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# WELL-POSEDNESS AND BLOW UP FOR IBVP FOR SEMILINEAR PARABOLIC EQUATIONS AND NUMERICAL METHODS

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**Abstract** — We have studied the stability of finite-difference schemes approximating boundary value problems for parabolic equations with a nonlinear and nonmonotonic source of the power type. We have obtained simple sufficient input data conditions, in which the solution of the differential problem is globally stable for all $0 \leqslant t \leqslant +\infty$. It is shown that if these conditions fail, then the solution can blow up (go to infinity) in finite time. The lower bound of the blow up time has been determined. The stability of the solution of BVP for the nonlinear convection-diffusion equation has been investigated. In all cases, we used the method of energy inequalities based on the application of the Chaplygin comparison theorem for nonlinear differential equations, Bihari-type inequalities and their discrete analogs.

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## Introduction

Nonlinear parabolic equations play an important part in the mathematical modeling of applied problems. A large number of papers has been devoted to the study of these equations. Many authors (see, for example, [5, 6, 8, 10]) have shown that, in the general case of arbitrary initial data, there exist no global solutions of the Cauchy problem for nonlinear parabolic equations.

The existence of solutions to boundary-value problems for nonlinear parabolic equations has been studied in [2, 26, 27]. In these works, it has established that the boundary-value problem with homogeneous boundary conditions has exactly one solution with a finite life-time $T > 0$. However, the blow-up time has not been estimated. It was only indicated that this time depends on the norm of the initial function. In [12], the upper bound of the blow-up time was obtained. In [25], double-sided estimates of the blow-up time for the

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solution of Sobolev-type equations was obtained using the method of energy inequalities. To investigate the stability of the solution of nonlinear evolution equations, the dynamical systems approach [9] was also used.

To prove the existence of solutions to initial-boundary value problems for nonlinear parabolic equations, the method of nonlinear capacity developed by E. Mitidieri and S. Pokhozhaev (see, for example, [7, 16] and the references therein) as well as by other authors.

This work presents the results concerning the stability of solutions of both differential problems and corresponding finite-difference schemes for one-dimensional parabolic equations with a nonmonotonic source. The existence of a bounded global solution under conditions imposed only on the input data of the problem has been proved. In the case where these conditions are not fulfilled, the existence of a solution in a finite time has been proved proved. Herewith the lower bound of the time of possible blow-up of the solution has been obtained. This estimate shows that the blow-up time depends on both the norm of the initial function and other input data such as the domain measure [19] and coefficients of the equation.

For the finite-difference schemes approximating the above-mentioned problems we have also obtained estimates of the solutions for arbitrary $0 \leqslant t \leqslant +\infty$ under conditions imposed on the input data of the problem. The finiteness of the solution in a finite time in the case where these conditions are not fulfilled has been proved. All discrete conditions and estimates are consistent with the differential analogs.

To obtain estimates of the solutions Bihari-type inequalities and their differential analogs are often used [1, 4, 13, 15, 18]. In this paper, besides such estimates we use the auxiliary differential and discrete inequalities obtained by means of the Chaplygin comparison theorem [3]. Similar approach is used for investigating Cauchy problem for non-linear Schrödinger equation in [17].

Moreover in this paper using the energy inequalities technique, we have obtained estimates of the solutions and the times of their possible blow-up for multidimensional problems. Sufficient conditions for the stability of the solution of the initial-boundary value problem for the nonlinear convection-diffusion equation have been obtained.

We present numerical results of investigating the behavior of the approximate solution depending on the fulfillment and non-fulfillment of the conditions for the existence of a bounded solution of the initial-boundary value problem for the semilinear parabolic equation. The two-sided bounds of the blow-up time have been verified. The obtained experimental time are consistent with both the estimates obtained in this paper and with the results from [12].

## 1. Global stability of the solution of the semilinear homogeneous parabolic equation

In this section, we obtain *a priori* estimates for the solution of the initial-boundary value problem for the one-dimensional semilinear parabolic equation with a nonlinear nonmonotonic source and estimates of the global stability.

Consider the following problem:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\,\frac{\partial u}{\partial x}\right) + cu|u|^{p-1}, \quad x \in \Omega = \{x \ : \ 0 < x < l\}, \quad 0 < t \leqslant T, \tag{1.1}$$

$$u(0,t) = u(l,t) = 0, \quad 0 \leqslant t \leqslant T, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in \bar{\Omega}. \tag{1.3}$$

Here

$$c = \text{const} > 0, \quad p = \text{const} > 1, \tag{1.4}$$

$$k(x) \in C^1\left(\bar{\Omega}\right), \quad 0 < m_1 \leqslant k(x) \leqslant m_2 \quad \text{for all } x \in \Omega. \tag{1.5}$$

Let the operator $\mathcal{A}$ be given by

$$\mathcal{A}u = -\frac{\partial}{\partial x}\left(k(x)\,\frac{\partial u}{\partial x}\right).$$

This operator pertains the set $\mathscr{D}(\mathcal{A}) = \mathring{H}^1(\Omega) \cap H^2(\Omega)$ to $L_2(\Omega)$. It is easy to show that for the linear self-adjoint operator $\mathcal{A}$ the following inequality is fulfilled [24]:

$$(\mathcal{A}u, u)_{L_2(\Omega)} \geqslant \lambda \|u\|^2_{L_2(\Omega)} \quad \text{for all } u \in \mathscr{D}(\mathcal{A}), \quad \lambda = \frac{m_1 \pi^2}{l^2}. \tag{1.6}$$

Moreover, taking into account (1.6) we obtain

$$(\mathcal{A}u, u)_{L_2(\Omega)} \leqslant \|u\|_{L_2(\Omega)} \|\mathcal{A}u\|_{L_2(\Omega)} \leqslant \frac{1}{\sqrt{\lambda}} \sqrt{(\mathcal{A}u, u)_{L_2(\Omega)}} \|\mathcal{A}u\|_{L_2(\Omega)}.$$

Hence we have

$$\|\mathcal{A}u\|^2_{L_2(\Omega)} \geqslant \lambda\,(\mathcal{A}u, u)_{L_2(\Omega)} \quad \text{for all } u \in \mathscr{D}(\mathcal{A}). \tag{1.7}$$

For the functions $v \in \mathring{H}^1(\Omega)$ we get [20]

$$\|u\|^2_{C(\Omega)} \leqslant \gamma^2\,(\mathcal{A}u, u)_{L_2(\Omega)}, \quad \gamma = \frac{\sqrt{l}}{2\sqrt{m_1}}. \tag{1.8}$$

By $\mathscr{H}_{\mathcal{A}} = \mathring{H}^1(\Omega)$ we denote the energy space determined by the inner product $(v, w)_{\mathcal{A}} = (\mathcal{A}v, w)_{L_2(\Omega)}$.

Together with (1.1)–(1.3) we consider the following problem with perturbed initial data:

$$\frac{\partial \tilde{u}}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\,\frac{\partial \tilde{u}}{\partial x}\right) + c\tilde{u}|\tilde{u}|^{p-1}, \quad x \in \Omega, \quad 0 < t \leqslant T, \tag{1.9}$$

$$\tilde{u}(0,t) = \tilde{u}(l,t) = 0, \quad 0 \leqslant t \leqslant T, \tag{1.10}$$

$$\tilde{u}(x,0) = \tilde{u}_0(x), \quad x \in \Omega. \tag{1.11}$$

Suppose that

$$u_0(x), \tilde{u}_0(x) \in \mathscr{D}(\mathcal{A}). \tag{1.12}$$

### 1.1. *A priori* estimates of the solution for arbitrary $t \in [0, +\infty]$

To obtain global *a priori* estimates of the solution, we need the following

**Lemma 1.1.** *Let the nonnegative function $v = v(t)$ satisfy the following relations:*

$$\frac{\mathrm{d}v}{\mathrm{d}t} \leqslant av^q - bv, \quad v(0) = v_0, \tag{1.13}$$

*$a$, $b$ are positive constants, $q > 1$. Then, if*

$$\frac{av_0^{q-1}}{b} \leqslant 1, \tag{1.14}$$

*then the function $v(t)$ is bounded for all $0 \leqslant t \leqslant +\infty$, and the following estimate holds:*

$$v(t) \leqslant v_0 \tag{1.15}$$

*If condition (1.14) is not fulfilled, then the function $v(t)$ can blow up within a finite time.*

*Proof.* Consider the Cauchy problem for the Bernoulli equation

$$\frac{\mathrm{d}w}{\mathrm{d}t} = aw^q - bw, \quad w(0) = v_0. \tag{1.16}$$

By the Chaplygin comparison theorem [3] we have $v(t) \leqslant w(t)$. The solution of problem (1.16) is defined by the following formula:

$$w(t) = \frac{b^{1/(q-1)} v_0}{\left(av_0^{q-1} + \left(b - av_0^{q-1}\right) e^{(q-1)bt}\right)^{1/(q-1)}}. \tag{1.17}$$

Thus, taking into account condition (1.13), we complete the proof of the lemma. □

Now we prove the following

**Theorem 1.1.** *If the input data of problems (1.1)–(1.3) and (1.9)–(1.12) satisfy the inequality*

$$\frac{cl^{(p+3)/2} \max\left\{\|u_0\|_{\mathcal{A}}^{p-1}, \|\tilde{u}_0\|_{\mathcal{A}}^{p-1}\right\}}{2^{p-1}\pi^2 m_1^{(p+1)/2}} \leqslant 1, \tag{1.18}$$

*then for the solutions of problems (1.1)–(1.3) and (1.9)–(1.12), if any, for any $t$ the following estimates hold:*

$$\left\|u(t)\right\|_{\mathcal{A}} \leqslant \|u_0\|_{\mathcal{A}}, \quad \left\|\tilde{u}(t)\right\|_{\mathcal{A}} \leqslant \|\tilde{u}_0\|_{\mathcal{A}}. \tag{1.19}$$

*Proof.* We obtain the first estimate from (1.19). Multiply Eq. (1.1) by $2\mathcal{A}u$ and integrate the result over the domain $\Omega$. We get the following energy identity:

$$\frac{\mathrm{d}\|u\|_{\mathcal{A}}^2}{\mathrm{d}t} + 2\|\mathcal{A}u\|_{L_2(\Omega)}^2 = 2c\left(u|u|^{p-1}, \mathcal{A}u\right)_{L_2(\Omega)}. \tag{1.20}$$

For the right-hand side of (1.20), using The Schwarz and Cauchy inequalities and taking into account (1.6)–(1.8), we get

$$
\begin{aligned}
2c\left(u|u|^{p-1}, \mathcal{A}u\right)_{L_2(\Omega)} &\leqslant 2c\|u\|^{p-1}_{C(\Omega)}\|u\|_{L_2(\Omega)}\|\mathcal{A}u\|_{L_2(\Omega)} \leqslant \frac{2c\gamma^{p-1}}{\lambda^{1/2}}\|u\|^{p}_{\mathcal{A}}\|\mathcal{A}u\|_{L_2(\Omega)} \\
&\leqslant \frac{2c\gamma^{p-1}}{\lambda^{1/2}}\left(\varepsilon\|u\|^{2p}_{\mathcal{A}} + \frac{1}{4\varepsilon}\|\mathcal{A}u\|^2_{L_2(\Omega)}\right).
\end{aligned}
\tag{1.21}
$$

Substituting the last estimate into (1.20), in view of (1.7) and taking $\varepsilon = c\gamma^{p-1}/(2\lambda^{1/2})$, we obtain

$$
\frac{\mathrm{d}\|u\|^2_{\mathcal{A}}}{\mathrm{d}t} \leqslant \frac{c^2\gamma^{2(p-1)}}{\lambda}\gamma^{p-1}\|u\|^{2p}_{\mathcal{A}} - \lambda\|u\|^2_{\mathcal{A}}.
$$

Further we use Lemma 1.1. Let $v = \|u\|^2_{\mathcal{A}}$, $a = c^2\gamma^{2(p-1)}/\lambda$, $b = \lambda$, $q = p$. Under (1.18) the conditions of Lemma (1.14) are satisfied and, consequently, the first estimate from (1.19) holds. The second bound is proved similarly. □

Theorem 1.1 contains the sufficient condition (1.18) of finiteness of the energy norm of the solution for arbitrary $T \leqslant +\infty$.

**Remark 1.1.** We prove only the finiteness of the solution of the studied problem. However, suppose that the following (more strong than (1.14) and, consequently, than (1.18)) condition holds:

$$
b - av_0^{q-1} \geqslant \delta > 0.
$$

Then from (1.17) we can prove that the solution tends to zero as $t \to +\infty$.

### 1.2. Global stability and uniqueness of the solution

To study the stability of the solution of problem (1.1)–(1.3), subtract (1.1), (1.2) and (1.3) from (1.9), (1.10) and (1.11), respectively. Taking into account the mean value theorem

$$
\tilde{u}|\tilde{u}|^{p-1} - u|u|^{p-1} = p\left(\int_0^1 |u + \theta\bar{u}|^{p-1}\,\mathrm{d}\theta\right)\bar{u} = p\mathcal{P}(u,\tilde{u})\bar{u},
$$

we get the following perturbation problem $\bar{u} = \tilde{u} - u$:

$$
\frac{\partial\bar{u}}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial\bar{u}}{\partial x}\right) + pc\mathcal{P}(u,\tilde{u})\bar{u}, \quad x \in \Omega, \quad 0 < t \leqslant T, \tag{1.22}
$$

$$
\bar{u}(0,t) = \bar{u}(l,t) = 0, \quad 0 < t \leqslant T, \tag{1.23}
$$

$$
\bar{u}(x,0) = \bar{u}_0(x) = \tilde{u}_0(x) - u_0(x), \quad x \in \Omega. \tag{1.24}
$$

It is clear that the function $\mathcal{P}(u,\tilde{u})$ satisfies the inequality

$$
\left\|\mathcal{P}(u,\tilde{u})\right\|_{C(\Omega)} \leqslant \max\left\{\|u\|^{p-1}_{C(\Omega)}, \|\tilde{u}\|^{p-1}_{C(\Omega)}\right\}. \tag{1.25}
$$

Now we prove the following

**Theorem 1.2.** *Suppose that the input data of problems* (1.1)–(1.3) *and* (1.9)–(1.12) *satisfy the inequality*

$$\frac{cl^{(p+3)/2}\max\left\{\|u_0\|_{\mathcal{A}}^{p-1},\|\tilde{u}_0\|_{\mathcal{A}}^{p-1}\right\}}{2^{p-1}\pi^2 m_1^{(p+1)/2}} \leqslant \frac{1}{p}. \tag{1.26}$$

*Then the solution of problem* (1.1)–(1.3), *if any, is stable in the sense of the initial data for any* $t \in [0,T]$ *and the perturbation of the solution satisfies the following estimate:*

$$\left\|\tilde{u}(t) - u(t)\right\|_{\mathcal{A}} \leqslant \|\tilde{u}_0 - u_0\|_{\mathcal{A}}. \tag{1.27}$$

*Proof.* Multiplying both sides of Eq. (1.22) by $2\mathcal{A}\bar{u}$ and integrating the result over $\Omega$, we get the following energy identity:

$$\frac{\mathrm{d}\|\bar{u}\|_{\mathcal{A}}^2}{\mathrm{d}t} + 2\|\mathcal{A}\bar{u}\|_{L_2(\Omega)}^2 = 2pc\left(\mathcal{P}(u,\tilde{u})\bar{u}, \mathcal{A}\bar{u}\right)_{L_2(\Omega)}. \tag{1.28}$$

In view of (1.25), it follows for the right-hand side of the last relation that

$$\begin{aligned} 2pc\left(\mathcal{P}(u,\tilde{u})\bar{u}, \mathcal{A}\bar{u}\right)_{L_2(\Omega)} &\leqslant 2pc\left\|\mathcal{P}(u,\tilde{u})\bar{u}\right\|_{L_2(\Omega)}\|\mathcal{A}\bar{u}\|_{L_2(\Omega)} \\ &\leqslant 2pc\left\|\mathcal{P}(u,\tilde{u})\right\|_{C(\Omega)}\|\bar{u}\|_{L_2(\Omega)}\|\mathcal{A}\bar{u}\|_{L_2(\Omega)} \\ &\leqslant 2pc\max\left\{\|u\|_{C(\Omega)}^{p-1},\|\tilde{u}\|_{C(\Omega)}^{p-1}\right\}\|\bar{u}\|_{L_2(\Omega)}\|\mathcal{A}\bar{u}\|_{L_2(\Omega)}. \end{aligned} \tag{1.29}$$

Hence, in view of (1.6)–(1.8), (1.19) and the Cauchy inequality, we obtain

$$2pc\left(\mathcal{P}(u,\tilde{u})\bar{u}, \mathcal{A}\bar{u}\right)_{L_2(\Omega)} \leqslant \frac{2pc\gamma^{p-1}}{\sqrt{\lambda}}c_1^{p-1}\left(\varepsilon\|\bar{u}\|_{\mathcal{A}}^2 + \frac{1}{4\varepsilon}\|\mathcal{A}\bar{u}\|_{L_2(\Omega)}^2\right), \tag{1.30}$$

where $c_1 = \max\left\{\|u_0\|_{\mathcal{A}},\|\tilde{u}_0\|_{\mathcal{A}}\right\}$. Substituting the last estimate into (1.28), we get

$$\frac{\mathrm{d}\|\bar{u}\|_{\mathcal{A}}^2}{\mathrm{d}t} + 2\left(1 - \frac{pc\gamma^{p-1}c_1^{p-1}}{4\sqrt{\lambda}\varepsilon}\right)\|\mathcal{A}\bar{u}\|_{L_2(\Omega)}^2 \leqslant \frac{2pc\gamma^{p-1}}{\sqrt{\lambda}}c_1^{p-1}\varepsilon\|\bar{u}\|_{\mathcal{A}}^2.$$

Letting

$$\varepsilon = \frac{pc\gamma^{p-1}c_1^{p-1}}{2\sqrt{\lambda}}$$

and taking into account (1.7), we obtain

$$\frac{\mathrm{d}\|\bar{u}\|_{\mathcal{A}}}{\mathrm{d}t} \leqslant \frac{\left(pc\gamma^{p-1}c_1^{p-1}\right)^2 - \lambda^2}{2\lambda}\|\bar{u}\|_{\mathcal{A}}. \tag{1.31}$$

Hence, if we recall (1.26), we get the following relation

$$\frac{\mathrm{d}\|\bar{u}\|_{\mathcal{A}}}{\mathrm{d}t} \leqslant 0,$$

which completes the proof of the theorem. □

The application of the proven stability estimate (1.27) yields uniqueness of the solution of problem (1.1)–(1.3) (see, e.g., [15]).

**Remark 1.2.** Theorem 1.2 contains the sufficient conditions (1.26) on global stability of the solution of problem (1.1)–(1.3), i.e., the stability for any $T \leqslant +\infty$. However, if conditions (1.18) holds but relations (1.26) are not fulfilled, then from (1.31) we can obtain for the solution of problem (1.1)–(1.3) the following estimate of $\rho$-stability (for any finite $T$):

$$\left\|\tilde{u}(t) - u(t)\right\|_{\mathcal{A}} \leqslant e^{\kappa t}\|\tilde{u}_0 - u_0\|_{\mathcal{A}}\,, \quad \kappa = \frac{\left(pc\gamma^{p-1}c_1^{p-1}\right)^2 - \lambda^2}{2\lambda}.$$

## 2. Global stability of the finite-difference scheme

In the domain $\bar{Q}_T = \{(x,t) : 0 \leqslant x \leqslant l, 0 \leqslant t \leqslant T\}$, we introduce a uniform grid $\bar{\omega} = \bar{\omega}_h \times \bar{\omega}_\tau$, $\bar{\omega}_h = \{x_i = ih, i = 0,\ldots,N, hN = l, N \geqslant 3\} = \omega_h \cup \{x_0 = 0, x_N = l\}$; $\bar{\omega}_\tau = \{t_n = n\tau, n = 0,\ldots,N_0 \quad \tau N_0 = T\} = \omega_\tau \cup \{t_{N_0} = T\}$, $\omega = \omega_h \times \omega_\tau$.

On the grid introduced we approximate the differential problem (1.1)–(1.3) by the following difference problem:

$$y_t + A_h\hat{y} = cy|y|^{p-1}\,, \tag{2.1}$$

$$y(x,0) = u_0(x), \quad x \in \bar{\omega}_h, \quad \hat{y}_0 = 0, \quad \hat{y}_N = 0, \tag{2.2}$$

where

$$A_h y = -\left(a y_{\bar{x}}\right)_x, \quad a = 0,5\left(k_{i-1} + k_i\right). \tag{2.3}$$

Here and below we use the standard notation of the theory of difference schemes [20, 23]:

$$y = y_i^n = y\left(x_i, t_n\right); \quad \hat{y} = y^{n+1} = y\left(x_i, t_{n+1}\right); \quad y_t = \frac{\hat{y} - y}{\tau};$$

$$\left(a y_{\bar{x}}\right)_x = \frac{1}{h}\left(a_{i+1}\frac{\hat{y}_{i+1} - \hat{y}_i}{h} - a_i\frac{\hat{y}_i - \hat{y}_{i-1}}{h}\right).$$

To study the stability, we approximate the perturbed problem (1.9)–(1.11) by the similar difference scheme

$$\tilde{y}_t + A_h\hat{\tilde{y}} = c\tilde{y}|\tilde{y}|^{p-1}\,, \tag{2.4}$$

$$\tilde{y}(x,0) = \tilde{u}_0(x), \quad x \in \bar{\omega}_h, \quad \hat{\tilde{y}}_0 = 0, \quad \hat{\tilde{y}}_N = 0. \tag{2.5}$$

Subtracting Eqs. (1.1)–(1.3) from (1.9)–(1.11), respectively, and using the mean value theorem, we get the following problem for perturbation $\bar{y} = \tilde{y} - y$:

$$\bar{y}_t + A_h\hat{\bar{y}} = cp|y + \theta\bar{y}|^{p-1}\,\bar{y}, \quad \bar{y}\left(x,0\right) = \tilde{u}_0(x) - u_0(x), \quad \bar{y}_0^{n+1} = \bar{y}_N^{n+1} = 0, \quad 0 < \theta_i^n < 1. \tag{2.6}$$

It is clear from (2.6) that before investigating the stability, we must obtain the *a priori* estimates for $y$ and $\tilde{y}$.

**Definition 2.1 [20].** *the difference scheme* (2.1) — (2.2) *is called* unconditionally stable in the sense of the initial data *if for sufficiently small $\tau \leqslant \tau_0$, $h \leqslant h_0$ the following inequality holds:*

$$\|\tilde{y} - y\|_{1_h} \leqslant M_1\|\tilde{u}_0 - u_0\|_{2_h}\,, \tag{2.7}$$

*where $\|\cdot\|_{1_h}$ and $\|\cdot\|_{2_h}$ are some grid norms, $M_1$ is a constant independent of $\tau$, $h$, $y$, $\tilde{y}$ and the choice of input data of the problem.*

Let us introduce the inner products and the grid norms

$$\|y\|_C = \max_{1\leqslant i\leqslant N-1}|y_i|\,; \quad \|y\|_{\bar C} = \max_{0\leqslant i\leqslant N}|y_i|\,; \quad (y,v)_h = \sum_{i=1}^{N-1} y_i v_i h; \quad \|y\|_h = \sqrt{(y,y)_h};$$

$$\|y\|_{A_h} = \sqrt{(A_h y, y)_h}, \quad \|y_{\bar x}]|^2 = \sum_{i=1}^{N} h y_{\bar x,i}^2,$$

where $A_h = A_h^* > 0$ is a positive self-adjoint operator defined by (2.3).

The following grid analogs of embedding theorems hold [20,22,23]:

**Lemma 2.1.** *For an arbitrary grid function $y(x)$ given on a uniform grid $\bar\omega_h$ and vanished at $x=0$, $x=l$, the following inequalities hold:*

$$\|y\|_h \leqslant \frac{1}{\sqrt{\lambda_h}}\|y\|_{A_h}, \quad \|y\|_{A_h} \leqslant \frac{1}{\sqrt{\lambda_h}}\|A_h y\|, \quad \|y\|_h \leqslant \frac{1}{\lambda_h}\|A_h y\|, \quad \lambda_h = \frac{9m_1}{l^2}, \tag{2.8}$$

$$\|y\|_C \leqslant \gamma\|y\|_{A_h}, \quad \gamma = \frac{\sqrt{l}}{2\sqrt{m_1}}. \tag{2.9}$$

*Proof.* Using the estimate [22] $\|y_{\bar x}]|^2 \geqslant \frac{9}{l^2}\|y\|_h^2$ and the properties of the coefficient $k(x)$, we get the first inequality from (2.8)

$$\|y\|_{A_h}^2 = \left(a, y_{\bar x}^2\right] \geqslant m_1 \|y_{\bar x}]|^2 \geqslant \lambda_h \|y\|_h^2. \tag{2.10}$$

The next two inequalities follow from the relations

$$\lambda_h\|y\|_h^2 \leqslant \|y\|_{A_h}^2 = -\left(A_h y, y\right) \leqslant \|A_h y\|\|y\|_h \leqslant \frac{1}{\sqrt{\lambda_h}}\|A_h y\|\|y\|_{A_h}.$$

To prove the last inequality from (2.9), we have to use the estimate [20] $\|y\|_C \leqslant \frac{\sqrt{l}}{2}\|y_{\bar x}]|$ and inequality (2.10). □

**Theorem 2.1.** *Suppose that the input data of the problem satisfy the condition*

$$c_{2h} + \frac{\tau\lambda_h c_{2h}^2}{2} \leqslant 1, \quad c_{2h} = \frac{2^{1-p} c l^{\frac{p+3}{2}}}{9 m_1^{\frac{p+1}{2}}} \max\left\{\|u_0\|_{A_h}^{p-1}, \|\tilde u_0\|_{A_h}^{p-1}\right\}. \tag{2.11}$$

*Then for the solutions of the finite-difference schemes* (2.1)–(2.2), (2.4)–(2.5) *the following* a priori *estimates hold for any $T \in [0,\infty]$:*

$$\max_{t\in\bar\omega_\tau}\left\|y(t)\right\|_{A_h} \leqslant \|u_0\|_{A_h}, \quad \max_{t\in\bar\omega_\tau}\left\|\tilde y(t)\right\|_{A_h} \leqslant \|\tilde u_0\|_{A_h}. \tag{2.12}$$

*Proof.* Taking the inner product of (2.1) with $2\tau A_h\hat y$, we get the energy identity

$$\tau^2\|y_t\|_{A_h}^2 + 2\tau\|A_h\hat y\|^2 + \|\hat y\|_{A_h}^2 = \|y\|_{A_h}^2 + 2\tau c\left(y|y|^{p-1}, A_h\hat y\right)_h. \tag{2.13}$$

Taking into account

$$y = \hat y - \tau y_t, \tag{2.14}$$

we have for the inner product from (2.13)

$$2\tau c\left(y|y|^{p-1}, A_h\hat{y}\right)_h = 2\tau c\left(\hat{y}|y|^{p-1}, A_h\hat{y}\right)_h - 2\tau^2 c\left(y_t|y|^{p-1}, A_h\hat{y}\right)_h. \tag{2.15}$$

Using embedding (2.8) and (2.9) we estimate the first term from the right-hand side of (2.15) as follows:

$$2\tau c\left(\hat{y}|y|^{p-1}, A_h\hat{y}\right)_h \leqslant 2\tau c\gamma^{p-1}\|y\|_{A_h}^{p-1}\|\hat{y}\|_h\|A_h\hat{y}\|_h \leqslant 2\tau c_{3h}\|y\|_{A_h}^{p-1}\|A_h\hat{y}\|_h^2, \tag{2.16}$$

where $c_{3h} = c\gamma^{p-1}/\lambda_h$. Similarly we estimate the second term

$$2\tau^2 c\left(y_t|y|^{p-1}, A_h\hat{y}\right)_h \leqslant \tau^2\|y_t\|_{A_h}^2 + \tau^2\lambda_h c_{3h}^2\|y\|_{A_h}^{2(p-1)}\|A_h\hat{y}\|_h^2. \tag{2.17}$$

Substituting the obtained estimates into (2.13), we get

$$2\tau R_{1h}\|A_h\hat{y}\|_h^2 + \|\hat{y}\|_{A_h}^2 \leqslant \|y\|_{A_h}^2, \tag{2.18}$$

where $R_{1h} = 1 - c_{3h}\|y\|_{A_h}^{p-1} - c_{3h}^2\lambda_h\tau/2$. Since $R_{1h} = R_{1h}^1 \geqslant 0$, from (2.18) it follows that

$$\left\|y^1\right\|_{A_h} \leqslant \|u_0\|_{A_h}.$$

Further, by induction we have

$$\left\|y^{n+1}\right\|_{A_h} \leqslant \|y^n\|_{A_h} \leqslant \ldots \leqslant \|u_0\|_{A_h}.$$

The second estimate from (2.12) is proved in the same way. □

**Theorem 2.2.** *Let the input data of the problem satisfy the conditions*

$$pc_{2h} + \frac{\tau\left(pc_{2h}\right)^2\lambda_h}{2} \leqslant 1. \tag{2.19}$$

*Then the difference scheme is globally stable in the energy norm $A_h$ and for any $t \in [0,\infty)$ the following* a priori *estimate holds:*

$$\max_{t\in\omega_\tau}\left\|\tilde{y}(t) - y(t)\right\|_{A_h} \leqslant \|\tilde{u}_0 - u_0\|_{A_h}. \tag{2.20}$$

*Proof.* Taking the inner product on both sides of Eq. (2.6) with $2\tau A_h\hat{\bar{y}}$, we obtain the following energy identity:

$$\tau^2\|\bar{y}_t\|_{A_h}^2 + 2\tau\left\|A_h\hat{\bar{y}}\right\|_h^2 + \left\|\hat{\bar{y}}\right\|_{A_h}^2 = \|\bar{y}\|_{A_h}^2 + 2\tau cp\left(|y+\theta\bar{y}|^{p-1}\bar{y}, A_h\hat{\bar{y}}\right)_h. \tag{2.21}$$

Using the equality $y = \hat{y} - \tau y_t$, we write for the inner product from the right-hand side of (2.21) in the form

$$2\tau cp\left(|y+\theta\bar{y}|^{p-1}\bar{y}, A_h\hat{\bar{y}}\right)_h = 2\tau cp\left(|y+\theta\bar{y}|^{p-1}, \hat{\bar{y}}A_h\hat{\bar{y}}\right)_h - 2\tau^2 cp\left(|y+\theta\bar{y}|^{p-1}, \bar{y}_t A_h\hat{\bar{y}}\right)_h.$$

Taking into account the inequality $|y+\theta\bar{y}| \leqslant c_{4h}$, $c_{4h} = \max\left\{\|u_0\|_{A_h}, \|\tilde{u}_0\|_{A_h}\right\}$, using embedding (2.8) and (2.9), similarly to (2.16) and (2.17), we get

$$2\tau cp\left(|y+\theta\bar{y}|^{p-1}, \hat{\bar{y}}A_h\hat{\bar{y}}\right)_h \leqslant 2\tau c_{3h}pc_{4h}^{p-1}\left\|A_h\hat{\bar{y}}\right\|_h^2, \tag{2.22}$$

$$2\tau^2 cp\left(|y+\theta\bar{y}|^{p-1}, \bar{y}_t A_h \hat{\bar{y}}\right)_h \leqslant \tau^2\|y_t\|^2_{A_h} + \tau^2\lambda_h c^2_{3h} p^2 c^{2(p-1)}_{4h}\left\|A_h\hat{\bar{y}}\right\|^2_h. \tag{2.23}$$

Substituting the obtained estimates into (2.21), we get

$$2\tau R_{2h}\left\|A_h\hat{\bar{y}}\right\|^2 + \left\|\hat{\bar{y}}\right\|^2_{A_h} \leqslant \|\bar{y}\|^2_{A_h}, \tag{2.24}$$

where $R_{2h} = 1 - c_{3h}c^{p-1}_{4h}p - c^2_{3h}c^{2(p-1)}_{4h}\lambda_h\tau p^2/2$.

Further, in the same way as in the proof of Theorem 2.1, we obtain the required estimate. □

## 3. Monotonicity of the finite-difference scheme

Write the problem for perturbation (2.6) in the following canonical form [20]:

$$A^n_i\bar{y}^{n+1}_{i-1} - C^n_i\bar{y}^{n+1}_i + B^n_i\bar{y}^{n+1}_{i+1} = -F^n_i, \quad \bar{y}^{n+1}_0 = \bar{y}^{n+1}_N = 0, \tag{3.1}$$

where $A^n_i = B^n_i = \frac{\tau}{h^2}$; $C^n_i = 1 + \frac{2\tau}{h^2}$; $F^n_i = \left(1 + \tau cp|y_i + \theta\bar{y}_i|^{p-1}\right)\bar{y}^n_i$.In accordance with the definition of [15] the difference scheme (3.1) is monotone if the condition $\tilde{u}_0 - u_0 \geqslant 0$ yields $\bar{y}^{n+1}_i \geqslant 0$ for any $n = 0, ..., N_0$.

Here we show that under the conditions of Theorem 2.1 the difference scheme (3.1) is unconditionally stable. Below we need the following statement.

**Lemma 3.1 [20].** *Let the following positivity conditions of the coefficients hold:*

$$A^n_i > 0, \quad B^n_i > 0, D^n_i = C^n_i - A^n_i - B^n_i \geqslant 0.$$

*Then from the inequality* $F^n_i \geqslant 0 \quad (F^n_i \leqslant 0)$ *it follows that* $\bar{y}^{n+1}_i \geqslant 0 \quad \left(\bar{y}^{n+1}_i \leqslant 0\right)$.

**Theorem 3.1.** *Let the conditions of Theorem 2.1 be satisfied. Then the difference scheme* (3.1) *is*

*Proof.* It is sufficient to show that $F^n_i \geqslant 0$ for any $n$. Let $\bar{y}^0_i = \tilde{u}_0 - u_0 \geqslant 0$. Then $F^0_i \geqslant 0$ and by Lemma 3.1 $\bar{y}^1_i \geqslant 0$.Similar arguments can be extended to any $n$. Consequently, $F^n_i \geqslant 0$ for any $n$. The case of $F^n_i \leqslant 0$ can be proved in the same manner. □

## 4. Stability for the possible blow-up of the solution of the homogeneous semi-linear parabolic equation

Above we supposed that the input data of problems (1.1)–(1.3) and (1.9)–(1.11) satisfy condition (1.18). In that case, we were able to get estimates of the solution for arbitrary $t \leqslant +\infty$. In this Section, we obtain *a priori* estimates for the solution and investigate its stability when condition (1.18) is not satisfied. In this case, we get estimates only for the solution for finite time $T < +\infty$. To obtain the bound of blow-up time, we can use relation (1.17). But in this case, this bound will be rough. In particular, when diffusion is absent ($k_1 = 0$, $\lambda = 0$) from (1.17) it follows that the solution can blow up immediately at $t \geqslant 0$. So we use a different technique based on the Bihari lemma [1].

**Lemma 4.1.** *Let the nonnegative function $v = v(t)$ satisfy for all $t \in [0, T]$ the following relations:*

$$\frac{\mathrm{d}v}{\mathrm{d}t} \leqslant av^q, \quad v(0) = v_0, \tag{4.1}$$

*where $a$ is a positive constant. Then for $t \in [0, T_{\mathrm{cr}})$ the following inequality holds:*

$$v(t) \leqslant \frac{v_0}{\left(1 - (q-1)av_0^{q-1}t\right)^{1/(q-1)}}, \tag{4.2}$$

*where*

$$T_{\mathrm{cr}} = \frac{1}{(q-1)av_0^{q-1}}. \tag{4.3}$$

## 4.1. *A priori* estimates of the solution

First, we obtain the *a priori* estimates for the solutions of problems (1.1)–(1.3) and (1.9)–(1.11) when condition (1.18) is not satisfied.

**Theorem 4.1.** *If conditions* (1.18) *are not satisfied, then for the solutions of problems* (1.1)–(1.3) *and* (1.9)–(1.11), *if any, the following estimates hold for $t \in [0, T_{\mathrm{cr}}^{(1)})$:*

$$\left\|u(t)\right\|_{\mathcal{A}} \leqslant \frac{\lambda^{1/(2(p-1))}\left\|u_0\right\|_{\mathcal{A}}}{\left(\lambda - 0.5(p-1)c^2\gamma^{2(p-1)}\left\|u_0\right\|_{\mathcal{A}}^{2(p-1)}t\right)^{1/(2(p-1))}}, \tag{4.4}$$

$$\left\|\tilde{u}(t)\right\|_{\mathcal{A}} \leqslant \frac{\lambda^{1/(2(p-1))}\left\|\tilde{u}_0\right\|_{\mathcal{A}}}{\left(\lambda - 0.5(p-1)c^2\gamma^{2(p-1)}\left\|\tilde{u}_0\right\|_{\mathcal{A}}^{2(p-1)}t\right)^{1/(2(p-1))}}, \tag{4.5}$$

*where*

$$T_{\mathrm{cr}}^{(1)} = \frac{2\lambda}{(p-1)c^2\gamma^{2(p-1)}\max\left\{\left\|u_0\right\|_{\mathcal{A}}^{2(p-1)}, \left\|\tilde{u}_0\right\|_{\mathcal{A}}^{2(p-1)}\right\}}. \tag{4.6}$$

*Proof.* Applying to the right-hand side of identity (1.20) the Schwarz and Cauchy inequalities and estimates (1.6) and (1.8), we have

$$\begin{aligned}2c\left(u|u|^{p-1}, \mathcal{A}u\right)_{L_2(\Omega)} &\leqslant 2c\left\|u|u|^{p-1}\right\|_{L_2(\Omega)}\|\mathcal{A}u\|_{L_2(\Omega)} \leqslant 2c\|u\|_{C(\Omega)}^{p-1}\|u\|_{L_2(\Omega)}\|\mathcal{A}u\|_{L_2(\Omega)} \\ &\leqslant 2\frac{c\gamma^{p-1}}{\sqrt{\lambda}}\|u\|_{\mathcal{A}}^{p}\|\mathcal{A}u\|_{L_2(\Omega)} \leqslant 2\frac{c\gamma^{p-1}}{\sqrt{\lambda}}\left(\varepsilon\|u\|_{\mathcal{A}}^{2p} + \frac{1}{4\varepsilon}\|\mathcal{A}u\|_{L_2(\Omega)}^2\right).\end{aligned}$$

Substituting the last bound into identity (1.20) and choosing $\varepsilon = \frac{c\gamma^{p-1}}{4\sqrt{\lambda}}$, we get

$$\frac{\mathrm{d}\|u\|_{\mathcal{A}}^2}{\mathrm{d}t} \leqslant \frac{c^2\gamma^{2(p-1)}}{2\lambda}\|u\|_{\mathcal{A}}^{2p}.$$

Hence, by Lemma 4.1 we obtain estimate (4.4). Inequality (4.5) is proved in the same way. □

Theorem 4.1 yields the estimate of the solution of the initial-boundary value problem (1.1)–(1.3) for the finite time $T_{\mathrm{cr}}^{(1)}$ defined by relation (4.6) when the solution can blow up. If the solution of problem (1.1)–(1.3) goes to infinity, then the lower bound of the blow-up time is given by (4.6), i.e., the blow-up time $T_{\mathrm{bu}}$ satisfies the inequality $T_{\mathrm{bu}} \geqslant T_{\mathrm{cr}}^{(1)}$.

From 4.1 we have the following

**Corollary 4.1.** *For* $t \in [0, T_{\mathrm{cr}}^{(1)} - \delta]$ *the solutions of problems* (1.1)–(1.3) *and* (1.9)–(1.11) *are bounded in the norm of* $\mathscr{H}_{\mathcal{A}}$ *and the following estimates hold:*

$$\left\|u(t)\right\|_{\mathcal{A}} \leqslant c_2 \quad \textit{and} \quad \left\|\tilde{u}(t)\right\|_{\mathcal{A}} \leqslant c_2, \quad c_2 = 2\left(\frac{2m_1^{p/2}\pi}{\delta(p-1)cl^{(p+1)/2}}\right)^{1/(p-1)}. \tag{4.7}$$

This corollary shows the finiteness of the solution of problem (1.1)–(1.3) on a finite time interval. Below Corollary 4.1 will used to obtain the stability estimates.

### 4.2. Stability of the solution

Now consider problem (1.22)–(1.24) for the perturbation of the solution.

**Theorem 4.2.** *If conditions* (1.18) *are not satisfied, then for* $t \in [0, T_{\mathrm{cr}}^{(1)} - \delta_1]$ *the solution of problem* (1.1)–(1.3)*, if any, is stable in the sense of the initial data, and for its perturbation the following estimate holds:*

$$\left\|\tilde{u}(t) - u(t)\right\|_{\mathcal{A}} \leqslant e^{\mu t}\|\tilde{u}_0 - u_0\|_{\mathcal{A}}, \quad \mu = \frac{p^2c^2c_2^2l^{p+1}}{2^{2p-1}m_1^p\pi^2}. \tag{4.8}$$

*Proof.* Applying to the right-hand side of identity (1.28) relations (1.29) and taking into account (1.6)–(1.8) and Corollary 4.1, we get

$$2pc\left(\mathcal{P}(u,\tilde{u})\bar{u}, \mathcal{A}\bar{u}\right)_{L_2(\Omega)} \leqslant 2\frac{pc\gamma^{p-1}}{\sqrt{\lambda}}c_2\|\bar{u}\|_{\mathcal{A}}\|\mathcal{A}\bar{u}\|_{L_2(\Omega)}.$$

Hence, using the Cauchy inequality, we have

$$2pc\left(\mathcal{P}(u,\tilde{u})\bar{u}, \mathcal{A}\bar{u}\right)_{L_2(\Omega)} \leqslant 2\frac{pcc_2\gamma^{p-1}}{\sqrt{\lambda}}\left(\varepsilon\|\bar{u}\|_{\mathcal{A}}^2 + \frac{1}{4\varepsilon}\|\mathcal{A}\bar{u}\|_{L_2(\Omega)}^2\right).$$

Substituting the last relation into identity (1.28) and choosing $\varepsilon = \frac{pcc_2\gamma^{p-1}}{4\sqrt{\lambda}}$, we obtain

$$\frac{\mathrm{d}\|\bar{u}\|_{\mathcal{A}}^2}{\mathrm{d}t} \leqslant \frac{(pcc_2)^2\gamma^{2(p-1)}}{2\lambda}\|\bar{u}\|_{\mathcal{A}}^2,$$

whence the statement of the theorem follows. □

## 5. Stability of the finite-difference scheme when the solution blows up

### 5.1. *A priori* estimates of the stability when the diffusion degenerates

If conditions (2.11) are not satisfied, then the solution can increase indefinitely in a finite time. In this case, the corresponding *a priori* estimates, expressing finiteness of the solution of the finite-difference scheme and its stability, can only be proved for some finite time $t \leqslant T_{\mathrm{cr}}$.

Below we need a grid analog of the Bihari inequality [4].

**Lemma 5.1.** *Let $m > 1$ and the following inequalities be satisfied:*

$$0 \leqslant v_0 \leqslant c \quad (c > 0), \quad v_n \leqslant c + \sum_{k=0}^{n-1} a_k v_k^m, \quad (n = 1, 2, \dots), \tag{5.1}$$

*where sequences $v_k \geqslant 0$, $a_k \geqslant 0$ $(k = 0, 1, 2, ...)$. Then the following inequality holds:*

$$v_n \leqslant \frac{c}{\left(1 - (m-1)c^{m-1}\sum_{k=0}^{n-1} a_k\right)^{\frac{1}{m-1}}}, \quad (n = 1, 2, \dots), \tag{5.2}$$

*provided that*

$$\sum_{k=0}^{n-1} a_k < \frac{1}{(m-1)c^{m-1}}. \tag{5.3}$$

**Theorem 5.1.** *Let*

$$T < \min\left\{T_{\mathrm{cr}h}, \tilde{T}_{\mathrm{cr}h}\right\}, \quad T_{\mathrm{cr}h} = \frac{1}{c_{5h}c_{6h}}, \quad \tilde{T}_{\mathrm{cr}h} = \frac{1}{c_{5h}\tilde{c}_{6h}}, \tag{5.4}$$

$$c_{5h} = \frac{c_{3h}^2 \lambda_h (p-1)}{2}, \quad c_{6h} = \|u_0\|_{A_h}^{2(p-1)}, \quad \tilde{c}_{6h} = \|\tilde{u}_0\|_{A_h}^{2(p-1)}. \tag{5.5}$$

*Then for the solutions of the finite-difference problems* (2.1), (2.2) *and* (2.4), (2.5) *the following* a priori *estimates hold:*

$$\|y^n\|_{A_h} \leqslant \frac{\|u_0\|_{A_h}}{(1 - c_{5h}c_{6h}t_n)^{\frac{1}{2(p-1)}}}, \quad n = 0, 1, \dots, N_0, \tag{5.6}$$

$$\|\tilde{y}^n\|_{A_h} \leqslant \frac{\|\tilde{u}_0\|_{A_h}}{(1 - c_{5h}\tilde{c}_{6h}t_n)^{\frac{1}{2(p-1)}}}, \quad n = 0, 1, \dots, N_0. \tag{5.7}$$

*Proof.* In the energy inequality (2.13) we estimate the inner product from the right-hand side using the Schwarz and Cauchy inequalities and embedding (2.8),(2.9)

$$2\tau c\left(y|y|^{p-1}, A_h\hat{y}\right)_h \leqslant 2\tau\|A_h\hat{y}\|^2 + \frac{\tau c_{3h}^2\lambda_h}{2}\|y\|_{A_h}^{2p}.$$

Substituting the last estimate into (2.13), we get the following recurrence relation:

$$\left\|y^{n+1}\right\|_{A_h}^2 \leqslant \|y^n\|_{A_h}^2 + \tau c_{5h}\|y\|_{A_h}^{2p} \leqslant \dots \leqslant \left\|u^0\right\|_{A_h}^2 + c_{5h}\sum_{k=0}^{n}\tau\left\|y^k\right\|_{A_h}^{2p}. \tag{5.8}$$

Now, using Lemma 5.1 with $v_n = \|y^n\|_{A_h}^2$, $c = \|u_0\|_{A_h}^2$, $a_k = \tau c_{5h}$, $m = p$, we obtain estimate (5.6). Inequality (5.7) is proved in the same way. □

Now suppose that the stronger condition holds

$$T < \frac{1 - \delta^{-2(2p-1)}}{c_{5h}\overline{c}_{6h}}, \quad \delta > 1. \tag{5.9}$$

In this case, estimates (5.6) and (5.7) are the form of

$$\|y^n\|_{A_h} \leqslant \delta\|u_0\|_{A_h} = c_{7h}, \quad \|\tilde{y}^n\|_{A_h} \leqslant \delta\|\tilde{u}_0\|_{A_h} = \tilde{c}_{7h}. \tag{5.10}$$

**Theorem 5.2.** *Let condition* (5.9) *be satisfied. Then the finite-difference scheme* (2.1)–(2.2) *is* $\rho$*-stable in th sense of the initial data and for all* $t \in \bar{\omega}_\tau$ *the following estimate holds:*

$$\left\|\tilde{y}(t) - y(t)\right\|_{A_h} \leqslant e^{Tc_{8h}} \|\tilde{u}_0 - u_0\|_{A_h}. \tag{5.11}$$

*Proof.* Consider again the energy identity (2.21). The inner product from the right-hand side of (2.21) is estimated as follows:

$$2\tau cp \left(|y + \theta\bar{y}|^{p-1}\,\bar{y}, A_h\hat{\bar{y}}\right)_h \leqslant 2\tau\left\|A_h\hat{\bar{y}}\right\|_h^2 + \tau c_{8h}\|\bar{y}\|_{A_h}^2, \tag{5.12}$$

where $c_{8h} = \frac{\left(cp\bar{c}_{7h}^{p-1}\right)^2}{2\lambda_h}$, $\bar{c}_{7h} = \max\{c_{7h}, \tilde{c}_{7h}\}$. Substituting estimate (5.12) into the energy identity (5.12), we obtain the relation

$$\left\|\hat{\bar{y}}\right\|_{A_h}^2 \leqslant (1 + 2\tau c_{8h})\|\bar{y}\|_{A_h}^2,$$

which can be rewritten in the form

$$\left\|\bar{y}^{n+1}\right\|_{A_h} \leqslant e^{\tau c_{8h}}\|\bar{y}^n\|_{A_h} \leqslant \ldots \leqslant e^{c_{8h}T}\|\bar{u}_0\|_{A_h}.$$

□

## 6. Stability of the solution of the nonhomogeneous semilinear parabolic equation

This question is very important for two reasons. Firstly, the initial-boundary value problem with nonzero boundary conditions is reduced to problems for such equations. The influence of such boundary conditions on the generation of blowing-up solutions is an interesting problem and deserves careful consideration. Secondly, only from the estimates of stability in the sense of the right-hand side and consistency of the scheme it follows that the approximate solution converges to an exact one (Lax theorem). Note that even if the difference scheme is homogeneous, the problem for error of the method always contains a nonhomogeneous equation. In this case, the right-hand side is the truncation error. To be definite, assume that $p = 3$.

Now we consider the following initial-boundary value problem for the nonhomogeneous semi-linear parabolic equation:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\,\frac{\partial u}{\partial x}\right) + cu^3 + f(x,t), \quad f(x,t) \geqslant 0, \quad x \in \Omega, \quad 0 < t \leqslant T, \tag{6.1}$$

$$u(0,t) = u(l,t) = 0, \quad 0 \leqslant t \leqslant T, \tag{6.2}$$

$$u(x,0) = u_0(x), \quad x \in \bar{\Omega}. \tag{6.3}$$

Together with (6.1)–(6.3) we consider the problem with perturbed initial data and the right-hand side

$$\frac{\partial \tilde{u}}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\,\frac{\partial \tilde{u}}{\partial x}\right) + c\tilde{u}^3 + \tilde{f}(x,t), \quad \tilde{f}(x,t) \geqslant 0, \quad x \in \Omega, \quad 0 < t \leqslant T, \tag{6.4}$$

$$\tilde{u}(0,t) = \tilde{u}(l,t) = 0, \quad 0 \leqslant t \leqslant T, \tag{6.5}$$

$$\tilde{u}(x,0) = \tilde{u}_0(x), \quad x \in \bar{\Omega}. \tag{6.6}$$

As before, assume that conditions (1.4), (1.5), (1.12) are satisfied and

$$\max\left\{\max_{0<t\leqslant T}\left\|f(t)\right\|_{L_2(\Omega)}, \max_{0<t\leqslant T}\left\|\tilde{f}(t)\right\|_{L_2(\Omega)}\right\} \leqslant F. \tag{6.7}$$

In this case, for some additional conditions (to be formulated below) we obtain the global estimates of the solutions of problems (6.1)–(6.3) and (6.4)–(6.6).

### 6.1. *A priori* estimates for arbitrary $t \in [0, +\infty]$

To obtain the estimates for any $t \in [0, +\infty]$, we need the following

**Lemma 6.1.** *Let $v(t)$ be a nonnegative function and $a$, $b$, and $r$ be positive constants. Suppose that for $t \in [0, T]$, where $T \leqslant +\infty$, the following relation is satisfied:*

$$\frac{\mathrm{d}v}{\mathrm{d}t} \leqslant av^2 - bv + r, \quad v(0) = v_0. \tag{6.8}$$

*Under the conditions*

$$D^2 = b^2 - 4ar > 0 \tag{6.9}$$

*and*

$$\frac{2av_0}{b+D} \leqslant 1, \tag{6.10}$$

*the following inequality holds for any $t \in [0, T]$:*

$$v(t) \leqslant \frac{b+D}{2a}. \tag{6.11}$$

*Proof.* Consider the Cauchy problem for the Riccati equation

$$\frac{\mathrm{d}w}{\mathrm{d}t} = aw^2 - bw + r, \tag{6.12}$$

$$w(0) = v_0. \tag{6.13}$$

According to the Chaplygin comparison theorem, $v(t) \leqslant w(t)$.

The solution of the Cauchy problem (6.12)–(6.13) is given by

$$w(t) = w_1 + \frac{e^{-Dt} D\left(v_0 - w_1\right)}{D - a\left(v_0 - w_1\right)\left(1 - e^{-Dt}\right)}, \quad w_1 = \frac{b+D}{2a}.$$

Taking into account condition (6.10), from the last relation we get estimate (6.11). □

We use Lemma 6.1 to prove the following

**Theorem 6.1.** *Let the solutions of problems* (6.1)–(6.3) *and* (6.4)–(6.6) *exist. If the input data of problems* (6.1)–(6.3) *and* (6.4)–(6.6) *satisfy the relations*

$$\lambda^2 - 24c\gamma^2F^2 > 0 \tag{6.14}$$

*and*

$$\frac{12c\gamma^2 \max\left\{\|u_0\|_{\mathcal{A}}^2, \|\tilde{u}_0\|_{\mathcal{A}}^2\right\}}{\lambda + \sqrt{\lambda^2 - 24c\gamma^2 F^2}} \leqslant 1, \tag{6.15}$$

*then the solutions of problems* (6.1)–(6.3) *and* (6.4)–(6.6) *are bounded for any* $t \in [0, T]$ *and the following estimates holds:*

$$\|u\|_{\mathcal{A}}^2 \leqslant c_3, \quad \|u\|_{\mathcal{A}}^2 \leqslant c_3, \qquad c_3 = \frac{\lambda + \sqrt{\lambda^2 - 24c\gamma^2 F^2}}{12c\gamma^2}. \tag{6.16}$$

*Proof.* Multiplying both sides of equation (6.8) by $2\mathcal{A}u$ and integrating the result over $\Omega$, we obtain the following energy identity:

$$\frac{\mathrm{d}\|u\|_{\mathcal{A}}^2}{\mathrm{d}t} + 2\|\mathcal{A}u\|_{L_2(\Omega)}^2 = 2c\left(u^3, \mathcal{A}u\right)_{L_2(\Omega)} + 2\left(f, \mathcal{A}u\right)_{L_2(\Omega)}. \tag{6.17}$$

To estimate the first term of the right-hand side of (6.17), we use (1.21) with $p = 3$

$$2c\left(u^3, \mathcal{A}u\right)_{L_2(\Omega)} \leqslant 6c\gamma^2\|u\|_{\mathcal{A}}^4. \tag{6.18}$$

Applying the Schwarz and Cauchy inequalities to the second term of the right-hand side of (6.17), we have

$$2\left(f, \mathcal{A}u\right)_{L_2(\Omega)} \leqslant 2\|f\|_{L_2(\Omega)}\|\mathcal{A}u\|_{L_2(\Omega)} \leqslant 2\left(\varepsilon\|f\|_{L_2(\Omega)}^2 + \frac{1}{4\varepsilon}\|\mathcal{A}u\|_{L_2(\Omega)}^2\right).$$

Substituting the last inequality and (6.18) into (6.16), we get

$$\frac{\mathrm{d}\|u\|_{\mathcal{A}}^2}{\mathrm{d}t} + 2\left(1 - \frac{1}{4\varepsilon}\right)\|\mathcal{A}u\|_{L_2(\Omega)}^2 \leqslant 6c\gamma^2\|u\|_{\mathcal{A}}^4 + 2\varepsilon\|f\|_{L_2(\Omega)}^2. \tag{6.19}$$

Choosing in the last relation $\varepsilon = 1/2$ and taking into account estimates (1.6) and (6.7), we obtain the inequality

$$\frac{\mathrm{d}\|u\|_{\mathcal{A}}^2}{\mathrm{d}t} \leqslant 6c\gamma^2\|u\|_{\mathcal{A}}^4 - \lambda\|u\|_{\mathcal{A}}^2 + F^2. \tag{6.20}$$

Let $v = \|u\|_{\mathcal{A}}^2$, $a = 6c\gamma^2$, $b = \lambda$, $r = F^2$. Then from (6.20) and Lemma 6.1 we get the statement of the theorem. □

Note that conditions (6.7) determine either the finite or the infinite interval $[0, T]$, for which estimates (6.16) hold.

In the case of homogeneous equations (6.1) and (6.4), i.e., when $f(x,t) \equiv 0$ and $\tilde{f}(x,t) \equiv 0$, and, consequently, $F = 0$, condition (6.15) and estimates (6.16) are reduced to (1.18) and (1.19), respectively, with $p = 3$.

### 6.2. Global stability of the solution in the sense of the initial data and the right-hand side

Consider the following problem for the perturbation of the solution:

$$\frac{\partial \bar{u}}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\,\frac{\partial \bar{u}}{\partial x}\right) + 2\mathcal{P}(u,\tilde{u})\bar{u} + \bar{f}(x,t), \quad x \in \Omega, \quad 0 < t \leqslant T, \tag{6.21}$$

$$\bar{u}(0,t) = \bar{u}(l,t) = 0, \quad 0 < t \leqslant T, \tag{6.22}$$

$$\bar{u}(x,t) = \bar{u}_0(x) = \tilde{u}_0(x) - u_0(x), \quad x \in \bar{\Omega}, \tag{6.23}$$

where $\bar{f}(x,t) = \tilde{f}(x,t) - f(x,t)$.

**Theorem 6.2.** *Let conditions* (6.14) *and* (6.15) *be satisfied and the solutions of problems* (6.1)–(6.3) *and* (6.4)–(6.6) *be existent. Then the solution of problem* (6.1)–(6.3) *is stable in the sense of the initial data and the right-hand side for all* $t \in [0,T]$ *and its perturbation satisfies the following estimate:*

$$\left\|\tilde{u}(t) - u(t)\right\|_{\mathcal{A}}^2 \leqslant \|\tilde{u}_0 - u_0\|_{\mathcal{A}}^2 + 2\int_0^t \left\|\tilde{f}(s) - f(s)\right\|_{L_2(\Omega)}^2 ds. \tag{6.24}$$

*Proof.* Multiplying both sides of equation (6.21) by $2\mathcal{A}\bar{u}$ and integrating the result over $\Omega$, we get the following energy identity:

$$\frac{\mathrm{d}\|\bar{u}\|_{\mathcal{A}}^2}{\mathrm{d}t} + 2\|\mathcal{A}\bar{u}\|_{L_2(\Omega)}^2 = 4c\left(\mathcal{P}(u,\tilde{u})\bar{u}, \mathcal{A}\bar{u}\right)_{L_2(\Omega)} + 2\left(\bar{f}, \mathcal{A}\bar{u}\right)_{L_2(\Omega)}. \tag{6.25}$$

For the first term of the right-hand side of identity (6.25) estimate (1.29) holds. Thus, using (1.6)–(1.8), (6.16) and the Schwarz and Cauchy inequalities, we get

$$4c\left(\mathcal{P}(u,\tilde{u})\bar{u}, \mathcal{A}\bar{u}\right)_{L_2(\Omega)} \leqslant \frac{4cc_3\gamma^2}{\sqrt{\lambda}}\left(\varepsilon_1\|\bar{u}\|_{\mathcal{A}}^2 + \frac{1}{4\varepsilon_1}\|\mathcal{A}\bar{u}\|_{L_2(\Omega)}^2\right) \tag{6.26}$$

Now estimate the second term, using the Schwarz and Cauchy inequalities

$$2\left(\bar{f}, \mathcal{A}\bar{u}\right)_{L_2(\Omega)} \leqslant 2\left\|\bar{f}\right\|_{L_2(\Omega)}\|\mathcal{A}\bar{u}\|_{L_2(\Omega)} \leqslant 2\varepsilon_2\left\|\bar{f}\right\|_{L_2(\Omega)}^2 + \frac{1}{2\varepsilon_2}\|\mathcal{A}\bar{u}\|_{L_2(\Omega)}^2. \tag{6.27}$$

Substituting (6.26), (6.27), and (1.7) into (6.25) and choosing

$$\varepsilon_1 = \frac{2cc_3\gamma^2}{\sqrt{\lambda}}, \quad \varepsilon_2 = 1, \tag{6.28}$$

we get the inequality

$$\frac{\mathrm{d}\|\bar{u}\|_{\mathcal{A}}^2}{\mathrm{d}t} + \left(\lambda - \frac{8(cc_3\gamma^2)^2}{\lambda}\right)\|\bar{u}\|_{\mathcal{A}}^2 \leqslant 2\left\|\bar{f}\right\|_{L_2(\Omega)}^2. \tag{6.29}$$

Taking into account the values of $c_3$, $\gamma$ and $\lambda$, we obtain the following estimate:

$$\lambda - \frac{8(cc_3\gamma^2)^2}{\lambda} \geqslant \frac{7}{9}\lambda > 0,$$

This estimate and (6.29) lead to the statement of the theorem. □

Estimate (6.24) expresses the stability of the solution of problem (6.1)–(6.3) in the sense of the initial data and the right-hand side. Moreover, if the right-hand sides of equations (6.1) and (6.4) equal to zero, then estimate (6.24) is consistent with the estimate of the stability in the sense of the initial data (1.27).

### 6.3. *A priori* estimate of the solution when it can blow up

Here we obtain the *a priori* estimate for the solution of problem (6.1)–(6.3) in the general case, if conditions (6.14) and (6.15) are not satisfied. We need the following

**Lemma 6.2.** *Let the nonnegative function $v$ satisfy the following relations:*

$$\frac{\mathrm{d}v}{\mathrm{d}t} \leqslant av^2 + r, \quad v(0) = v_0, \tag{6.30}$$

$r = \mathrm{const} \geqslant 0$, $a = \mathrm{const} > 0$. *Then for $t \in [0, T_{\mathrm{cr}}^{(r)})$*

$$v(t) \leqslant \frac{v_0 + g_1 \operatorname{tg}(g_2 t)}{1 - g_1 \operatorname{tg}(g_2 t)\, v_0}, \quad g_1 = \sqrt{r/a}, \quad g_2 = \sqrt{ra}, \tag{6.31}$$

*where*

$$T_{\mathrm{cr}}^{(r)} = \frac{\pi}{2g_2} - \operatorname{arctg}(g_1 v_0). \tag{6.32}$$

*Proof.* Consider the solution of the problem

$$\frac{\mathrm{d}w}{\mathrm{d}t} = aw^2 + r, \quad w(0) = v_0 \tag{6.33}$$

Then $v(t) \leqslant w(t)$. The solution of problem (6.33) is given by

$$w(t) = g_1 \operatorname{tg}\left(g_2 t + \operatorname{arctg}(g_1 v_0)\right).$$

Hence, recalling the representation of the tangent of the sum of angles, we get the statement of the lemma. □

Choosing in (6.19), from the proof of Theorem 6.2 $\varepsilon = 1/4$ and denoting $v(t) = \|u(t)\|_{\mathcal{A}}^2$, $a = 6c\gamma^2$, $r = F^2$, we get an inequality of the form of (6.30), whence the following estimate of the solution of problem (6.1)–(6.3) is obtained:

$$\|u\|_{\mathcal{A}}^2 \leqslant \frac{\|u_0\|_{\mathcal{A}}^2 + F/(\sqrt{6c}\gamma) \operatorname{tg}\left(\sqrt{6c}F\gamma t\right)}{1 - \sqrt{6c}\gamma/F \operatorname{tg}\left(\sqrt{6c}F\gamma t\right) \|u_0\|_{\mathcal{A}}^2}. \tag{6.34}$$

This estimate holds for $0 \leqslant t < T_{\mathrm{cr}}^{(r)}$.

## 7. Stability of the finite-difference schemes for the nonhomogeneous equation

On the grid $\bar{\omega}$ introduced above we approximate the differential problem (6.1)–(6.3) by the difference problem

$$y_t + A_h \hat{y} = c\hat{y}y^2 + \varphi, \quad y(x,0) = u_0(x), \quad x \in \bar{\omega}_h, \quad \hat{y}_0 = 0, \quad \hat{y}_N = 0, \tag{7.1}$$

where $\varphi$ is a certain stencil functional on $f$ [20]. In particular, it can be chosen to be $\varphi = f$. Perturbing the input data (the initial condition and the right-hand side) in the finite-difference scheme, we get the perturbed problem

$$\tilde{y}_t + A_h \hat{\tilde{y}} = c\hat{\tilde{y}}\tilde{y}^2 + \tilde{\varphi}, \quad \tilde{y}(x,0) = \tilde{u}_0(x), \quad x \in \bar{\omega}_h, \quad \hat{\tilde{y}}_0 = 0, \quad \hat{\tilde{y}}_N = 0. \tag{7.2}$$

Now we obtain the *a priori* estimates expressing for any $t \in \bar{\omega}_\tau$, $T < \infty$ the finiteness of the difference solutions $y$, $\tilde{y}$ in the grid norm $\|\cdot\|_{A_h}$.

**Theorem 7.1.** *Suppose that the input data satisfy the condition*

$$c_{3h}\left(\|u_0\|^2_{A_h}+\sum_{t\in\omega_\tau}\frac{\tau}{\delta_3^2}\big\|\varphi(t)\big\|_h^2\right)+\frac{\delta_3^2}{2}\leqslant 1,\quad \delta_3=\text{const}>0. \tag{7.3}$$

$$c_{3h}\left(\|\tilde u_0\|^2_{A_h}+\sum_{t\in\omega_\tau}\frac{\tau}{\delta_3^2}\big\|\tilde\varphi(t)\big\|_h^2\right)+\frac{\delta_3^2}{2}\leqslant 1. \tag{7.4}$$

*Then for any $t\in\bar\omega_\tau$ the solutions of the finite-difference schemes* (7.1) *and* (7.2) *satisfy the following* a priori *estimates:*

$$\max_{t\in\bar\omega_\tau}\big\|y(t)\big\|^2_{A_h}\leqslant\|u_0\|^2_{A_h}+\frac{1}{\delta_3^2}\sum_{t\in\omega_\tau}\tau\big\|\varphi(t)\big\|_h^2=c_{9h}, \tag{7.5}$$

$$\max_{t\in\bar\omega_\tau}\big\|\tilde y(t)\big\|^2_{A_h}\leqslant\|\tilde u_0\|^2_{A_h}+\frac{1}{\delta_3^2}\sum_{t\in\omega_\tau}\tau\big\|\tilde\varphi(t)\big\|_h^2=\tilde c_{9h}. \tag{7.6}$$

*Proof.* Taking the inner product of equation (7.1) with $2\tau A_h\hat y$, we obtain the energy inequality

$$\tau^2\|y_t\|^2_{A_h}+2\tau\|A_h\hat y\|^2+\|\hat y\|^2_{A_h}=\|y\|^2_{A_h}+2\tau\left(cy^2\hat y+\varphi,A_h\hat y\right)_h.$$

Using the Schwarz inequality and the grid embedding (2.8),(2.9), we get the estimate

$$2\tau\left(cy^2\hat y,A_h\hat y\right)\leqslant 2\tau c\|y\|^2_{C_h}\|\hat y\|_h\|A_h\hat y\|_h\leqslant 2\tau c_{3h}\|y\|^2_{A_h}\|A_h\hat y\|^2_h. \tag{7.7}$$

Let $n=0$. Then, by Theorem 7.1 the last inequality can be rewritten in the form

$$\big\|y^1\big\|^2_{A_h}+\tau\big\|\delta_3A_hy^1-\delta_3^{-1}\varphi^0\big\|^2_h\leqslant\big\|y^0\big\|^2_{A_h}+\frac{\tau}{\delta_3^2}\big\|\varphi^0\big\|^2_h.$$

Hence, we get

$$\big\|y^1\big\|^2_{A_h}\leqslant\|u_0\|^2_{A_h}+\frac{\tau}{\delta_3^2}\big\|\varphi^0\big\|^2_h.$$

By induction we obtain the required estimates (7.5) and (7.6). □

Note that the finite interval $t\in[0,T]$ (where estimates (7.5), (7.6) hold) is determined by conditions (7.3) and (7.4).

Further we study the stability of the finite-difference scheme. Subtracting equation (7.1) from (7.2) and applying identity (2.14), we obtain the following problem for the perturbation of the solution $\bar y$:

$$\bar y_t+A_h\hat{\bar y}=c\left(\left(l_{1h}+y^2\right)\hat{\bar y}+\tau l_{1h}\bar y_t\right)+\tilde\varphi-\varphi, \tag{7.8}$$

$$\bar y(x,0)=\tilde u_0(x)-u_0(x),\quad \bar y_0^{n+1}=y_N^{n+1}=0,\quad l_{1h}=\hat{\tilde y}\left(\tilde y+y\right). \tag{7.9}$$

Let us prove the following

**Theorem 7.2.** *Suppose that the input data of the problem considered satisfy conditions* (7.3)*,* (7.4) *and the inequality*

$$c_{10h} + \frac{\tau\lambda_h c_{10h}}{2} + \varepsilon \leqslant 1, \quad c_{10h} = \frac{c\gamma^2 \bar{c}_{9h}}{\lambda_h}, \quad \bar{c}_{9h} = \max\left\{c_{9h}, \tilde{c}_{9h}\right\}. \tag{7.10}$$

*Then the finite-difference scheme is stable in the sense of the initial data and right-hand side in the energy norm* $\|\cdot\|_{A_h}$*. Moreover, for any* $t \in [0, T]$ *the following estimate holds:*

$$\max_{t\in\bar{\omega}_\tau}\left\|\tilde{y}(t) - y(t)\right\|_{A_h}^2 \leqslant \|\tilde{u_0} - u_0\|_{A_h}^2 + \frac{1}{2\varepsilon}\sum_{t\in\omega_\tau}\tau\left\|\varphi(t) - \tilde{\varphi}(t)\right\|_h^2, \tag{7.11}$$

*where* $\varepsilon > 0$ *is a constant satisfying condition* (7.10)*.*

*Proof.* Taking the inner product of both sides of (7.8) with $2\tau A_h\hat{\bar{y}}$, we obtain the energy identity

$$\tau^2\|\bar{y}_t\|_{A_h}^2 + 2\tau\left\|A_h\hat{\bar{y}}\right\|_h^2 + \left\|\hat{\bar{y}}\right\|_{A_h}^2 = \|\bar{y}\|_{A_h}^2 + 2\tau c\left(\left(l_{1h} + y^2\right)\hat{\bar{y}}, A_h\hat{\bar{y}}\right)_h + 2\tau^2 c\left(l_{1h}\bar{y}_t, A_h\hat{\bar{y}}\right)_h + 2\tau\left(\tilde{\varphi} - \varphi, A_h\hat{\bar{y}}\right)_h. \tag{7.12}$$

Similarly to (2.22), (2.23) we get the estimates

$$2\tau c\left(\left(l_{1h} + y^2\right)\hat{\bar{y}}, A_h\hat{\bar{y}}\right)_h \leqslant 2\tau c_{10h}\left\|A_h\hat{\bar{y}}\right\|_h^2,$$

$$2\tau^2 c\left(l_{1h}\bar{y}_t, A_h\hat{\bar{y}}\right)_h \leqslant \tau^2\|y_t\|_{A_h}^2 + \tau^2\lambda_h c_{10h}^2\left\|A_h\hat{\bar{y}}\right\|_h^2.$$

Applying the Schwarz and Cauchy inequalities to the last term on the right-hand side of (7.12), we obtain

$$2\tau\left(A_h\hat{\bar{y}}, \tilde{\varphi} - \varphi\right)_h \leqslant 2\tau\varepsilon\left\|A_h\hat{\bar{y}}\right\|_h^2 + \frac{\tau}{2\varepsilon}\|\tilde{\varphi} - \varphi\|_h^2. \tag{7.13}$$

Substituting the estimates obtained in (7.12) and taking into account the conditions of the theorem, we get the recurrence relation

$$\left\|\bar{y}^{n+1}\right\|_{A_h}^2 \leqslant \|\bar{y}^n\|_{A_h}^2 + \frac{\tau}{2\varepsilon}\|\tilde{\varphi}^n - \varphi^n\|_h^2. \tag{7.14}$$

Hence we obtain the required estimate (7.11). □

When conditions (7.10) are not satisfied, the solution can go to infinity for a finite time. Now we obtain the *a priori* estimate for this case. Below we need the following lemma regarding the nonlinear recurrence inequality (discrete analog of Lemma 6.2).

**Lemma 7.1.** *Let the grid function* $v^n = v(t_n) \geqslant 0$ *given on the grid* $\bar{\omega}_\tau$ *satisfy the inequality*

$$\frac{v^{n+1} - v^n}{\tau} \leqslant a v^n v^{n+1} + r, \quad v^0 = v_0, \tag{7.15}$$

*where* $a$*,* $r$ *are positive constants. Then*

$$v^n \leqslant \frac{v_0 + g_1\operatorname{tg}(g_2 t_n)}{1 - v_0 g_1\operatorname{tg}(g_2 t_n)}, \quad n = 0, 1, ..., N_0. \tag{7.16}$$

*for all* $0 \leqslant t_n < T_{\text{cr}}^{(r)}$*,*

$$T_{\text{cr}}^{(r)} = \frac{\pi}{2g_2} - \operatorname{arctg}(v_0 g_1), \quad g_1 = \sqrt{r/a}, \quad g_2 = \sqrt{ra}. \tag{7.17}$$

*Proof.* Write inequality (7.15) in the following form:

$$\frac{v^{n+1} - v^n}{1 + \frac{a}{r} v^n v^{n+1}} \leqslant \tau r. \tag{7.18}$$

Define $\sqrt{a/r} v^n = \operatorname{tg} \gamma^n$. Then taking into account the representation of the tangent of the difference of angles from (7.18), we get

$$\operatorname{tg}\left(\gamma^{n+1} - \gamma^n\right) \leqslant \tau r / g_1.$$

Hence,

$$\gamma^{n+1} \leqslant \gamma^n + \operatorname{arctg}\left(g_2 \tau\right) \leqslant \gamma^0 + n \operatorname{arctg}\left(g_2 \tau\right) \leqslant \gamma^0 + g_2 t_{n+1}. \tag{7.19}$$

Since $\gamma^n = \operatorname{arctg}\left(g_1 v^n\right)$, from (7.19) we obtain

$$\operatorname{arctg}\left(g_1 v^{n+1}\right) \leqslant \operatorname{arctg}\left(g_1 v^0\right) + g_2 t_{n+1}.$$

Hence we get inequality (7.16). □

**Remark 7.1.** Note that the discrete estimate (7.17) of the time of the possible blow up of the solution is consistent with differential (6.32) due to the fact that the grid inequality (7.15) is exactly consistent with differential (6.30). Indeed, in [14] was shown that the difference scheme

$$\frac{v^{n+1} - v^n}{\tau} = \left( \frac{1}{v^{n+1} - v^n} \int\limits_{v^n}^{v^{n+1}} \frac{dv}{f(v)} \right)^{-1}, \quad v^0 = v_0, \tag{7.20}$$

is exactly consistent with the problem

$$\frac{dv}{dt} = f(v), \quad v(0) = v_0. \tag{7.21}$$

Letting in (7.20) $f(v) = av^2 + r$, we obtain the finite-difference equation $v^{n+1} - v^n = \left(a v^n v^{n+1} + r\right)\tau$.

Now we prove the following

**Theorem 7.3.** *Suppose that the input data of the problem satisfy conditions* (7.3). *Then for all* $t \in \bar{\omega}_\tau$,

$$T < T_{\mathrm{cr}}^{(r)}, \quad a = 4c\gamma^2, \quad r = \frac{1}{2} \max_{t \in \bar{\omega}_\tau} \left\| \varphi(t) \right\|_h^2,$$

*for the solution of the finite-difference scheme* (7.1) *the following estimate holds:*

$$\left\| y(t) \right\|_{A_h}^2 \leqslant \frac{\|u_0\|_{A_h}^2 + \sqrt{r/a} \operatorname{tg}\left(\sqrt{ar} t\right)}{1 - \|u_0\|_{A_h}^2 \sqrt{r/a} \operatorname{tg}\left(\sqrt{ar} t\right)}. \tag{7.22}$$

*Proof.* Take the inner product of both sides of equation (7.1) with $2\tau A_h \hat{y}$ and apply the formula of summing by parts (the first difference Green formula [20])

$$\left( (a y_{\bar{x}})_x, v \right) = - \left( a y_{\bar{x}}, v_{\bar{x}} \right].$$

We obtain the following energy inequality:

$$\tau^2\|\bar{y}_t\|^2_{A_h}+2\tau\big\|A_h\hat{y}\big\|^2_h+\big\|\hat{y}\big\|^2_{A_h}=\|\bar{y}\|^2_{A_h}-2\tau c\left(y^2_{(-1)}\hat{y}_{\bar{x}}+\hat{y}y_{0,5}y_{\bar{x}},a\hat{y}_x\right]_h+2\tau\left(\varphi,A_h\hat{y}\right)_h. \quad (7.23)$$

Here $y_{(-1)}=y^n_{i-1}$. Using embedding (2.8) and (2.9) and the Schwarz and Cauchy inequalities, we get

$$-2\tau c\left(y^2_{(-1)}\hat{y}_{\bar{x}}+\hat{y}y_{0,5}y_{\bar{x}},a\hat{y}_x\right]_h\leqslant 4\tau c\gamma^2\|y\|^2_{A_h}\|\hat{y}\|^2_h\,,$$

$$2\tau\left(\varphi,A_h\hat{y}\right)_h\leqslant\frac{\tau}{2}\|\varphi\|^2_h+2\tau\|A_h\hat{y}\|^2_h\,.$$

Substituting the last relation into (7.22), we obtain an inequality of the form (7.15), where $v^n=\|y^n\|^2_{A_h}$, $v^0=\|u_0\|^2_{A_h}$. Finally, applying Lemma 7.1 we complete the proof of the theorem. □

## 8. Monotonicity of the difference scheme for the nonhomogeneous equation

Let us obtain the problem for perturbation (7.8)–(7.9) in the form (3.1), where $A_i=B_i=\frac{\tau}{h^2}$; $C_i=1+\frac{2\tau}{h^2}-\tau c\left(l_{1h}+y^2\right)-\tau cl_{1h}$; $F_i=\overline{y}^n_i\left(1-\tau cl_{1h}\right)+\tau\left(\tilde{\varphi}-\varphi\right)$.

Using Lemma 3.1, we show that scheme (7.8), (7.9) is conditionally stable up to a time $T<T^{(r)}_{cr}$.

**Theorem 8.1.** *Under the condition* $1-\tau c\left(2l_{1h}+y^2\right)>0$ *the difference scheme* (7.8), (7.9) *is stable.*

The proof is similar to that of Theorem 3.1.

## 9. Estimates of the solution of the initial-boundary value problem for a multidimensional semilinear parabolic equation

Let $\Omega=\left\{\mathbf{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n:\ 0<x_k<l_k\right\}$, $\partial\Omega$ be the boundary of $\Omega$ and $\bar{\Omega}=\Omega\cup\partial\Omega$ be the closure of $\Omega$. In $Q_T=\Omega\times(0,T]$, consider the following initial-boundary value problem:

$$\frac{\partial u}{\partial t}=\operatorname{div}\left(k(\mathbf{x})\operatorname{grad}u\right)+cu|u|^{p-1},\quad(\mathbf{x},t)\in Q_T, \quad (9.1)$$

$$u(\mathbf{x},t)=0,\quad(\mathbf{x},t)\in\partial\Omega\times(0,T], \quad (9.2)$$

$$u(\mathbf{x},0)=u_0(\mathbf{x}),\quad\mathbf{x}\in\bar{\Omega}. \quad (9.3)$$

Suppose that

$$c=\text{const}>0,\quad p>1, \quad (9.4)$$

$$0<m_1\leqslant k(\mathbf{x})\leqslant m_2,\quad\mathbf{x}\in\Omega. \quad (9.5)$$

Define the operator $\mathcal{A}$ by

$$\mathcal{A}u=-\operatorname{div}\left(k(\mathbf{x})\operatorname{grad}u\right).$$

This operator takes the set $\mathscr{D}(\mathcal{A}) = \mathring{H}^1(\Omega) \cap H^2(\Omega)$ to $L_2(\Omega)$. It can be shown that for the linear self-adjoint operator $\mathcal{A}$ inequalities (1.6) and (1.7) hold with the constant

$$\lambda = m_1 \pi^2 \sum_{k=1}^{n} \frac{1}{l_k^2}$$

Introduce a Hilbert space $\mathscr{H}_{\mathcal{A}} = \mathring{H}^1(\Omega)$ with the inner product $(v, w)_{\mathcal{A}} = (\mathcal{A}v, w)_{L_2(\Omega)}$. Suppose that

$$u_0(\mathbf{x}) \in \mathscr{D}(\mathcal{A}). \tag{9.6}$$

It is known that for any function $u \in \mathring{H}^1(\Omega)$ the following inequality holds [11]:

$$\|u\|_{L_{\frac{2q}{q-2}}(\Omega)} \leqslant M \|\mathrm{grad} u\|_{L_2(\Omega)} \tag{9.7}$$

where $M = \beta_1 \mathrm{mes}^{\frac{1}{n}-\frac{1}{q}} \Omega = \beta_1 \left( \prod_{k=1}^{n} l_k \right)^{\frac{1}{n}-\frac{1}{q}}$ and

$$\begin{aligned} q \geqslant n, \quad & \beta_1 = \frac{2(n-1)}{n-2} \quad && \text{for} \quad n \geqslant 3, \\ q > 2, \quad & \beta_1 = \frac{2(q-1)}{q-2} \quad && \text{for} \quad n = 2, \\ q \geqslant 2, \quad & \beta_1 = 2 \quad && \text{for} \quad n = 1. \end{aligned} \tag{9.8}$$

Moreover, it is clear that

$$\|u\|_{\mathcal{A}}^2 \geqslant m_1 \|\mathrm{grad} u\|_{L_2(\Omega)}^2 . \tag{9.9}$$

To obtain the *a priori* estimates for the solution of problem (9.1)–(9.3), let us multiply both sides of equation (9.1) by $2\mathcal{A}u$ and integrate the result over $\Omega$. Thus, we get the following energy identity:

$$\frac{\mathrm{d}\|u\|_{\mathcal{A}}^2}{\mathrm{d}t} + 2\|\mathcal{A}u\|_{L_2(\Omega)}^2 = 2c \left( u|u|^{p-1}, \mathcal{A}u \right)_{L_2(\Omega)} . \tag{9.10}$$

Applying the Schwarz and Cauchy inequalities to the right-hand side of (9.10), we obtain

$$2c \left( u|u|^{p-1}, \mathcal{A}u \right)_{L_2(\Omega)} \leqslant 2c \left( \varepsilon \left\| u|u|^{p-1} \right\|_{L_2(\Omega)}^2 + \frac{1}{4\varepsilon} \|\mathcal{A}u\|_{L_2(\Omega)}^2 \right) . \tag{9.11}$$

It is obvious that

$$\left\| u|u|^{p-1} \right\|_{L_2(\Omega)}^2 = \|u\|_{L_{2p}(\Omega)}^{2p} . \tag{9.12}$$

Substituting (9.11) and (9.12) into (9.10) and using (9.7)–(9.9), we get

$$\frac{\mathrm{d}\|u\|_{\mathcal{A}}^2}{\mathrm{d}t} + 2\left(1 - \frac{c}{4\varepsilon}\right) \|\mathcal{A}u\|_{L_2(\Omega)}^2 \leqslant 2c \left( \frac{M}{\sqrt{m_1}} \right)^{2p} \varepsilon \|u\|_{\mathcal{A}}^{2p} , \tag{9.13}$$

where

$$\begin{aligned} 1 < p \leqslant \frac{n}{n-2}, \quad & \text{for} \quad n \geqslant 3, \\ p > 1, \quad & \text{for} \quad n = 1, 2. \end{aligned} \tag{9.14}$$

Let us denote $v(t) = \left\|u(t)\right\|_{\mathcal{A}}^2$, $q = p$, $a = 2c\left(M/\sqrt{m_1}\right)^{2p}\varepsilon$. In (9.13), we set $\varepsilon = c/2$. Taking into account (1.7) and denoting $b = \lambda$, from (9.13), we get the relation of the form (1.13). Using Lemma 1.1, we obtain that under the condition

$$\frac{c^2\beta_1^{2p}\left(\prod\limits_{k=1}^{n} l_k\right)^{2p/n-p+1}\|u_0\|_{\mathcal{A}}^{2(p-1)}}{m_1^{p+1}\pi^2\sum\limits_{k=1}^{n}\prod\limits_{\substack{i=1\\ i\neq k}}^{n} l_i^2} \leqslant 1 \tag{9.15}$$

the solution of problem (9.1)–(9.3) is bounded in the norm of $\mathscr{H}_{\mathcal{A}}$ for all $t \in [0,T]$, $T \leqslant +\infty$, and satisfies the inequality

$$\left\|u(t)\right\|_{\mathcal{A}} \leqslant \|u_0\|_{\mathcal{A}}\,. \tag{9.16}$$

If condition (9.15) is not satisfied then we use Lemma 4.1 and choose $\varepsilon = c/4$ in (9.13). In this case, for the solution of problem (9.1)–(9.3) the following estimate holds:

$$\left\|u(t)\right\|_{\mathcal{A}} \leqslant \frac{\|u_0\|_{\mathcal{A}}}{\left(1 - 0.5(p-1)c^2\beta_1^{2p}\left(\prod\limits_{k=1}^{n} l_k\right)^{2p/n-p+1} m_1^{-p}\|u_0\|_{\mathcal{A}}^{2(p-1)}\,t\right)^{1/(2(p-1))}} \tag{9.17}$$

for $t \in [0, T_{\mathrm{cr}})$. Here $T_{\mathrm{cr}}$ is determined by (4.3). And if the solution blows up, then the blow-up time satisfies the inequality $T_{\mathrm{bu}} \geqslant T_{\mathrm{cr}}$.

## 10. Numerical results

Since the problem is nonlinear, the computing experiment is the only way to test the theoretical estimates of the blow-up time.

In rectangular $\bar{Q}_T$ with $l = \pi$, consider the following initial-boundary value problem with a constant diffusion coefficient $k(x) = k = 1$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + cu|u|\,, \tag{10.1}$$

$$u(0,t) = u(l,t) = 0; \quad u(x,0) = 2\sin x. \tag{10.2}$$

To solve (10.1),(10.2) approximately, we use the finite-difference scheme

$$y_t + A_h\hat{y} = cy|y|\,, \tag{10.3}$$

$$y(x,0) = u_0(x), \quad x \in \bar{\omega}_h, \quad \hat{y}_0 = 0, \quad \hat{y}_N = 0. \tag{10.4}$$

First, we choose $c = 0.2$, when the conditions of Theorems 1.1 and 2.1 are satisfied. Figure 10.1(a) presents the function $y_i^n$ for various times. It is seen that the solution is bounded. This fact is consistent with the *a priori* estimate (2.12) and Remark 1.1.

If $c = 1.18$ the conditions of Theorems 1.1 and 2.1 are not satisfied and the solution can blow up in a finite time. In Section 4, for the differential solution the lower bound of the blow-up time (4.6) is obtained. For problem (10.1), (10.2), $T_{\mathrm{PDE}} = 0.291$. For the difference problem the lower bound of the blow-up time is determined by (5.4) and $T_{\mathrm{FDS}} = 0.265$.

In [12], the upper bound of the blow-up time is obtained for the differential problem (10.1), (10.2)

$$T_L = \left[(2\alpha+1)\|u_0\|_{L_2}^2 / \alpha^2(2\alpha+2)\right]\left[G(u_0) - \frac{1}{2}(u_0, Au_0)\right]^{-1}, \qquad (10.5)$$

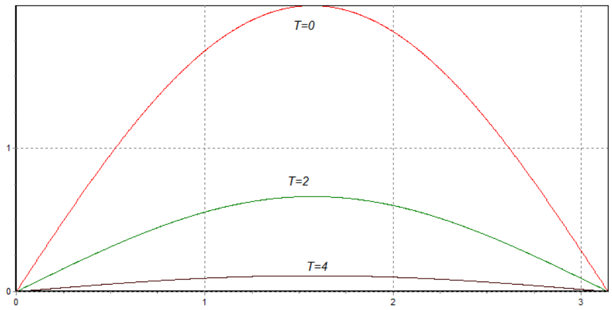
where

$$G(u) = \int_0^1 \left(\rho^p cu|u|^{p-1}, u\right) d\rho = \frac{c}{p+1}\|u\|_{L_{p+1}}^{p+1}.$$

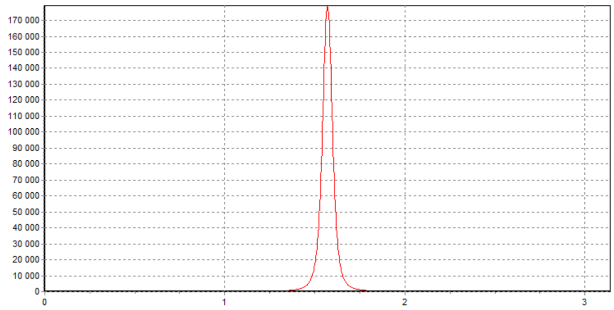
The argument $\alpha$ satisfies the relation $2(\alpha+1)G(u) = \left(u, cu|u|^{p-1}\right)$, whence we get $\alpha = (p+1-2c)/(2c)$. For problem (10.1),(10.2) at $c = 1,18$ the condition of Theorem 1 from [12] is satisfied

$$G(u_0) - \frac{1}{2}(u_0, Au_0) = 1,054 > 0.$$

This condition ensures that the solution blows up. Substituting the values of the arguments into (10.5), we get $T_L \approx 49,175$.




(a)




(b)

Fig. 10.1. The function $y_i^n$

Figure 10.1(b) presents the solution of the finite-difference scheme (10.3), (10.4) at the time $T_{\text{exp}} = 0.642$. One can see from Fig. 10.1(b) that at this time the solution blows up ( [21]).

Thus, for $T_{\text{exp}}$, $T_{\text{PDE}}$, $T_{\text{FDS}}$, $T_L$, the following inequality holds:

$$T_{\text{FDS}} < T_{\text{PDE}} < T_{\text{exp}} < T_L.$$

This is consistent with the theoretical results, i.e., the solution is bounded until $T_{\text{FDS}}$, and the blow-up time $T_{\text{exp}}$ comes to $T_L$.

Different approaches to the study of the nonlinear differential problems with blow-up solutions present an interesting problem of minimizing the interval $[T_{\text{PDE}}, T_L]$.

**Remark 10.1.** All results can be generalized to the case of initial-boundary value problems for differential equations of the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x,t)\frac{\partial u}{\partial x}\right) + c(x,t)u|u|^{p-1} + f(x,t). \qquad (10.6)$$

In particular, if the coefficients of the equation are continuous with respect to $t$ and $\frac{\partial k(x,t)}{\partial t} < 0$, then the results remain the same.

**Remark 10.2.** If the sufficient stability condition (1.18) is not satisfied, then the solution can blow up in a finite time. Non-fulfilment of this condition can be due to not only the small perturbation of the initial condition $u_0$, but also the small perturbation of the coefficients $k(x)$, $c$ and the interval $[0; l]$. Since the last parameters for mathematical modeling can be given approximately, an interesting problem arises. This problem concerns the stability of the solution in the sense of perturbation of the coefficients of the equation and the domain of the problem. Note also that condition (1.18) connects all the input data of the problem. This condition can control the well-posedness of the mathematical model.

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