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ORDINARY DIFFERENTIAL EQUATIONS

On the Existence of Linear Pfaff Systems with Lower Characteristic Sets of Positive Lebesgue *m*-Measure

N. A. Izobov, S. G. Krasovskii, and A. S. Platonov
Institute of Mathematics, National Academy of Sciences, Minsk, Belarus
Mogilev State University, Mogilev, Belarus

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Abstract—We prove the existence of an n-dimensional completely integrable Pfaff system with multidimensional time of dimension $m \geq 2$, with bounded infinitely differentiable coefficients, and with the set of lower characteristic vectors of its solutions having positive Lebesgue m-measure.

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We consider n-dimensional linear Pfaff systems

$$\frac{\partial x}{\partial t_i} = A_i(t)x, \quad x \in \mathbb{R}^n, \quad n \geq 2, \quad i = 1, \ldots, m, \quad t = (t_1, \ldots, t_m) \in \mathbb{R}_+^m, \quad m \geq 2, \qquad (1_{mn})$$

with continuously differentiable coefficient matrices satisfying the complete integrability condition [1, pp. 14–24; 2, pp. 16–26]

$$\frac{\partial A_i(t)}{\partial t_j} + A_i(t)A_j(t) = A_j(t)A_i(t) + \frac{\partial A_j(t)}{\partial t_i}, \qquad i, j = 1, \ldots, m, \qquad t \in \mathbb{R}_+^m, \qquad (2)$$

in $\mathbb{R}_+^m = \{t \in \mathbb{R}^m : \ t \geq 0\}$.

Let $\lambda[x] \in \mathbb{R}^m$ and $p[x] \in \mathbb{R}^m$ be some characteristic [3] and lower characteristic [3] vectors of a nontrivial solution $x : \mathbb{R}_+^m \to \mathbb{R}^n \backslash \{0\}$ of system (1_{mn}); these vectors are m-dimensional analogs of the corresponding characteristic Lyapunov exponent and lower Perron exponent. In addition, the first of them—the characteristic vector $\lambda[x]$—is determined by the relations

$$L_x(\lambda[x]) \equiv \overline{\lim_{t \to \infty}} \frac{\ln \|x(t)\| - (\lambda[x], t)}{\|t\|} = 0, \qquad L_x(\lambda[x] - \varepsilon e_i) > 0 \qquad \forall \varepsilon > 0,$$
$$e_i = (\underbrace{0, \ldots, 0, 1}_{i}, 0, \ldots, 0) \in \mathbb{R}_+^m, \qquad i = 1, \ldots, m,$$

and the second—the lower characteristic vector $p[x]$—is determined by the conditions

$$l_x(p[x]) \equiv \underline{\lim_{t \to \infty}} \frac{\ln \|x(t)\| - (p[x], t)}{\|t\|} = 0, \qquad (3_1)$$
$$l_x(p[x] + \varepsilon e_i) < 0 \qquad \forall \varepsilon > 0, \qquad i = 1, \ldots, m. \qquad (3_2)$$

The sets $\Lambda_x = \bigcup \lambda[x]$ and $P_x = \bigcup p[x]$ are referred to as the characteristic set [3] and the lower characteristic set [4], respectively, of the nontrivial solution $x(t)$ of system (1_{mn}); they are bounded closed convex [2] and concave [4] curves, respectively, in the case of two-dimensional time $t \in \mathbb{R}^2$.

It is known that the set of characteristic exponents (e.g., see [5, p. 15]) of an n-dimensional ordinary differential system consists of at most n distinct numbers; i.e., such a system has at most n nontrivial solutions with pairwise distinct Lyapunov exponents. There is no counterpart of the previous result for the analog of the Lyapunov characteristic exponent of a nontrivial solution of an ordinary differential system, that is, for the characteristic set Λ_x of a nontrivial solution $x : \mathbb{R}^m_+ \to \mathbb{R}^n$ of a linear completely integrable Pfaff system (1_{mn}). Namely, it was proved in [6] that there exists an n-dimensional system (1_{2n}) with two-dimensional time $t \in \mathbb{R}^2_+$ that has a countable set $\{x_1(t), \ldots, x_k(t), \ldots\}$ of nontrivial solutions $x_k : \mathbb{R}^2_+ \to \mathbb{R}^n \backslash \{0\}$ with pairwise distinct characteristic sets Λ_{x_k} ($\neq \Lambda_{x_j}$ for arbitrary $k \neq j$). In addition, it was proved in [7] that each completely integrable Pfaff system (1_{22}) has at most countably many solutions with pairwise distinct characteristic sets. Therefore, the entire characteristic set $\Lambda(A_1, A_2) = \bigcup_{x \neq 0} \Lambda_x$ of system (1_{22}) has zero plane Lebesgue measure.

The situation is different for the Perron lower exponent and its analog, the lower characteristic set. In the case of ordinary differential systems (1_{1n}), there exist systems whose set of Perron lower exponents has positive Lebesgue measure [8]. The existence of n-dimensional completely integrable Pfaff linear systems (1_{2n}) with two-dimensional time and (1_{3n}) with three-dimensional time and lower characteristic sets $\Pi(A_1, A_2)$ of positive plane measure and $\Pi(A_1, A_2, A_3)$ of positive measure in the space $\mathbb{R}^3$ was proved in [4] and [9], respectively. We encounter the problem on the existence of systems (1_{mn}) with a similar property and with time $t \in \mathbb{R}^m_+$ of arbitrary dimension m.

The present paper deals with the construction of an n-dimensional linear completely integrable system (1_{mn}) with m-dimensional time $t \in \mathbb{R}^m_+$, $m \geq 2$, with bounded infinitely differentiable coefficients in $\mathbb{R}^m_+$, and with lower characteristic set

$$\Pi(A_1, \ldots, A_m) = \bigcup_{x \neq 0} P_x$$

of positive Lebesgue measure in the space $\mathbb{R}^m$. To this end, we first construct the lower characteristic set of the sum of special scalar exponentials.

Lemma. *The lower characteristic set P_E of the function*

$$E(t) = \sum_{i=1}^{m} e^{-a_i t_i}, \quad a_i = \text{const}, \quad m \geq 2, \quad t_i \geq 0,$$

coincides with the set

$$\Pi_E \equiv \left\{ p \in \mathbb{R}^m_- : \ S(p) \equiv \sum_{i=1}^{m} \frac{p_i}{a_i} = -1 \right\}.$$

Proof. First, let us prove the inequality $p \leq 0$ for any lower characteristic vector $p \in P_E$. Suppose the contrary: $p_k > 0$ for some $k \in \{1, \ldots, m\}$. Then we have the inequalities

$$l_E(p) \leq \varliminf_{\|t\| = t_k \to +\infty} \frac{\ln(m - 1 + e^{-a_k t_k}) - p_k t_k}{t_k} = -p_k < 0$$

contradicting the first condition $l_E(p) = 0$ in the definition of a lower characteristic vector p.

Now let us prove the inclusion

$$\Pi_E \subset P_E \tag{4}$$

for the part of the plane $\sum_{i=1}^m p_i/a_i = -1$ in the m-dimensional quadrant $\mathbb{R}^m_- = \{p \in \mathbb{R}^m : p \leq 0\}$ of the space $\mathbb{R}^m$. Let

$$p \in \Pi_E, \quad t \in \mathbb{R}^m_+ \backslash \{0\}, \quad \text{and} \quad a_k t_k = \min_{i=1,\ldots,m} \{a_i t_i\}$$

for some $k \in \{1, \ldots, m\}$. Then, by virtue of the already proved inequalities $-p_i \geq 0$, we have the estimates

$$\ln E(t) - (p, t) \geq -a_k t_k - p_k t_k - \sum_{i \neq k}^{m} p_i \frac{a_k}{a_i} t_k = a_k t_k \biggl(-1 - \sum_{i=1}^{m} \frac{p_i}{a_i} \biggr) = 0,$$

which imply that $l_E(p) \geq 0$. At the same time, on the ray

$$T_a = \{t \in \mathbb{R}_+^m \backslash \{0\} : \ a_i t_i = a_k t_k, \ i = 1, \dots, m\}, \qquad k \in \{1, \dots, m\},$$

we have the opposite estimate

$$[\ln E(t) - (p,t)]_{t \in T_a} \leq \ln m - a_k t_k \left(-1 - \sum_{i=1}^{m} \frac{p_i}{a_i} \right) = \ln m,$$

which justifies the inequality $l_E(p) \leq 0$ and hence the desired relation $l_E(p) = 0$.

For a vector $p \in \Pi_E$, let us prove the second necessary condition

$$l_E(p + \varepsilon e_k) < 0, \qquad e_k = (\underbrace{0, \dots, 0, 1}_{k}, 0, \dots, 0) \qquad \forall \varepsilon > 0.$$

Indeed, on the ray T_a, we have the estimate

$$[\ln E(t) - (p + \varepsilon e_k, t)]_{t \in T_a} \leq \ln m - \varepsilon t_k,$$

which implies the desired inequality

$$l_E(p + \varepsilon e_k) \leq -\frac{\varepsilon}{a_k r}, \qquad r \equiv \left(\sum_{i=1}^{m} \frac{1}{a_i^2} \right)^{1/2}, \qquad k = 1, \dots, m.$$

The proof of the inclusion (4) is complete.

Now let us prove the coincidence of the sets Π_E and P_E. To this end, we suppose the contrary: there exists a vector $p \in P_E$ such that $p \notin \Pi_E$. By virtue of the above-proved inequality $q \leq 0$, we have $p \leq 0$ for any lower characteristic vector $q \in P_E$; therefore, we have the following cases for the quantity

$$S(p) \equiv \sum_{i=1}^{m} \frac{p_i}{a_i}.$$

1. $S(p) \in (-1, 0]$.
2. $S(p) < -1$.

In case 1, on the ray T_a, we have the relations

$$\begin{aligned} [\ln E(t) - (p,t)]|_{t \in T_a} &= \ln m - a_k t_k (1 + S(p)), \qquad 1 + S(p) > 0, \qquad t_k \geq 0, \\ \|t\||_{t \in T_a} &= a_k t_k r, \qquad k \in \{1, \dots, m\}, \qquad 1 + S(p) > 0, \qquad t_k \geq 0, \end{aligned}$$

which imply that $l_E(p) \leq -(1 + S(p)) r^{-1} < 0$. This inequality shows that the case $S(p) \in (-1, 0]$ is impossible.

Consider case 2. Obviously, there exists a vector $q \in \Pi_E$ for which the conditions $p \leq q \leq 0$ and $p \neq q$ are satisfied and the inequality $p_l < q_l$ holds for some $l \in \{1, \dots, m\}$. Then an arbitrary vector $t \in \mathbb{R}_+^m \backslash \{0\}$ such that

$$a_k t_k = \min_{i=1,\dots,m} \{a_i t_i\}, \qquad k \in \{1, \dots, m\},$$

satisfies the inequalities

$$\begin{aligned} \ln E(t) - (p,t) - \varepsilon t_l &= \ln E(t) - (q,t) + (q - p, t) - \varepsilon t_l \geq -a_k t_k \left(1 + \sum_{i=1}^{m} \frac{q_i}{a_i} \right) + (q_l - p_l - \varepsilon) t_l \\ &= (q_l - p_l - \varepsilon) t_l \geq 0, \qquad t \in \mathbb{R}_+^m, \end{aligned}$$

for $\varepsilon \in (0, q_l - p_l)$. They imply the inequality $l_E(p + \varepsilon_l) \geq 0$ for all sufficiently small $\varepsilon > 0$, whence we find that the vector p is not a lower characteristic vector of the function $E : \mathbb{R}_+^m \to \mathbb{R}_+ \backslash \{0\}$;

therefore, the second case $S(p) < -1$ is impossible. We have thereby shown that each vector $p \in P_E$ belongs to the set Π_E. This, together with the above-proved inclusion (4), implies that $P_E = \Pi_E$. The proof of the lemma is complete.

The following assertion establishes the existence of the desired system.

Theorem. *For arbitrary positive integers $m \geq 2$ and $n \geq 2$, for real numbers $\alpha_1 \leq \alpha_2 \leq 0$, and for a real vector $a \in \mathbb{R}^m$ with positive components, there exists a completely integrable Pfaff system (1_{mn}) with bounded infinitely differentiable coefficients in $\mathbb{R}^m_+$ and with lower characteristic set*

$$\Pi(A_1, \dots, A_m) = \left\{ p \in \mathbb{R}^m_- : \ \alpha_1 \leq \sum_{i=1}^{m} \frac{p_i}{a_i} \leq \alpha_2 \right\}$$

of positive Lebesgue measure in the space $\mathbb{R}^m$.

Proof. 1. The construction of the desired two-dimensional system (1_{m2}) and its general solution. Following [4], for the case in which $\alpha_1 < \alpha_2$ we first construct a perfect set P_0 on $\Delta = [0, 1]$ similar to the Cantor perfect set [10, p. 50] by using the quantities

$$\varepsilon_n = \exp[(\alpha_1 - \alpha_2) \exp 2^{n+1}], \qquad n \in \mathbb{N}.$$

We divide the original closed interval $\Delta^{(1)}_0 = \Delta$ of zero rank and of length 1 into two closed intervals $\Delta^{(1)}_1 = [0, \varepsilon_1]$ and $\Delta^{(2)}_1 = [1 - \varepsilon_1, 1]$ of length ε_1 of the first rank and one interval $\delta^{(1)}_1 = (\varepsilon_1, 1 - \varepsilon_1)$ of the same rank. In a similar way, we split any closed interval $\Delta^{(m)}_n$, $m \in \{1, \dots, 2^n\}$, of length ε_n and rank n into two closed intervals $\Delta^{(2m-1)}_{n+1}$ and $\Delta^{(2m)}_{n+1}$ of length ε_{n+1} and rank $(n + 1)$ whose left and right endpoints, respectively, coincide with those of $\Delta^{(m)}_n$, and one interval

$$\delta^{(m)}_{n+1} = \Delta^{(m)}_n \backslash (\Delta^{(2m+1)}_{n+1} \cup \Delta^{(2m)}_{n+1})$$

of rank $(n + 1)$. We continue this process infinitely; then Δ contains exactly 2^n intervals $\Delta^{(m)}_n$, $m = 1, \dots, 2^n$, and 2^{n-1} intervals $\delta^{(m)}_n$, $m = 1, \dots, 2^{n-1}$, for each $n \in \mathbb{N}$.

By $\alpha^{(m)}_n$, $m = 1, \dots, 2^n$, we denote the midpoint of $\Delta^{(m)}_n$ and introduce the set

$$P_0 = \bigcap_{n=1}^{+\infty} \bigcup_{n=1}^{2^n} \Delta^{(m)}_n,$$

which, by [10, p. 50], has nonzero Lebesgue measure.

On the closed interval Δ, we define the Cantor step function $\Theta_0 : \Delta \to [0, 1]$ with intervals $\delta^{(m)}_n$ of constant values. According to [10, p. 200], this function is a continuous nondecreasing function on $[0, 1]$ and has the range $[0, 1] = \{\Theta_0(\alpha) : \ \alpha \in P_0\}$. Following [4], we introduce the new continuous nondecreasing function

$$\Theta(\alpha) = |\alpha_2| + (|\alpha_1| - |\alpha_2|)\Theta_0(\alpha) : \ [0, 1] \to [|\alpha_2|, |\alpha_1|].$$

Throughout the following, to preserve the conventional above-introduced notation of the dimension $m \geq 2$ of the time space $\mathbb{R}^m_+$ and the superscript $m \in \{1, \dots, 2^n\}$ on the intervals $\Delta^{(m)}_n$ (and their midpoints $\alpha^{(m)}_n$) and the intervals $\delta^{(m)}_n$ of rank n used in the construction of the Cantor perfect set $P_0 \subset [0, 1]$ and to avoid related confusion, in the proof of this theorem, we denote the dimension m of the spaces $\mathbb{R}^m$ and $\mathbb{R}^m_+$ by $m_0 \geq 2$; i.e., we consider the spaces $\mathbb{R}^{m_0}$ and $\mathbb{R}^{m_0}_+$ and the m_0-dimensional time $t = (t, \dots, t_{m_0}) \in \mathbb{R}^{m_0}_+$.

We split the domain $\mathbb{R}^{m_0}_+$ of the m_0-dimensional time t by the planes

$$\tau(t) \equiv a_1 t_1 + \dots + a_{m_0} t_{m_0} = e^n, \qquad n \in \mathbb{N},$$

into the domains

$$\{t \in \mathbb{R}^{m_0}_+ : \ e^n \leq \tau(t) < e^{n+1}\}, \qquad n \in \mathbb{N}.$$

These domains are successively denoted by $\Pi_n^{(m)}$ so as to ensure that, for any fixed $n \in \mathbb{N}$, the index m takes all values $1, \ldots, 2^n$ and the right boundary of the domain $\Pi_n^{(m)}$ coincides with the left boundary of the domain $\Pi_n^{(m+1)}$; in addition, the right boundary of the domain $\Pi_n^{(2^n)}$ coincides with the left boundary of the domain $\Pi_{n+1}^{(1)}$. In turn, we split any domain $\Pi_n^{(m)}$ with left closed boundary $\tau(t) = \tau_{mn}$ into the subdomains

$$\tilde{\Pi}_n^{(m)} = \{t \in \Pi_n^{(m)} :\ \tau_{mn} \le \tau(t) < \tau_{mn}\sqrt{e} \equiv \tau'_{mn}\}, \qquad \tilde{\tilde{\Pi}}_n^{(m)} = \Pi_n^{(m)} \backslash \tilde{\Pi}_n^{(m)}.$$

In addition, we split the last subdomain $\tilde{\tilde{\Pi}}_n^{(m)}$ into the subdomains

$$\bar{\Pi}_n^{(m)} = \{t \in \tilde{\tilde{\Pi}}_n^{(m)} :\ \tau'_{mn} \le \tau(t) < \tau_{mn} e^{3/4} \equiv \tau''_{mn}\}, \qquad \bar{\bar{\Pi}}_n^{(m)} = \tilde{\tilde{\Pi}}_n^{(m)} \backslash \bar{\Pi}_n^{(m)}.$$

By straightforward computations, we represent the left boundary τ_{mn} of the domain $\tilde{\Pi}_n^{(m)}$ thus defined as

$$\tau_{mn} = \exp(2^n + m - 3), \qquad m = 1, \ldots, 2^n, \qquad n \in \mathbb{N}. \tag{5}$$

Let us introduce the analog

$$e_{01}(\tau, \tau_1, \tau_2) = \begin{cases} \exp\{-\ln^{-2}(\tau/\tau_1) \times \exp[-\ln^{-2}(\tau/\tau_2)]\} & \text{if} \quad \tau_1 < \tau < \tau_2 \\ i - 1 & \text{if} \quad \tau = \tau_i,\ i = 1, 2, \end{cases}$$

of the infinitely differentiable Gelbaum–Olmsted function [11, p. 54 of the Russian translation] on the interval $[\tau_1, \tau_2]$. By using the function e_{01} and by following [4], we introduce two new functions $f[\tau(t)]$ and $F[\tau(t)]$, $t \in \mathbb{R}_+^{m_0}$. We introduce a bounded infinitely differentiable function $f[\tau(t)]$ that is equal to $|\alpha_2|$ for $t \in \tilde{\Pi}_n^{(m)}$ and for arbitrary $n \in \mathbb{N}$ and $m = 1, \ldots, 2^n$ and also for $\{t \in \mathbb{R}_+^{m_0} :\ 0 \le \tau(t) < 1\}$. In the domain $\tilde{\tilde{\Pi}}_n^{(m)}$, we define this function as follows:

$$f[\tau(t)] = \begin{cases} |\alpha_2| + [\Theta(\alpha_n^{(m)}) - |\alpha_2|] e_{01}(\tau(t), \tau'_{mn}, \tau''_{mn}) & \text{for} \quad t \in \bar{\Pi}_n^{(m)} \\ \Theta(\alpha_n^{(m)}) + [|\alpha_2| - \Theta(\alpha_n^{(m)})] e_{01}(\tau(t), \tau''_{mn}, e\tau_{mn}) & \text{for} \quad t \in \bar{\bar{\Pi}}_n^{(m)}. \end{cases}$$

Obviously, this function has the property $f :\ [0, +\infty) \to [|\alpha_2|, |\alpha_1|]$.

We define a bounded infinitely differentiable function $F : [0, +\infty) \to [0, 1]$ by the relations

$$F[\tau(t)] = \begin{cases} \alpha_1^{(1)} & \text{if} \quad 0 \le \tau(t) \le 1 \\ \alpha_n^{(m)} & \text{if} \quad \tau'_{mn} \le \tau(t) < e\tau_{mn}, \\ F(\tau_{mn}) + [F(\tau'_{mn}) - F(\tau_{mn})] e_{01}(\tau(t), \tau_{mn}, \tau'_{mn}) & \text{if} \quad \tau_{mn} < \tau(t) < \tau'_{mn}. \end{cases}$$

Both functions have bounded partial derivatives of any order with respect to all variables $t_1, \ldots, t_{m_0}$, which follows from the corresponding properties of the functions $e_{01}(\tau(t), \tau_1, \tau_2)$.

By using the functions $E(t)$, $f[\tau(t)]$, and $F[\tau(t)]$, we define the two-dimensional vector function

$$x(t, c) = (c_1 [E(t)]^{f[\tau(t)]},\ [c_1 F(\tau(t)) + c_2][E(t)]^{|\alpha_2|}) \in \mathbb{R}^2, \qquad c = (c_1, c_2) \in \mathbb{R}^2, \qquad t \in \mathbb{R}_+^{m_0}, \tag{6}$$

of an arbitrary constant vector $c \in \mathbb{R}^2$ and the m_0-dimensional time vector $t \in \mathbb{R}_+^{m_0}$; this function coincides in form with the function $x(t, c)$ in [4] but substantially differs from the latter not only in the number of independent variables $t_1, \ldots, t_{m_0}$ but also in the definition of the basic functions

$$E(t), \quad f[\tau(t)], \quad F[\tau(t)] \quad \text{and} \quad \tau(t).$$

This function is a general solution of the two-dimensional linear partial differential system

$$\frac{\partial x}{\partial t_i} = A_i(t) x, \quad x \in \mathbb{R}^2, \quad t \in \mathbb{R}_+^{m_0}, \quad i = 1, \ldots, m_0, \tag{7}$$

with bounded infinitely differentiable matrices

$$A_i(t) = \begin{pmatrix} \dfrac{\partial f[\tau(t)]}{\partial t_i} \ln E(t) + f[\tau(t)] E^{-1}(t) \dfrac{\partial E(t)}{\partial t_i} & 0 \\ \dfrac{\partial F[\tau(t)]}{\partial t_i} [E(t)]^{|\alpha_2| - f[\tau(t)]} & |\alpha_2| E^{-1}(t) \dfrac{\partial E(t)}{\partial t_i} \end{pmatrix}, \quad t \in \mathbb{R}_+^{m_0}, \quad i = 1, \dots, m_0. \tag{8}$$

The proof of the infinite differentiability and boundedness of these matrices in $\mathbb{R}_+^m$ as well as of the complete integrability of the constructed two-dimensional system $(1_{m_0 2})$ is similar to the corresponding proofs in [4].

2. Construction of a subset of positive Lebesgue m_0-measure in the entire lower characteristic set of the two-dimensional linear system $(1_{m_0 2})$. By the definition of the set P_0, for any $\alpha \in P_0$ and $n \in \mathbb{N}$, there exists an $m = m_n(\alpha) \in \{1, \dots, 2^n\}$ such that

$$|\alpha_n^{(m)} - \alpha| \le \frac{\varepsilon_n}{2}, \qquad m = m_n(\alpha), \qquad n \in \mathbb{N}. \tag{9}$$

We introduce the concise notation

$$\tau''_{m_n(\alpha),n} \equiv \eta_n(\alpha) \quad \text{and} \quad \alpha_n^{(m_n(\alpha))} \equiv \alpha_n^m(\alpha);$$

by (5), $\eta_n(\alpha)$ satisfies the estimate

$$\eta_n(\alpha) \le e\tau_{m_n(\alpha),n}(\alpha) < \exp 2^{n+1}, \qquad n \in \mathbb{N}. \tag{10}$$

To estimate $|F(\eta_n(\alpha)) - \alpha|$ from above, we use the inequality

$$1 \ge \frac{1}{m_0} E(t) \ge \exp\left[-\frac{\tau(t)}{m_0}\right], \qquad t \in \mathbb{R}_+^{m_0}, \tag{11}$$

and the inequality $f(\tau(t)) \le |\alpha_1|$. Inequalities (10) and (11) imply also the inequality

$$1 \ge \frac{1}{m_0} E(t)\bigg|_{r(t)=\eta_n(\alpha)} > \exp\left(-\frac{1}{m_0} \exp 2^{n+1}\right) > \exp(-\exp 2^{n+1}), \qquad n \in \mathbb{N}. \tag{12}$$

Therefore, by the definition of the function $F(\tau(t))$ and the estimate (12), we have the inequalities

$$\begin{aligned} 2|F(\eta_n(\alpha)) - \alpha| = 2|\alpha_n^{(m)}(\alpha) - \alpha| \underset{(9)}{\le} \varepsilon_n = e^{(\alpha_1 - \alpha_2)\exp 2^{n+1}} \le \left[\frac{1}{m_0} E(t)\bigg|_{\tau(t)=\eta_n(\alpha)}\right]^{|\alpha_1| - |\alpha_2|} \\ \le \left[\frac{1}{m_0} E(t)\bigg|_{\tau(t)=\eta_n(\alpha)}\right]^{f(\eta_n(\alpha)) - |\alpha_2|} \le [E(t)|_{\tau(t)=\eta_n(\alpha)}]^{f(\eta_n(\alpha)) - |\alpha_2|}, \end{aligned} \tag{13}$$

where the penultimate inequality holds by virtue of the left estimate in (11).

By virtue of inequality (13), the considered solution $x(t, a)$ with the initial vector $a = (c_1, -\alpha c_1)$, $\alpha \in P_0$, $c_1 = \text{const} \ne 0$, satisfies the inequalities

$$\begin{gathered} \|x(t,a)\||_{\tau(t)=\eta_n(\alpha)} \le 2|c_1| [E(t)|_{\tau(t)=\eta_n(\alpha)}]^{f(\eta_n(\alpha)} = 2|c_1| [E(t)|_{\tau(t)=\eta_n(\alpha)}]^{\Theta(\alpha_n^{(m)})}, \\ m = m_n(\alpha) \in \{1, \dots, 2^n\}, \qquad n \in \mathbb{N}, \qquad t \in \mathbb{R}_+^{m_0}. \end{gathered} \tag{14}$$

It follows from inequalities (9) that

$$\alpha_n^{(m_n(\alpha))} \to \alpha \in P_0 \subset [0, 1], \qquad n \to +\infty.$$

Therefore, by virtue of the continuity of the function $\Theta(\alpha): [0,1] \to [|\alpha_2|, |\alpha_1|]$, we have

$$\Theta(\alpha_n^{(m_n(\alpha))}) \to \Theta(\alpha), \qquad \alpha \in P_0, \qquad n \to +\infty. \tag{15}$$

Now let us show that the set P_x of lower characteristic vectors of the considered solution $x(t,a)$ contains the set

$$Q_\alpha \equiv \left\{ p \in \mathbb{R}_-^{m_0}: \sum_{i=1}^{m_0} \frac{p_i}{a_i} = -\Theta(\alpha) \right\}.$$

To this end, in accordance with conditions (3_1) and (3_2), we should prove the relation $l_x(p) = 0$ and the inequalities $l_x(p + \varepsilon e_k) < 0$, $k = 1, \ldots, m_0$, for any vector $p \in Q_\alpha$. Let us obtain the latter inequalities. From (14), we have

$$\begin{aligned} l_x(p+\varepsilon e_k) &\le \varliminf_{n\to\infty} \frac{\Theta(\alpha_n^{(m)})\ln E(t) - (p,t) - \varepsilon t_k}{\|t\|}\Bigg|_{\substack{t\in T_\alpha \\ \tau(t)=\eta_n(\alpha)}} \\ &= \varliminf_{n\to\infty} \frac{1}{r}\left\{ -\left[\Theta(\alpha_n^{(m)}) - \sum_{i=1}^{m_0}\frac{p_i}{a_i}\right] - \frac{\varepsilon}{a_k}\right\} \\ &= -\frac{\varepsilon}{ra_k} + \varliminf_{n\to\infty}\frac{1}{r}[\Theta(\alpha) - \Theta(\alpha_n^{(m)})]|_{m=m_n(\alpha)} \\ &= -\frac{\varepsilon}{ra_k} < 0, \quad r \equiv \left(\sum_{i=1}^{m_0}\frac{1}{a_i^2}\right)^{1/2}, \quad \forall \varepsilon > 0, \quad k = 1,\ldots,m; \end{aligned}$$

here the last equality in the chain holds by virtue of property (15) and the continuity of the function $\Theta: [0,1] \to [|\alpha_2|, |\alpha_1|]$.

To prove property (3_1), let us first justify the inequality $l_x(p) \ge 0$, $p \in Q_\alpha$. Let the limit $l_x(p)$ be realized along a sequence $\{t(k)\}$ with the property $\|t(k)\| \to +\infty$ as $k \to \infty$. Obviously, this sequence has the property $\tau(t(k)) \to +\infty$ as $k \to \infty$, and without loss of generality, we can assume that there exist limits

$$f[\tau(t(k))] \to f_0 \in [|\alpha_2|, |\alpha_1|], \qquad F[\tau(t(k))] \to F_0 \in [0,1], \qquad k \to \infty.$$

Again without loss of generality, one can assume that there exists a fixed index $l \in \{1, \ldots, m_0\}$ such that

$$a_l t_l(k) = \min_{i=1,\ldots,m_0}\{a_i t_i(k)\} \qquad \forall k \in \mathbb{N}. \tag{16}$$

[If this condition does not hold for all $k \in \mathbb{N}$ but holds for infinitely many $k \in \mathbb{N}$, then we can rarefy the original sequence $\{t(k)\}$ by deleting the elements for which inequality (16) fails; the successive numbering of terms of the remaining infinite sequence gives a new sequence $\{t(k)\}$ for which condition (16) holds for all $k \in \mathbb{N}$.]

Finally, without loss of generality we assume that only one of the following two possible cases takes place.

1. $t(k) \in \tilde{\Pi}_{n(k)}^{(m(k))}$ for all $k \in \mathbb{N}$.

2. $t(k) \in \tilde{\tilde{\Pi}}_{n(k)}^{(m(k))}$ for all $k \in \mathbb{N}$.

Consider the first case. By using the estimate

$$\|x(t(k),c)\| \ge |c_1|[E(t(k))]^{|\alpha_2|}, \qquad t(k) \in \tilde{\Pi}_{n(k)}^{(m(k))}, \qquad k \in \mathbb{N}, \tag{17}$$

for the norm of the solution $x(t,c)$ and similar considerations from the proof of the lemma and by taking into account the inequalities $-p_i \ge 0$ and $\Theta(\alpha) \ge |\alpha_2|$, we obtain the estimates

$$\begin{aligned} l_x(p) &\ge \varliminf_{k\to\infty}\frac{|\alpha_2|\ln E(t(k)) - (p,t(k))}{\|t(k)\|} \ge \varliminf_{k\to\infty}\frac{a_l t_l(k)[-|\alpha_2| - S(p)]}{\|t(k)\|} \\ &\ge \varliminf_{k\to\infty}\frac{a_l t_l(k)[\Theta(\alpha) - |\alpha_2|]}{\|t(k)\|} \ge 0, \qquad p \in Q_\alpha. \end{aligned} \tag{18}$$

Now consider the second case. In this case, by the construction of the function $F[\tau(t)]$, for $F_0 = \alpha$, we have the relations

$$F[\tau(t(k))] = \alpha_{n(k)}^{(m(k))} \to \alpha, \qquad k \to \infty. \tag{19}$$

The construction of the function $f[\tau(t)]$ implies the inequalities

$$|\alpha_2| \le f[\tau(t(k))] \le \Theta(\alpha_{n(k)}^{(m(k))}), \qquad t(k) \in \tilde{\tilde{\Pi}}_{n(k)}^{(m(k))}, \qquad k \in \mathbb{N}. \tag{20}$$

Therefore, by virtue of the inequalities

$$0 < \frac{1}{m_0} E(t) \le 1, \qquad t \in \mathbb{R}_+^{m_0},$$

and inequality (20), the norm of the solution $x(t,c)$ can be estimated as

$$\|x(t(k),c)\| \ge |c_1|[E(t(k))]^{f[\tau(t(k))]} \ge |c_1| m_0^{|\alpha_2|-|\alpha_1|}[E(t(k))]^{\Theta(\alpha_{n(k)}^{(m(k))})}, \quad t(k) \in \tilde{\tilde{\Pi}}_{n(k)}^{(m(k))}, \quad k \in \mathbb{N}.$$

This, together with the inequalities $p_i \ge 0$, condition (16), and the limit relation (19), provides the desired inequalities

$$\begin{aligned} l_x(p) &\ge \varliminf_{k\to\infty} \frac{\Theta(\alpha_{n(k)}^{(m(k))}) \ln E(t(k)) - (p, t(k))}{\|t(k)\|} \ge \varliminf_{k\to\infty} \frac{-a_l t_l(k)[\Theta(\alpha_{n(k)}^{(m(k))}) + S(p)]}{\|t(k)\|} \\ &\ge \varliminf_{k\to\infty} [-a_l|\Theta(\alpha_{n(k)}^{(m(k))}) - \Theta(\alpha)|] = 0, \qquad p \in Q_\alpha. \end{aligned}$$

It remains to consider the subcase $F_0 \ne \alpha$ of the second case. Then, for sufficiently large k, the norm of the solution $x(t,c)$ satisfies the estimate

$$\|x(t(k),c)\| \ge \frac{1}{2}|c_1|\,|F_0 - \alpha|[E(t(k))]^{|\alpha_2|}, \qquad t(k) \in \tilde{\tilde{\Pi}}_{n(k)}^{(m(k))}, \qquad k_0 \le k \in \mathbb{N},$$

similar to the estimate (17) in the first case. By using this estimate and by performing considerations similar to the proof of the estimate (18), we obtain the desired inequality $l_x(p) \ge 0$ in this subcase as well.

Thus, the inequality $l_x(p) \ge 0$ has been proved for all $p \in Q_\alpha$. To prove the desired relation $l_x(p) = 0$ in property (3_1), we suppose the contrary: there exists a $p \in Q_\alpha$ such that $l_x(p) > 0$. Then we arrive at a contradiction in the following way:

$$\begin{aligned} l_x(p + \varepsilon e_i) &= \varliminf_{t\to+\infty} \frac{\ln\|x(t)\| - (p,t) - \varepsilon_i t_i}{\|t\|} \ge \varliminf_{t\to+\infty} \frac{\ln\|x(t)\| - (p,t)}{\|t\|} + \varliminf_{t\to+\infty} \left(-\varepsilon_i \frac{t_i}{\|t\|}\right) \\ &= l_x(p) - \varepsilon_i > 0 \qquad \forall i = 1, \ldots, m_0, \end{aligned}$$

for all $\varepsilon_i \in (0, l_x(p)/2)$.

We have thereby proved the inclusion

$$Q_\alpha \subset P_{x(\cdot,\alpha)}, \quad a = (c_1, -\alpha c_1) \ne 0, \qquad \forall \alpha \in P_0$$

and hence the inclusion

$$Q(m_0) \equiv \left\{ p \in \mathbb{R}_-^{m_0} : \ \alpha_1 \le \sum_{i=1}^{m_0} \frac{p_i}{a_i} \le \alpha_2 \right\} \subset \Pi(A_1, \ldots, A_{m_0}).$$

3. CONSTRUCTION OF THE ENTIRE CHARACTERISTIC SET OF THE LINEAR SYSTEM ($1_{M_0 2}$)

Now let us prove the relation

$$\Pi(A_1, \ldots, A_{m_0}) = Q(m_0).$$

To this end, we first prove the inequality $p \le 0$ for any lower characteristic vector

$$p \in \Pi(A_1, \ldots, A_{m_0}).$$

Suppose the contrary: there exists a vector p with a component $p_l > 0$. The vector p is a lower characteristic vector of some nontrivial solution $x(t, c)$ given by relation (6). Since the functions $f[\tau(t)]$ and $F[\tau(t)]$ satisfy the inequalities

$$|\alpha_2| \le f[\tau(t)] \le |\alpha_1|, \qquad 0 \le F[\tau(t)] \le 1, \qquad t \in \mathbb{R}_+^{m_0}, \tag{21}$$

it follows that the solution $x(t, c)$ is bounded,

$$\|x(t, c)\| \le 2|c_1| m_0^{|\alpha_1|} + |c_2| m_0^{|\alpha_2|}, \qquad t \in \mathbb{R}_+^{m_0}.$$

Therefore, in the direction $t_i = 0$, $i \ne l$, $t_l \to +\infty$, we have an inequality contradicting condition (3_1) in the definition of a lower characteristic vector p. The proof of the inequality $p \le 0$ is complete.

Now let us prove the second desired inequality

$$S(p) \ge \alpha_1 \qquad \forall p \in \Pi(A_1, \ldots, A_{m_0}). \tag{22}$$

We again suppose the contrary: there exists a vector $p \in \Pi(A_1, \ldots, A_{m_0})$ such that $S(p) < \alpha_1$. Let $x(t, c)$ be a nontrivial solution (6) for which p is a lower characteristic vector. By virtue of the inequality $E(t)/m_0 \le 1$, the norm of this solution admits the estimates

$$\begin{aligned} &\|x(t, c)\| \ge |c_1| m_0^{|\alpha_2|} [E(t)/m_0]^{f[\tau(t)]} \ge |c_1| m_0^{\alpha_1 - \alpha_2} [E(t)]^{|\alpha_1|}, \qquad c_1 \ne 0, \qquad t \in \mathbb{R}_+^{m_0}, \\ &\|x(t, c)\| = |c_2| [E(t)]^{|\alpha_2|} \ge |c_2| m_0^{\alpha_1 - \alpha_2} [E(t)]^{|\alpha_1|}, \qquad c_1 = 0, \qquad c_2 \ne 0, \qquad t \in \mathbb{R}_+^{m_0}. \end{aligned} \tag{23}$$

Some components p_i of the vector p can be zero. Without loss of generality, one can assume that the first s of them are negative and all the remaining components are zero,

$$p_i < 0, \qquad i = 1, \ldots, s, \qquad s \in \{1, \ldots, m_0\}, \qquad p_{s+1} = \cdots = p_{m_0} = 0. \tag{24}$$

We fix some $l \in \{1, \ldots, s\}$ and choose a number ε satisfying the conditions

$$0 < 2\varepsilon \max_{i,j} \{a_i / a_j\} < \alpha_1 - S(p), \qquad 2\varepsilon < \min_{i=1,\ldots,s} \{|p_i|\}. \tag{25}$$

The following cases are possible for an arbitrary vector $t \in \mathbb{R}_+^{m_0}$:

$$\min_{i=1,\ldots,s} \{a_i t_i\} \equiv a_l t_l \le \frac{1}{2} \max_{i=1,\ldots,s} \{a_i t_i\} \equiv \frac{1}{2} a_q t_q, \qquad l \ne q, \tag{26}$$

$$a_l t_l > \frac{1}{2} a_q t_q. \tag{27}$$

In case (26), from the estimate (23), relation (24), and the second condition in (25), we have the inequalities

$$\begin{aligned} r(t) &\equiv \ln \|x(t, c)\| - (p, t) - \varepsilon t_s \ge d + \alpha_1 a_l t_l - \sum_{i=1}^{s} \frac{p_i}{a_i} a_i t_i - \varepsilon t_q \\ &\ge d + a_l t_l [\alpha_1 - S(p)] - \frac{p_q}{a_q} (a_q t_q - a_l t_l) - \varepsilon t_q \\ &\ge d + a_l t_l [\alpha_1 - S(p)] - \left(\frac{1}{2} \frac{p_q}{a_q} + \frac{\varepsilon}{a_q} \right) a_q t_q \ge d = \text{const}, \qquad t \in \mathbb{R}_+^{m_0}. \end{aligned}$$

In case (27), from the same estimates (23), relation (24), and the first condition in (25), we have the inequalities

$$r(t) \geq d + a_l t_l[\alpha_1 - S(p)] - \varepsilon t_q \geq d + a_l t_l[\alpha_1 - S(p) - 2\varepsilon a_l/a_q] \geq d = \text{const}, \qquad t \in \mathbb{R}_+^{m_0}.$$

We have thereby proved the assertion

$$\alpha_1 - S(p) > 0 \;\Rightarrow\; r(t) \geq d = \text{const}, \qquad t \in \mathbb{R}_+^{m_0},$$

and hence the inequality

$$l_x(p + \varepsilon e_s) \geq 0, \qquad \varepsilon > 0,$$

which contradicts condition (3_2) in the definition of a lower characteristic vector $p \in P_x$. The proof of the desired inequality (22) is thereby complete.

Finally, let us prove the last desired inequality

$$S(p) \leq \alpha_2, \qquad p \in \Pi(A_1, \ldots, A_{m_0}). \tag{28}$$

Just as above, we suppose the contrary: there exists a vector $p \in \Pi(A_1, \ldots, A_{m_0})$ such that the opposite inequality $S(p) > \alpha_2$ holds. The vector p is a lower characteristic vector of some nontrivial solution $x(t,c)$ given by relation (6). This, together with inequalities (21), implies that such a solution can be estimated as

$$\begin{aligned}\|x(t,c)\| &\leq |c_1| m_0^{|\alpha_1|}[E(t)/m_0]^{|\alpha_2|} + (|c_1| + |c_2|)[E(t)]^{|\alpha_2|} \\ &\leq (2|c_1| m_0^{|\alpha_1|} + |c_2|)[E(t)]^{|\alpha_2|}, \qquad t \in \mathbb{R}_+^{m_0}.\end{aligned}$$

By using them and the first property in (3_1) of the definition of a lower characteristic vector p of a solution $x(t,c)$, we obtain the following contradiction:

$$0 = l_x(p) \leq \varliminf_{t\to\infty} \left.\frac{|\alpha_2| \ln E(t) - (p,t)}{\|t\|}\right|_{t\in T_a} = \frac{1}{r}[\alpha_2 - S(p)] < 0.$$

The proof of inequality (28) is complete. We have thereby proved the theorem in the two-dimensional case $n = 2$.

4. CONSTRUCTION OF AN N-DIMENSIONAL SYSTEM (1_{M_0N}) WITH M_0-DIMENSIONAL TIME AND WITH LOWER CHARACTERISTIC SET OF POSITIVE LEBESGUE M_0-MEASURE

To give a complete proof of the theorem (in the case of an arbitrary $n > 2$), we supplement the constructed system $(1_{m_0 2})$ with general solution $x(t,c)$ given by (6) by a completely integrable system such that, in the general solution $z(t,C)$ of the new system (1_{mn}), the general solution $x(t,c)$ of system $(1_{m_0 2})$ with a nonzero vector $c \in \mathbb{R}^2$ is dominant, and in the case in which $c = 0$ and hence $x(t,0) \equiv 0$, the supplementing system of order $(n-2)$ has a lower characteristic set lying in the lower characteristic set of the two-dimensional system $(1_{m_0 2})$. By following [4], we choose such a supplementing system in the form

$$\frac{\partial y}{\partial t_i} = \frac{|\alpha_1|}{E(t)} \frac{\partial E(t)}{\partial t_i} y, \qquad y \in \mathbb{R}^{n-2}, \qquad t \in \mathbb{R}_+^{m_0}, \qquad i = 1, \ldots, m_0, \tag{29}$$

with a new function $E(t)$. Obviously, this system has bounded infinitely differentiable coefficient matrices

$$B_i(t) = \frac{|\alpha_1|}{E(t)} \frac{\partial E(t)}{\partial t_i} E_{n-2}, \qquad i = 1, \ldots, m_0,$$

is completely integrable, and has the general solution

$$y(t,d) = (d_1, \ldots, d_{n-2})[E(t)]^{|\alpha_1|}, \quad t \in \mathbb{R}_+^{m_0}, \tag{30}$$

with an arbitrary constant vector $d \in \mathbb{R}^{n-2}$. The same properties hold for the complete system (1_{mn}) consisting of two block-systems $(1_{m_0 2})$ and (29), and its general solution $z(t, C)$ has the form

$$z(t,C) = (x(t,c), y(t,d)) \in \mathbb{R}^n, \qquad C = (c,d) \in \mathbb{R}^n, \qquad t \in \mathbb{R}^{m_0}_+. \tag{31}$$

This final block-diagonal system has the form

$$\frac{\partial z}{\partial t_i} = \operatorname{diag}[A_i(t), B_i(t)]z, \qquad z \in \mathbb{R}^n, \qquad t \in \mathbb{R}^{m_0}_+, \qquad i = 1, \ldots, m_0. \tag{32}$$

By the above-proved lemma, any nontrivial solution $y(t,d)$ of system (29) given by (30) has the lower characteristic set $\{p \in \mathbb{R}^{m_0}_- : \ S(p) = \alpha_1\}$, which belongs to the lower characteristic set of system $(1_{m_0 2})$. Therefore, any solution

$$z(t,C) = (0, y(t,d)) \in \mathbb{R}^n$$

of system (30) corresponding to the vector $C = (0,d) \in \mathbb{R}^n \backslash \{0\}$ has the same lower characteristic set. But for the case in which $c \neq 0$, from the representations (6), (30), and (31) and from the estimate

$$0 < E(t)/m_0 \leq 1, \qquad t \in \mathbb{R}^{m_0}_+,$$

we obtain the inequalities

$$1 \leq \frac{\|z(t,C)\|}{\|x(t,c)\|} \leq 1 + \frac{\|y(t,d)\|}{\|x(t,c)\|} \leq 1 + \frac{\|d\|}{|c_1|}\,[E(t)]^{|\alpha_1| - f[\tau(t)]} \leq 1 + \frac{\|d\|}{|c_1|} m_0^{|\alpha_1|}, \qquad t \in \mathbb{R}^{m_0}_+,$$

for $c \neq 0$ and

$$1 \leq \frac{\|z(t,C)\|}{\|x(t,c)\|} \leq 1 + \frac{\|d\|}{|c_2|}\,[E(t)]^{|\alpha_1| - |\alpha_2|} \leq 1 + \frac{\|d\|}{|c_2|} m_0^{|\alpha_1|}, \qquad t \in \mathbb{R}^{m_0}_+,$$

for $c_1 = 0$ and $c_2 \neq 0$. These inequalities can be represented by the single inequality

$$1 \leq \frac{\|z(t,C)\|}{\|x(t,c)\|} \leq 1 + \frac{\|d\| m_0^{|\alpha_1|}}{|c_1| + |c_2|(1 - \operatorname{sgn}|c_1|)}, \qquad t \in \mathbb{R}^{m_0}_+,$$

which implies the equivalence of norms of the solutions $z(t,C)$ and $x(t,c)$ for the case in which $c \neq 0$.

Therefore, the lower characteristic set of the n-dimensional system (32) with m_0-dimensional time coincides with the lower characteristic set of the two-dimensional system $(1_{m_0 2})$ with time of the same arbitrary dimension $m_0 \geq 2$.

For $\alpha_1 = \alpha_2 = \alpha \leq 0$, the desired system $(1_{m_0 n})$ has a form similar to the corresponding system in [4], and by a lemma in the present paper, its lower characteristic set coincides with the set $\{p \in \mathbb{R}^m_- : \ S(p) = \alpha\}$. The proof of the theorem is complete.

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