} = (6,8,9) \quad \text{and} \quad \tilde{n} = (11,12,15)$$

Then the fuzzy equation [8] will have a double infinity of real or imaginary solutions.

The reduced support of the new Triangular fuzzy coefficients, which depends of the alpha-cuts, formed with the extreme and the center are:

$$\tilde{m}(\alpha) = (6 + (8-6)\cdot\alpha, 8, 9 + (8-9)\cdot\alpha) = (6 + 2\cdot\alpha, 8, 9 - \alpha)$$

$$\tilde{n}(\alpha) = (11 + (12-11)\cdot\alpha, 12, 15 + (12-15)\cdot\alpha) = (11 + \alpha, 12, 15 - 3\alpha)$$

The next operation is calculate the roots $x_{ij}(\alpha)$ and $x'_{ij}(\alpha)$ of [16]. Thus, for example, $x_{13}(\alpha)$ and $x'_{13}(\alpha)$ are obtained from the equation:

$$x2-(6+2\cdot\alpha)\cdot x+(15-3\cdot\alpha)=0$$

Using an informatics' program, like Mathematica, we easy and quickly find the two Approximation's matrices for any value of the alpha-cut α. For $\alpha=0$ we have:

$$X(0)=\begin{pmatrix} 3+1.414\cdot i & 3+1.732\cdot i & 3+2.449\cdot i \\ 6.236 & 6 & 5 \\ 7.541 & 7.372 & 6.791 \end{pmatrix} \quad \text{and} \quad X'(0)=\begin{pmatrix} 3-1.414\cdot i & 3-1.732\cdot i & 3-2.449\cdot i \\ 1.764 & 2 & 3 \\ 1.459 & 1.628 & 2.209 \end{pmatrix}$$

As we have said $x_{ij}(\alpha)+x'_{ij}(\alpha)=m_i(\alpha)$, and therefore $x_{11}(0)+x'_{11}(0)=m_1(0)=m_1=6$. Also the addition of the terms of the second row and the same column are $m_2=8$, and the identically occurs for the third row $m_3=9$. Thus the second matrix $X'(\alpha)$ is easily obtained from the first, $X(\alpha)$.

In the same way for $\alpha=0.5$ and $\alpha=0.75$ we calculate the matrices:

$$X(0.5)=\begin{pmatrix} 4.3666 & 4 & 3.5+1.118\cdot i \\ 6.121 & 6 & 5.581 \\ 6.812 & 6.712 & 6.386 \end{pmatrix} \quad \text{and} \quad X(0.75)=\begin{pmatrix} 5.271 & 5.186 & 4.896 \\ 6.062 & 6 & 5.803 \\ 6.420 & 6.365 & 6.190 \end{pmatrix}$$

We see that, as α approaches to 1, the eight external terms of the matrix approaches to the central term $x_{22}=6$. Logically, for $\alpha=1$ we obtain the matrix X(1) formed only by the number 6. This is the crisp matrix. Moreover, we can see for $\alpha=0.75$ there is no complex term.

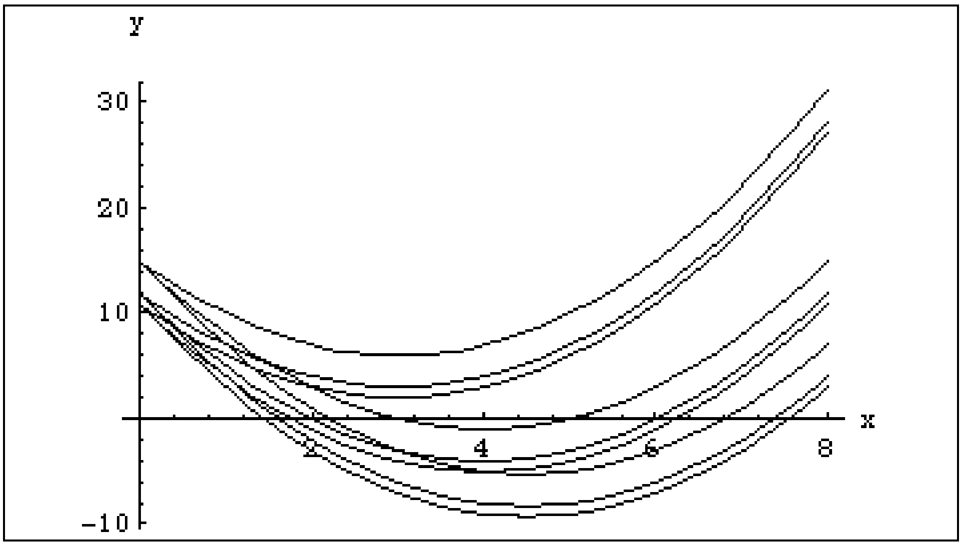

Fig. 2

If we wish to understand better the problem we are working with, we must make a graph of the family of nine parabolas $y=x^2-m_i(\alpha)\cdot x+n_j(\alpha)$, where i,j=1,2,3. For example, if $\alpha=0$ it results the *Fig.2*:

Obviously, the second degree equation $x^2-m_i(\alpha)\cdot x+n_j(\alpha)=0$ indicates the intersection points of the parabolas with the X axis, y=0. If the solutions of these equations are real and different, $x\neq x'$, then the parabolas cuts the X axis in two points and the parabolas are secant to X. If instead of being different are equal the two real roots, x=x', then the parabola will be tangent to the X axis. Finally, if the two roots are complex then will not be any intersection and the parabola will be above the X axis.

We wish now to obtain the minimum value k of the alpha-cuts with the finality that all the roots of our second degree equation [8], depending on α, are real. Therefore, if we choose $\alpha\geq k$ all the parabolas will be secant to the X axis.

Taking the equation [20] and the fuzzy coefficients $\tilde{m}$ =(6, 8, 9) and $\tilde{n}$ =(11, 12, 15) results:

$$(6+2\cdot k)^2=4\cdot(15-3\cdot k) \quad , \quad (3+k)^2=15-3\cdot k \quad , \quad k^2+9\cdot k-6=0$$

Since 0≤α≤1 the unique solution is with the positive sign:

$$k = (-9 + \sqrt{105})/2 \approx 0.6234753$$

We corroborate this value by proving that for α=0´6234 there is any complex roots in his Approximation's matrix, while for α=0´6235 all his roots are real:

$$X(0.6234) = \begin{pmatrix} 4.8504 & 4.686 & 3.6234 + 0.0278 \cdot i \\ 6.092 & 6 & 5.6942 \\ 6.6211 & 6.5424 & 6.2888 \end{pmatrix} \text{ and } X(0.6235) = \begin{pmatrix} 4.8508 & 4.6864 & 3.6394 \\ 6.092 & 6 & 5.6943 \\ 6.6209 & 6.5423 & 6.2887 \end{pmatrix}$$

Considering all the 9 terms of this last matrix we observe that the minimum, central and maximum values are respectively the elements of the secondary diagonal: x_{13}=3.6394, x_{22}=6 and x_{31}=6.6209. From this we deduce that the first root x will move from x_{13} to x_{31}, being x_{22} the value of the maximum presumption. Therefore we obtain the first fuzzy root and in a similar way, because the fact that the addition of the two roots are ($m_1(k)$, m_2, $m_3(k)$)=(7.247, 8, 8.3765), the second fuzzy root. These fuzzy solutions are:

$$\tilde{x} = (3.6394, 6, 6.6209) \text{ and } \tilde{x}' = (1.7556, 2, 3.6076)$$

Returning to *Fig.1* we can see graphically the deduction of the minimum value k. We must draw first the parabola $n=(m/2)^2$, which come from equation [20], and then obtain the rectangle M'N'P'Q' whose sides are parallel to the primitive rectangle MNPQ.

6. Conclusions

With the aim to preserve the mathematical properties of an economic polynomial function, such as increasing growth and convexity, it's necessary that the resultant equation has no complex roots. In this first analysis we have parted from a second degree equation with crisp coefficients and real roots. Then we have extended these coefficients to a triangular fuzzy numbers. Finally we have found an alpha-cut k, given by [20], that is his minimum value.

In the example above the equation was x^2-8x+12=0, whose roots were x=6 and x'=2. Extending the coefficients to the fuzzy numbers $\tilde{m}$ =(6, 8, 9) and $\tilde{n}$ =(11, 12, 15) we have obtained complex roots for certain alpha-cuts. Then we have found the minimum value k=0.6234753, and have reduced the coefficients to the new fuzzy numbers $\tilde{m}$ (k)=(7.247, 8, 8.376) and $\tilde{n}$ (k)=(11.6235, 12, 13.1295). Taking α>k the roots will be all real numbers and varying according to $\tilde{x}$ =(3.6394, 6, 6.6209) and $\tilde{x}$ '=(1.7556, 2, 3.6076).

We can assure now that if we work with these reduced fuzzy coefficients, then our particular benefit's function $f(x)=x^3-12\cdot x^2+36\cdot x+158$ will always have a maximum and a minimum.

7. References

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