REAL TIMES NUMERICAL FEEDBACK OPTIMAL CONTROLS TO PARABOLIC BOUNDARY PROBLEM

Aleksey V. Borzenkov[1)], Oleg L. Konovalov[2)], Ivan Ariko[3)]

[1)]Department of Computer Systems and Networks, Belarus State University of Informatics and Radioelectronics (BSUIR), 220027, P.Brovka str., 6, Minsk , Belarus, e-mail: *borzenkov_a@mail.ru*

[2)]Department of Applied Mathematics and Informatics, Belarus State University (BSU), 220050, Fr.Scorina pr.,4, Minsk , Belarus, e-mail: *KonovalovOL@ bsu.by*

[3)]Department of Computer Systems and Networks, Belarus State University of Informatics and Radioelectronics (BSUIR), 220027, P.Brovka str., 6, Minsk, Belarus, e-mail: *ariko@fromru.com*

Abstract. In our paper we consider model that has been formulated in the form of feedback control problem and simulates many processes in environmental branches. Now we present a results of numerical experience in MatLab VI to our algorithms. Algorithms construct numerical feedback controls in real time. Effect of real time has been received by means of method Monte-Carlo to integrate partial differential equations. We make some remarks about using that method.

1. Problem formulation

We consider [1], [2] the following parabolic system under uncertain measurable function $\omega(t)$, $t \in T = [t_0, t^*]$ with no probabilistic information available and boundary controls $u(t)$, $t \in T$:

$$\begin{cases} \dfrac{\partial \varphi(t,x)}{\partial t} = L_x \varphi(t,x) + \omega(t), & x \in (x_0, x^*),\ t \in (t_0, t^*); \\ A \cdot \dfrac{\partial \varphi}{\partial x}(t, x^*) = \varphi(t, x_0) = u(t); & t \in (t_0, t^*];\ \varphi(t_0, x) = 0; \quad x \in [x_0, x^*]; \end{cases} \tag{1}$$

Where $L_x \varphi(t,x) = \dfrac{\partial}{\partial x}\left(A \cdot \dfrac{\partial \varphi}{\partial x}\right) + B \cdot \dfrac{\partial \varphi}{\partial x} + C \cdot \varphi$, $x \in (x_0, x^*)$ is *PDE* operator and its real coefficients $A(t,x), B(t,x), C(t,x)$ are functions under restrictions [3], that provides an existence of system solutions in generalized sense. The components of *PDE* system (1) – system state $\varphi(t,x)$, $(t,x) \in \Omega = [x_0, x^*] \times [t_0, t^*]$, boundary controls $u(t)$, $t \in T$, and perturbation function $\omega(t)$, $t \in T$ – are also constrained:

$$b_* \le \varphi(t, x^*) \le b^*;\ (t,x) \in \Omega; \tag{2}$$

$$d_* \le u(t) \le d^*;\ t \in T; \quad \omega_* \le \omega(t) \le \omega^*;\ t \in T; \tag{3}$$

So, the following feedback optimal control problem is considered in initial formulation *(P)*:

$$(P)\ \text{maximize}\ J(u) = \int_{t_0}^{t^*} \big(u(\varphi(t, x^*)\big) dt = \int_{t_0}^{t^*} u(t)\, dt \ , \ \text{subject to (1)-(3).}$$

Following our papers [1], [2] instead of constraints (2), that are too difficult to numerical realization, we firstly consider constraints

$$\varphi(\tilde{t}_k, x^*) = b_k;\ \tilde{t}_k \in (t_0, t^*);\ k \in K = \{k_1, k_2, ..., k'\} \tag{4}$$

to fixed points $\tilde{t}_k \in (t_0, t^*);\ k \in K$. So, instead of problem *(P)*, now we consider the following feedback optimal control problem *(P1):*

$$(P1)\ \text{maximize}\ J(u) = \int_{t_0}^{t^*} u(t)\, dt \ , \ \text{subject to (1), (3), (4).}$$

In our paper we shall specify numerical results received earlier [4]. Note, that problem *(P)* firstly was formulated [5] and simulates underground irrigation water regime. Note, also, that all our results are based on technique to linear programming problem (LP) and linear control problem, that initially has been developed by Gabasov R., Kirillova F.M. [6], [7] to ordinary differential equations (ODE). Then we have adapted these results to partial differential equations (PDE). We present a of numerical experiment results of solving problem *(P1)* in pulse functions class and measurable functions class by dual technique.

2. Main definitions and theorems

Let's consider, that the partitition of the time segment $T=[t_0, t^*]$ is determined into segments $T_j=\{t: \xi_{j-1} \le t \le \xi_{j-1}+1=\xi_j\}$, $j \in J=\{1,2,...,n\}$, $n>\bar{k}$, $\xi_0=t_0$, $\xi_n=t^*$. Let's consider, that the control influence is $u(t)=u(t_j)=u_j=const$, $\forall t \in T_j$, $j \in J$ and satisfies to constraints (2), (6). The following designated control will be called an admissible pulse control $u=\{u_j=u(t_j), j \in J\}$. Admissible pulse control $u^0=\{u_j^0, j \in J\}$, which satisfies to unequation $J(u^0) \ge J(u)$ will be called an optimal impulse control. In the terms of pulse controls instead of conjugate differential system (5) we shall consider its difference - differential analog without uncertain disturbances $\omega(t,x)$ - conjugate system

$$\begin{cases} \psi_k(t_{j-1}, x^*) = \psi_k(t_j, x^*) + L_x^* \psi(t_j, x^*), \quad x \in (x_0, x^*), t \in (t_0, t^*); \\ A \cdot \dfrac{\partial \psi_k(t_j, x^*)}{\partial x} - B \cdot \psi_k(t_j, x^*) = f(t); \ A \cdot \dfrac{\partial \psi_k(t_j, x_0)}{\partial x} - B \cdot \psi_k(t_j, x_0) = 0; t \in (t_0, t^*] \\ \psi_k(t^*, x) = 0; \ x \in [x_0, x^*];; \end{cases} \tag{5}$$

and instead of initial differential system (5) we shall consider its difference - differential analog

$$\begin{cases} \varphi(t_{j+1}, x) = \varphi(t_j, x) + L_x \varphi(t_j, x), \quad x \in (x_0, x^*), t \in (t_0, t^*); \\ \varphi(t_j, x_0) = A \cdot \dfrac{\partial \varphi(t_j, x^*)}{\partial x} = u(t_j); t \in (t_0, t^*]; \quad \varphi(t_0, x) = 0; \ x \in [x_0, x^*]; \end{cases} \tag{6}$$

So, now we considered determined optimal control problem:

(P2) maximize $J(u) = \langle e, u \rangle$, over $d_* \le u \le d^*$, subject to (5) and the state constraints (4).

Let's separately notice, that solution of determined optimization problem *(P2)* will be called as program optimal control. The problem *(P2)* now is a special form of *LP*. Here $e = \left(e(t_j)=1, t_j \in T\right)$. Let's define the support of problem *(P2)* as ordered set of indexes $J_{\sup} = \{t_1, t_2, ..., t_{\bar{k}}\} = \{t_j \in (t_0, t^*), j=1,2,...,\bar{k}\}$ that provides the linear independence of vectors $\psi_j = (\psi_k(t_j, x^*), k \in K), j=1,2,...,\bar{k}$, where function $\psi_k(t_j, x^*), k \in K, j \in K$ are the solutions of the system (5) with right part $f(t) = p_k(t): p_k(t)=1, t=t_k, p_k(t)=0, t \ne t_k$. Define the matrix $\Psi_{\sup}^{matr} = \begin{bmatrix} \psi_k(t_j, x^*), j \in J_{\sup} \\ k \in K \end{bmatrix}$, $J_{n\sup} = J \setminus J_{\sup}$, that will be called as support matrix. There is a one-to-one correspondence between support $J_{\sup}$ and the function $\Psi(t_j, x^*), t_j \in (t_0, t^*), j=1,2,...,\bar{k}$ $t_j \in (t_0, t^*)$ $t_j \in (t_0, t^*), j=1,2,...,\bar{k}$, that is the solution of conjugate difference - differential system (5) with right part $f(t) = \sum_{k \in K} \lambda_k \cdot p_k(t)$. Here vector of potentials $\lambda = (\lambda_k, k \in K)$ - is the solution of the following system of linear equations: $\lambda \cdot \Psi_{\sup}^{matr} = s$, $s = e^{\sup}$, $e^{\sup} = (e_j = 1, j \in J_{\sup})^T$. Using the solution $\Psi(t_j, x^*), t_j \in (t_0, t^*), j=1,2,...,\bar{k}$ of conjugate system (5), we define vector of co-control $\delta = (\delta_j = \delta(t_j) = \Psi(t_j, x^*) - 1, j \in J)$ and following function:

$$H(\Psi(t_j, x^*), u(t_j), t_j) = \left(\Psi(t_j, x^*) - 1\right) \cdot u(t_j), \ t_j \in (t_0, t^*), j \in J.$$

Theorem 1. (The maximum principle). For optimality of the admissible pulse control u^0 there is necessary and sufficiently an existence of such a support $J^0_{\sup}$, that on corresponding them solution Ψ^0 of the conjugate system (7) the following conditions are fulfilled:

$$H\left(\Psi^0(t_j, x^*), u^0(t_j), t_j\right) = \underbrace{max}_{d_* \le v \le d^*} H\left(\Psi^0(t_j, x^*), v, t_j\right), \forall j \in J \qquad (7)$$

Let's define psevdo-control vector

$\chi = (\chi_j = \chi(t_j), j \in J)$ $: \chi_j = d_*\ if\ \delta_j > 0;\ \chi_j = d^*\ if\ \delta_j < 0;\ d_* < \chi_j < d^*\ if\ \delta_j = 0,\ j \in J_{n\sup}:$ $\Psi^{matr}_{\sup} \cdot \chi_{\sup} = b_k, \chi_{\sup} = (\chi_j, j \in J_{\sup}), k \in K$. In that terms we may reformulate above theorem.

Theorem 2. (Sufficiently optimality conditions). If for support indexes $j \in J_{\sup}$ are fulfilled the following conditions $\chi(t_j) = d_*\ \ if\ \ \delta(t_j) > 0, \chi(t_j) = d^*\ \ if\ \ \delta(t_j) < 0, d_* \le \chi(t_j) \le d^*\ \ if\ \ \delta(t_j) = 0$

then $u^0 = \{u^0_j = u^0(t_j) = \chi(t_j),\ j \in J\}$ is an optimal program pulse control.

Notice, that problem (*P2*) in the terms of pulse controls may be rewritten in dual *LP* form:

(*P3*) minimize $I(\delta) = <\lambda, b> + <\delta, \bar{d}>$, where $\bar{d} = d_*\ \ if\ \ \delta_j \ge 0,\ \bar{d} = d^*\ \ if\ \ \delta_j \le 0$.

This problem was solved [1] to make numerical experience statistics. The statistic data presented in [4],[8] give us a reason to make a conclusion that the main resources of computer are used to numerical integration for PDE. Therefore for integration of differential system we use Monte-Carlo method. We shall note two properties of these method. It is possible to obtain functions value in required points instead of on all grid of decisions. It is possible to be satisfied with low accuracy. Monte-Carlo method was programmed by Ariko I. Theoretical aspects were adapted by Borzenkov A., Ariko I.

3. Numerical experiment to program and feedback optimal controls in pulse functions class

The investigated methods of optimization to pulse function class [1], [2] was programmed in MatLab 6.0 language. Calculations were executed by using processor AMD Duron 950, 128 Mb DDR, Microsoft Windows 2000 Professional, MatLab 6 Release 12. Below we bring two tables that contain the results of numerical experiment for those methods. The first table contains a statistics to optimal program controls. The second table contains a statistics to optimal feedback controls. To integrate PDE we use two methods. Classic method of finite differences to the four points scheme was used firstly. The generated diagonal matrix then was solved with standard procedure of MatLab. Limits of disturbances function do not more than two percent to maximum control value. The following designations are used in the tables: h – a step of time digitization PDE, dim – dimension of matrix to integrate conjugate differential systems, k –a number of problem state restrictions, n – a number of time steps, ΔJ - functional increment (to initial value), Int – a number of conjugate differential systems integrations, T_{int} – a common time of conjugate differential systems integration (percents to common solution time of optimization problem), T – a common solution time of optimization problem (seconds), T_{max} – a maximum time to construct feedback control during one step.

Table 1. Construction of program optimal controls in pulse function class.

h	dim	k	n	ΔJ %	Dual method with classic approach to *PDE*			Dual method with Monte-Carlo approach to *PDE*		
					Int	T_{int}, %	T	Int	T_{int}, %	T
0.5	507	3	20	24	3	86.2	0.29	3	83.9	0.012
0.2	4674	3	50	25.7	7	85	6.5	7	46.8	0.0915
0.05	10361	3	200	29	20	85.6	82.7	20	30	0.851

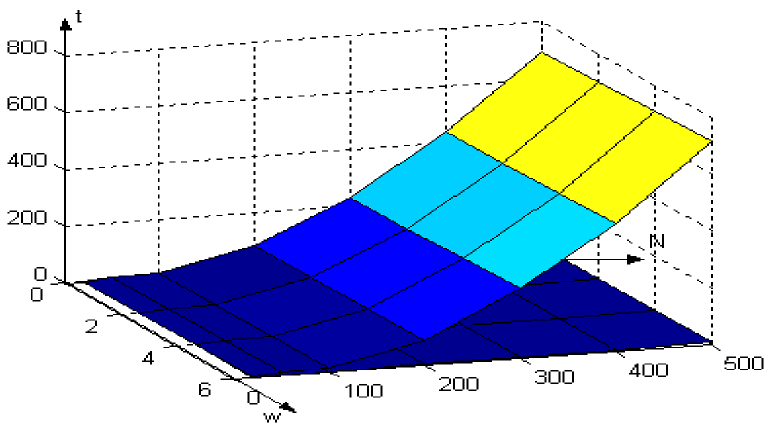
0.02	25961	3	500	30	35	85.5	693.9	30	15.2	6.16

Table 2. Construction of feedback optimal controls in pulse function class.

h	dim	k	n	ΔJ %	Dual method with classic approach to *PDE*			Dual method with Monte-Carlo approach to *PDE*		
					T_{max}	Int	$T_{int,}$ %	T_{max}	Int	$T_{int,}$ %
0.5	507	3	20	23	0.31	3.5	86	0.014	3.5	83.9
0.2	1824	3	50	26.7	7.1	7.3	85	0.11	7.3	46.8
0.05	9525	3	200	27	85.4	21.1	86	0.95	21.1	30

4. Numerical experiment to compare two approaches to integrate partial differential equations (PDE)

Figure1. Comparison a method of finite differences with Monte-Carlo method to integrate PDE



The figure shows time dependence to construct feedback control during one step from value of disturbances function and a number of time steps. The top surface shows the process with using finite differences method to the four points scheme to integrate PDE. The bottom surface – with using Monte-Carlo method. Here: ω – limits of disturbances function (in % to difference between d^*, d_* , ω = 0, 2, 4, 6 %), N – a number of time steps (N = 20, 50, 100, 200, 300, 400, 500), t – a maximum time to construct feedback control during one step. Here the surface of a of Monte Carlo method is close to zero because synthesis time with using this method on two order less than the same process with use a method of finite differences. For example to N=200 and ω =2% it will be 85 and 0.9 seconds accordingly. A time close by one second allows to make a conclusion about real times numerical feedback optimal controls to parabolic boundary problem.

5. Numerical experiment to program and feedback optimal controls in measurable function class.

Below we bring two tables. Results are entirely taken from our previous paper [4]. The first table contains a statistics to optimal program controls in measurable function class. The second table contains a statistics to optimal feedback controls. The investigated methods [1], [2] was programmed in MatLab 6.0 language. Calculations were executed by using AMD Duron 550, 128 Mb DDR. Classic method of finite differences to the four points scheme was used to integrate PDE. The generated diagonal matrix then was solved with MatLab. Limits of disturbances function do not more than two percent to maximum control value. The following designations are used in the tables: h – a step of time digitization, dim – dimension of matrix to integrate conjugate differential systems, k – a number of problem state restrictions, n – a number of time steps, ΔJ_c - functional increase (percents to functional optimal value in digitization's case), T – a common solution time of optimization problem (seconds), T_N – a solution time of Newton methods, Im – a middle number of Newton methods iterations to solve a nonlinear equations system, $T_{c\,m}$ – a middle time to construct feedback control, $T_{N\,m}$ – a middle solutions time of Newton method.

Table 3. Construction of program optimal controls in measurable function class.

H	k	n	ΔJ_C, %	I	T	T_N	T_N, %
0.5	3	20	3	1	0.84	0.05	6
0.2	3	50	2.3	1	2.63	0.059	2.2
0.05	3	200	3.1	1	46.52	0.07	0.15
0.02	3	500	2.9	1	690.3	0.09	0.01

Table 4. Construction of feedback optimal controls in measurable function class.

H	k	n	ΔJ_C, %	I_m	$T_{c\,m}$	$T_{N\,m}$	$T_{N\,m}$, %
0.5	3	20	1.2	1	0.8	0.01	1.25
0.2	3	50	1.3	1	1.1	0.012	1
0.05	3	200	1.19	1	15	0.015	0.1

Let's separately note, that studying an optimization problem in a class of measurable functions does not give an essential increment of quality criteria. Thus complexity of optimal control problem essentially grows.

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