A FAST ITERATIVE PROCEDURE TO SOLVE WKB EQUATION IN OPTICAL PLANAR WAVEGUIDES WITH GRADED-INDEX PROFILES

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Abstract. We propose a numerical method to determine effective indices and other related parameters of a planar graded-index waveguide. The secant method uses initial guess to iterate solutions of the WKB equation which is applied to waveguides characterization. The initial guess we choose is the normalized propagation constant of the waveguide. Effective indices, turning points, dispersion curves and cutoff frequencies of a guide with known index profile are obtained by simple iterations for a mode order m. The method is easy to implement and it gives results with good accuracy and may be compared to those existing in literature.

1. Introduction

The characteristics of the propagating modes in graded-index waveguides are obtained by solving Maxwell's equations subject to the appropriate boundary conditions. However, exact analytical solutions are available only for a limited class of profiles. In general, many approximate or numerical methods are used. Of the approximate methods, the WKB is widely used because of its simplicity and the clear physical explanation available, but its solution diverges near the turning point [1]. In his paper, Gedeon has demonstrated that, for monotonically decreasing index profiles, WKB analysis yields effective index values within about 0.01 percent of the exact value [2]. To find solutions to the implicit WKB equation, we use a procedure based on the iterative secant method which needs an initial value of the variable. We take the advantage from the physical interest of the problem and the initial guess can be the normalized values of the propagation constant b.

2. Theory

In graded-index waveguide (fig.1), it is assumed that the guide is produced in a substrate that occupies the region x≥0. For x<0 the medium is taken to be air. Thus the refractive index is defined everywhere by [2].

$$n(x) = \begin{cases} 1, & x < 0 \\ n(x), & x \geq 0 \end{cases} \tag{1}$$

For TE modes propagating in the y-direction, the field amplitude $E_{z,m}$ has the form [2]

$$\frac{d^2 E_{z,m}(x)}{dx^2} + \chi_m^2 . E_{z,m}(x) = 0 \tag{2}$$

where χ_m, the component of the wave vector in the x direction, is given by [2], [3], [4]

$$\chi_m = \sqrt{k_0^2 n^2(x) - \beta_m^2} \tag{3}$$

$k_0 = 2\pi/\lambda_0$, the free space propagation constant and β_m is the propagation constant of the m^{th} mode. The eigenvalues β_m of equation (2) can be found numerically for a given index profile defined as

$$n^2(x) = n_b^2 + 2n_b^2 \Delta n f(x) \tag{4}$$

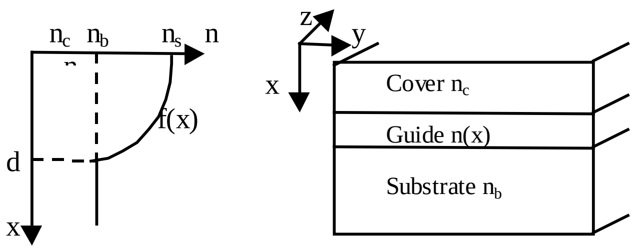

Fig. 1. *Planar graded-index waveguide and refractive index profile.*

Where n_b and n_c are the bulk and cover indices respectively, n_s the surface waveguide index, Δn the maximum of the increase in the refractive index and, d is the guide depth [2], [4]. According to equation (4), exponential index profile is given by

$$n^2(x) = n_b^2 + 2n_b\Delta n \exp\left(-\frac{x}{d}\right) \tag{5}$$

At a turning point $x=x_t$, the component $\chi_m(x_t)=0$ and equation (3) leads to

$$n^2(x_t) = \beta_m^2 / k_0^2 \tag{6}$$

For a given mode m, this expression defines the effective index and it is denoted as

$$n_e = \beta / k_0 = n(x_t) \tag{7}$$

Using this expression, we obtain the normalized propagation constant b which takes values between zero and unity [3], [4]. It is given by

$$b = \frac{n_e^2 - n_b^2}{n_s^2 - n_b^2} \tag{8}$$

Another basic guide parameter called the normalized frequency or film thickness V, is defined as

$$V = k_0 d\sqrt{n_s^2 - n_b^2} \tag{9}$$

When substrate and cover indices are different, a parameter characterizing this difference is defined and it is called the asymmetry constant [3], [4] which is defined as

$$a_E = \frac{n_b^2 - n_c^2}{n_s^2 - n_b^2} \tag{10}$$

3. WKB analysis

In most cases of physical interest, one finds that $\Delta n \ll 1$ so that $\chi_m(x)$ is a slowly varying function of position and, according to the WKB-condition [2], the simple function

$$E_{z,m}(x) \sim \chi_m^{-1/2} \cos\left[\int_{x_{t2}}^{x} \chi_m(x)dx - \phi_t\right] \tag{11}$$

becomes a good approximate eigenfunction of wave equation (2), in the interval $[x_{t2}, x_{t1}]$. The WKB analysis yields the eigenvalues n_e implicitly, as the solutions of the mode equation [2]

$$k_0 \int_{x_{t2}}^{x_{t1}} \sqrt{n^2(x) - n_e^2}\,dx = m\pi + \pi/4 + \phi_t \qquad m = 0,1,\ldots, N-1 \tag{12}$$

Where,

$$\phi_t = \begin{cases} \dfrac{\pi}{4}, & x_{t1} \succ 0 \\ arctg\left[\gamma^2\left(\dfrac{n_e^2 - n_c^2}{n_s^2 - n_e^2}\right)^{1/2}\right], & x_{t2} = 0 \end{cases} \qquad and \qquad \gamma = \begin{cases} 1, & TE \\ \dfrac{n_s}{n_c}, & TM \end{cases} \tag{13}$$

The term in LHS represents the variation of $\chi_m(x)$ in the oscillation region and the one in RHS corresponds to the phase shifts at the cover-guide interface and at the turning point. The mode spectrum of a waveguide is the set of n_e values calculated for the given refractive index profile.

4. Numerical calculations

To solve the equation (12), we take the general case of ϕ_t (i.e. $x_{t2}=0$), and the one half phase shift at cover-guide interface as $\pi/4$. For exponential profile, the first equation for TE modes in terms of n_e is

$$k_0\sqrt{2n_b\Delta n}\int_0^{x_t}\sqrt{\exp(-x/d) - \exp(-x_t/d)}\,dx = m\pi + \pi/4 + arctg\sqrt{\frac{n_e^2 - n_c^2}{n_s^2 - n_e^2}} \tag{14}$$

After integration and using the normalized propagation constant b, the implicit equation takes the form:

$$2k_0\sqrt{2n_b\Delta n}\left[\sqrt{1-b} - \sqrt{b}.ar\cos\left(\sqrt{b}\right)\right] = m\pi + \pi/4 + arctg\sqrt{\frac{n_b^2 + 2n_b\Delta nb - 1}{2n_b\Delta n(1-b)}} \tag{15}$$

The determination of b values yields the values of n_e using (8). The values of the turning points can be determined using (7) and, the dispersion curves b-v and the cutoff values using the equation

$$2V\left[\sqrt{1-b}-\sqrt{b}.ar\cos(\sqrt{b})\right]=m\pi+\frac{\pi}{4}+arctg\sqrt{\frac{a_E+b}{1-b}} \tag{16}$$

The secant method [5] starts computation from an initial guess and does iterations until a solution is obtained with the requested degree of accuracy. In its simple form, it can be given by

$$X_{i+1}=\frac{X_i\, f(x_{0'})-x_{0'}\, f(X_i)}{f(x_{0'})-f(X_i)} \tag{17}$$

where x_0 is the initial guess, X_i the i*th* iteration of the solution and f the function to solve.

As $0 \le b \le 1$, initial guess can takes all values in this range. However, it has been found that for b=1, when the effective index reaches the surface index value and the equation (15) becomes undeterminated. We used the limit $b \to 1$ instead of b=1. At the cutoff value (b=0), the effective index takes the value of the bulk index $n_e = n_b$. Also, we note that we can use the value of bulk index as initial guess in the equation (14).

5. Results and discussion

The figure 2 illustrates the zero order mode index versus b initial values for an asymmetric graded-index waveguide with exponential profile and with the following parameters: n_b=2.177, n_c=1, d=2.22726µm, Δn=0.08937 at λ=0.6328µm. The accurate zero mode index is 2.241583, it is obtained using b=0.5 as initial guess and the the minimum error is about $9.52.10^{-6}$ (see the table 1).

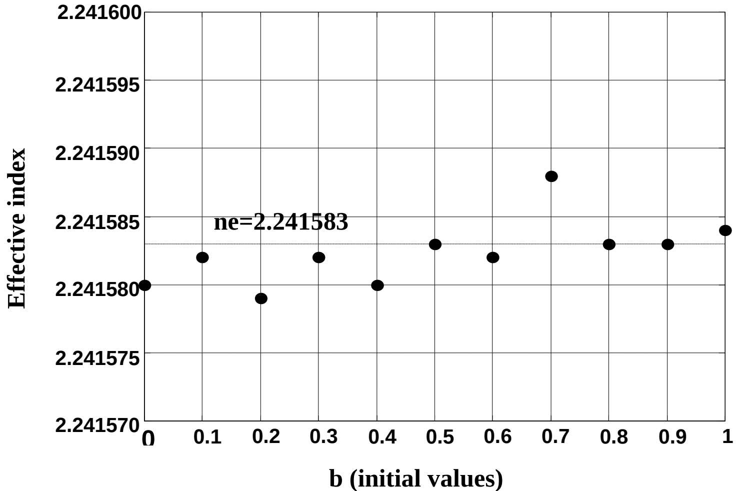

Fig. 2. *Zero order mode effectif index versus b (initial values) for the exponential profile. Accurate value of n_e=2.241583 obtained for an initial guess b=0.5.*

The figure 3 shows convergence process for each mode according to the initial guess. One can verify that convergence takes more time for low order modes than for high order modes. At the mode m=8, the convergence is imminent and, the solution (b=0.00157) is near the cutoff. At this mode, the range of b values is reduced to [0-0.3] and, if b≥0.3, no solution could be found. When order mode increases, the range of initial values decreases until it reaches the cutoff value. We used this fact to estimate the number N of guided modes and our program has detected 9 modes.

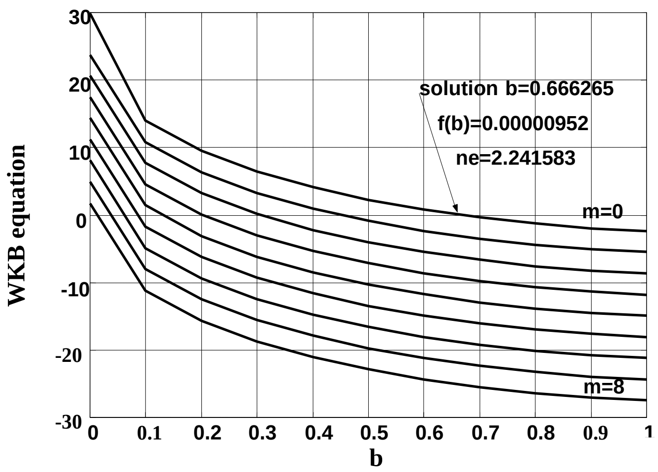

Fig. 3. *WKB equation solutions for each mode versus the values of b. Solutions are given when the curves crosses the zero axis.*

The dispersion curves are plotted using this iterative method with application of equation (16). The figure below shows the b-v curves for the discussed example. Also, we used this curves to determine cutoff frequencies and the number of guided modes.

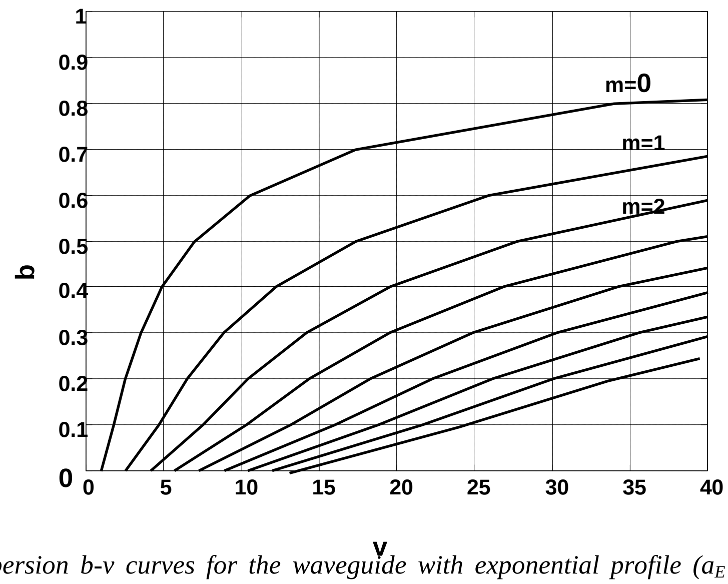

Fig. 4. *Dispersion b-v curves for the waveguide with exponential profile (a_E =8.7). Nine guided modes can be seen and V cutoff frequencies are readen on the zero axis when b=0*

6. Conclusion

In table 1, the mode spectrum calculated by our method is compared with exact theory and also with Gedeon results. The author has reported in his paper that his values are rounded off so that the error was $2.5x10^{-5}$. As reported in [1], WKB analysis is accurate to better than 0.00015, Our results show that the accuracy may be improved if initial guess takes values of the normalized propagation constant b during iteration process. In this work, results have not been rounded off and the accuracy is about 10^{-6}. Although WKB analysis is weak for the lower order modes, we have obtained results better than those found by Gedeon for the three first modes. Also, gain in time is obtained when the mode order increases. This fact leads to the better use of the method for multi-mode optical graded-index waveguides. Dispersion curves, values of turning points and cutoff frequencies are easily determined using the equations (15) and (16).

Table 1. *Mode spectrum of a graded-index waveguide with exponentiel profile (n_b=2.177, n_c=1, d=2.22726μm, Δn=0.08937) at λ=0.6328μm.*

m	Exact solution	WKB Gedeon	Our calculation b:initial guess	Best initial guess of b	Minimum error x10^{-6}
0	2.24135	2.24160	2.241583	0.5	9.52
1	2.22070	2.22078	2.220752	0.7	4.54
2	2.20680	2.20680	2.206817	0.2	4.33
3	2.19675	2.19675	2.196755	0.8	0.03
4	2.18940	2.18940	2.189402	0.9	0.03
5	2.18415	2.18415	2.184152	0.9	0.65
6	2.18050	2.18050	2.180498	1	1.09
7	2.17825	2.17825	2.178249	0.7	0.033
8	2.17715	2.17715	2.177155	0.2	2.28

7. References

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