STABILITY AND MONOTONICITY OF DIFFERENCE SCHEMES FOR NONLINEAR SCALAR CONSERVATION LAWS AND MULTIDIMENSIONAL QUASI-LINEAR PARABOLIC EQUATIONS

P. MATUS¹ AND S. LEMESHEVSKY²

Dedicated to Prof. P. Vabishchevich on the occasion of his 55th birthday

Abstract — We have proved the difference analogue of a Bihari-type inequality. Using this inequality, we study the stability in C and monotonicity of the difference schemes approximating initial-boundary value problems for nonlinear conservation laws and multi-dimensional parabolic equations. It has been shown that in the nonlinear case the stability and monotonicity are determined not only by the behavior of the approximate solution but also by its difference derivatives appearing in the nonlinear terms of the equation. The stability estimates are obtained without any assumptions about the properties of the solution and nonlinear coefficients of the differential problem. Here we use restrictions only on input data (initial and boundary conditions and the right-hand side).

The sufficient conditions of the shock wave generation is formulated for input data. For the Riemann problem two exact and stable difference schemes are analyzed.

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Introduction

Nonlinear hyperbolic conservation laws and parabolic equations play a central role in science and engineering and in the last few years their mathematical theory, as well as their numerical approximation, have made a significant progress [7, 8, 12]. Owing to the nonlinear nature of these equations such physical effects as shock waves, contact discontinuity, boundary layers, and others can appear beyond some finite time interval even when the input data are very smooth functions [12].

We are convinced that the shock wave generation is only determined by the behavior of the input data of the problem [12, 14]. In this connection, for one- and multidimensional nonlinear scalar conservation laws we study how the properties of initial and boundary

¹*Institute of Mathematics, NAS of Belarus, 11 Surganov Str., 220072 Minsk, Belarus, and Department of Mathematics, the John Paul II Catholic University of Lublin, Al. Raclawickie 14, 20-950 Lublin, Poland. E-mail: matus@im.bas-net.by*

²*Institute of Mathematics, NAS of Belarus, 11 Surganov Str., 220072 Minsk, Belarus. E-mail: svl@im.bas-net.by*

functions and the right-hand side influence the shock wave generation. Estimates of the time of shock wave generation for the one-dimensional case can be found in [3, 10].

In investigating difference schemes, the stability of the approximate solution with respect to small perturbations of the input data is one of the main issues. In the nonlinear case, the main distinguishing feature of the stability investigation is the need to study additionally all difference derivatives in the nonlinear terms of the difference equations [16].

The investigation of stability in the nonlinear case is a very difficult problem. In this case, either the *a priori* estimates are often obtained for linearized problem or certain restrictions on nonlinear coefficients or on the solution are imposed. The L_1 -stability of nonlinear hyperbolic systems has been proven in [4]. In [1, 2], the existence, uniqueness and stability of the difference solutions of one-dimensional viscous gas equations have been proved, as well as the error estimates have been derived. All results have been obtained “globally” with respect to the time and without *a priori* assumptions on the solution of gas motion equations [1, 2]. L^∞ -stability for isentropic gas has been proven in [13, 17]. In [17], the conditions for the initial and boundary data have been derived. These conditions ensure stability of the difference scheme.

This paper presents the results of investigating the stability of the solutions of difference schemes approximating nonlinear multi-dimensional scalar conservation laws and quasi-linear parabolic equations. The stability estimates have been proved without any assumptions imposed on the solution and nonlinear coefficients of the differential problems. Only restrictions imposed on input data are used. In contrast to the linear problem, the *a priori* estimates obtained hold, in general, only up to a certain finite instant of time. Of great importance for these investigations is the discrete Bihari-type inequality [11, 23]. A special case of this inequality is the Gronwall inequality used often for investigating the stability and convergence of the difference schemes for time-dependent problems in the linear case. In this paper, we also analyze the stability of the difference schemes approximating the Riemann problem for the Burgers equation.

In investigating the monotonicity, positiveness of the coefficients of the difference scheme or the fulfillment of the maximum principle are often required [28, 29]. In the nonlinear case, such requirements should be imposed not on the difference scheme but on the problem for a perturbation $\delta y = \tilde{y} - y$, where $\tilde{y}$ is the solution of the problem with perturbed input data [8]. Consequently, in this case monotone behavior of the difference derivatives of the approximate solution is required [21, 22]. Note that the monotonicity is necessary for computing problems with highly varying input data [9].

1. Differential problem and its properties

In this section, we consider problems for the first-order hyperbolic equations. We analyze how the input data influence the properties of the considered problems solutions.

1.1. The mixed problem for the one-dimensional transport equation. Consider the initial-boundary value problem for the quasi-linear transport equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = f(t, x), \quad 0 < x \leq l, \quad 0 < t \leq T, \quad (1.1)$$

$$u(t, 0) = \mu(t), \quad 0 < t \leq T, \quad u(0, x) = u_0(x), \quad 0 \leq x \leq l. \quad (1.2)$$

Further assume that $u_0(x) \geq 0$ for all $0 < x \leq l$, $u_0(0) = \mu(0) > 0$ and $\mu(t) > 0$, $f(t, x) \geq 0$

for all $0 < x \leq l$ and $0 < t \leq T$. Under these conditions the solution $u(t, x)$ of problem (1.1), (1.2) is nonnegative $u(t, x) \geq 0$ for all $0 < x \leq l$ and $0 \leq t \leq T$.

Note that the right-hand side $f(t, x)$ of equation (1.1) in general can change the sign. At the same time, if $f(t, x) \geq -M$, $M > 0$, and $u_0(x) \geq m_1 > 0$, $\mu(t) \geq m_2 > 0$ for all $0 < x \leq l$ and $0 < t \leq T$ (M, m_1, m_2 are constants), then the solution $u(t, x)$ of problem (1.1), (1.2) is positive only for some interval $[0, T^*]$. In this case, the upper bound of the time interval T^* depends on the constants M, m_1, m_2 , namely, $T^* = \min\{m_1, m_2\}/M$. More accurate bounds of T^* can be specified by obtaining lower bounds for $u(t, x)$.

To investigate the properties of the solution of the original problem, it is often necessary to estimate its derivative. Differentiation of equation (1.1) with respect to x yields to the Cauchy problem for the Riccati equation with respect to the derivative $v = \partial u / \partial x$

$$\frac{dv}{dt} = -v^2 + F(t), \quad F(t) = \frac{\partial f(t, x(t))}{\partial x}, \quad (1.3)$$

$$v(0) = v_0 = \begin{cases} \mu_1(t) = (f(t, 0) - \mu'(t))/\mu(t), & x(t) = 0, \quad 0 < t \leq T, \\ u'_0(x(0)), & 0 \leq x(0) \leq l, \quad t = 0, \end{cases} \quad (1.4)$$

where $d \cdot / dt$ is the derivative along the direction $dx/dt = u$, i.e., $du/dt = \partial u / \partial t + u \partial u / \partial x$. Further assume that $u_0(x)$ and $\mu_1(t)$ satisfy the consistency conditions $u'_0(0) = \mu_1(0) = (f(0, 0) - \mu'(0))/\mu(0)$.

Let us study how the input data (initial and boundary conditions, the right-hand side) influence the generation of a solution discontinuity (shock wave).

Further we use the following norms:

$$\begin{aligned} \|v(t)\|_{\bar{C}} &= \max_{0 \leq x(t) \leq l} \left| \frac{\partial u(t, x(t))}{\partial x} \right|, \quad \|F(t)\|_C = \max_{0 < x(t) \leq l} \left| \frac{\partial f(t, x(t))}{\partial x} \right|, \\ \|v_0\|_{C_\gamma} &= \max \left\{ \max_{0 < t \leq T} |\mu_1(t)|, \max_{0 \leq x \leq l} |u'_0(x)| \right\}. \end{aligned}$$

1.1.1. The initial conditions influence the shock wave generation. If $\partial f(t, x) / \partial x = 0$ and $\mu_1(t) = 0$ for all $0 < x \leq l$, $0 < t \leq T$, then the function $v(t)$ is defined as follows:

$$v(t) = \frac{v_0}{1 + tv_0}. \quad (1.5)$$

It is readily seen that if $u'_0(x) \geq 0$ for all $0 < x \leq l$, then the derivative of the solution of problem (1.1), (1.2) is always bounded, i.e., $\|\partial u(t, x) / \partial x\|_{\bar{C}} \leq \|u'_0(x)\|_{\bar{C}}$. If the derivative of the initial function $u_0(x)$ is negative at least at one point x^* then the derivative of the solution of problem (1.1), (1.2) $v = \partial u(t, x) / \partial x$ goes to infinity at the time-point $t^* = -1/u'_0(x^*)$, i.e., a shock wave has been generated. If the number of such points is more than one, then, denoting by $\mathcal{N} = \{x \geq 0 : u'_0(x) < 0\}$ the set of points where the derivative of the initial function is negative, we can determine the moment of shock wave generation as follows:

$$t^* = \frac{1}{\max_{x \in \mathcal{N}} |u'_0(x)|}.$$

Thus, if the right-hand side of equation (1.1) is independent of x and the function μ from the boundary conditions is such that $\mu'(t) = f(t, x)$, we get that a shock wave is generated only when $u'_0(x)$ is negative at least at one point.

1.1.2. *The boundary condition and the right-hand side influence the shock wave generation.* Now suppose that the initial condition satisfies $u'_0(x) \geq 0$ for all $x \geq 0$. Moreover, assume that $f(t, x) = 0$ for all $0 < x \leq l$, $0 < t \leq T$. Then the derivative $v = \partial u(t, x)/\partial x$ satisfies the following problem:

$$\frac{dv}{dt} = -v^2, \quad v(0) = v_0 = \begin{cases} -\mu'(t)/\mu(t), & x(t) = 0, \quad 0 < t \leq T, \\ u'_0(x(0)), & 0 \leq x(0) \leq l, \quad t = 0. \end{cases}$$

In this case, the solution $v(t)$ is also defined by (1.5). Since $u'_0(x) \geq 0$ for all $0 \leq x \leq l$ it follows that only the boundary condition can generate a shock wave. If $\mu'(t) \leq 0$ for all $0 < t \leq T$, then $v_0 \geq 0$ on the boundary $\gamma = \{(t, x) : x = 0, 0 < t \leq T\} \cup \{(t, x) : 0 \leq x \leq l, t = 0\}$, and a shock wave is not generated. But if the function $\mu'(t) > 0$ at least at one point and $t_1 = \min_{\mu(t) > 0} \{0 < t \leq T\}$, then $v(t)$ can go to infinity at the time-point:

$$t_1^* = \mu(t_1)/\mu'(t_1), \quad t_1^* > t_1.$$

Now let $u'_0(x) \geq 0$, $0 \leq x \leq l$, $\mu'(t) \leq 0$, $0 < t \leq T$, and $\partial f(t, x(t))/\partial x < 0$ at least at one point of the interval $0 < x \leq l$. In this case, the derivative of the solution of problem (1.1), (1.2) $v = \partial u(t, x)/\partial x$ satisfies the inhomogeneous Riccati equation (1.3) and the initial condition (1.4). Consider a simple example showing how the right-hand side influences the shock wave generation. Let

$$f(t, x) = -a^2x + f_1(t), \quad a = \text{const} \neq 0.$$

For such a right-hand side, problem (1.3), (1.4) can be solved exactly and the solution is determined by

$$v(t) = a \operatorname{tg} \left(-at + \operatorname{arctg} \frac{v_0}{a} \right).$$

In this case, a shock wave is generated at a time-point defined as follows:

$$t_2^* = \frac{\pi}{2|a|} + \frac{1}{|a|} \operatorname{arctg} \frac{\|v_0\|_{C_\gamma}}{|a|}.$$

We have considered how the input data influence the shock wave generation.

To obtain an estimate of the derivative in the general case, we can use the following lemma about integral Bihari-type inequalities and its corollary.

Lemma 1.1 [23]. *Let ϕ_i , $i = 1, 2$ be continuous and nondecreasing functions on $[0, \infty)$ and positive on $(0, \infty)$ such that $\phi = \phi_2/\phi_1$ is nondecreasing on $(0, \infty)$. Let u , g_i , $i = 1, 2$ be continuous and nonnegative functions on $[t_0, T]$ and $c > 0$ a constant. If*

$$u(t) \leq c + \int_{t_0}^t g_1(s) \phi_1(u(s)) ds + \int_{t_0}^t g_2(s) \phi_2(u(s)) ds, \quad (1.6)$$

then for $t \in [t_0, t_1]$

$$u(t) \leq \Phi_2^{-1} \left[\Phi_2(c_1) + \int_{t_0}^t g_2(s) ds \right], \quad (1.7)$$

where

$$\Phi_1(u) = \int_c^u \frac{dv}{\phi_1(v)}, \quad u > 0; \quad \Phi_2(u) = \int_{u_0}^u \frac{dv}{\phi_2(v)}, \quad u > 0, \quad u_0 > 0 \quad (1.8)$$

and t_1 is the largest number such that $t_1 \geq t_0$ and for $i = 1, 2$:

$$\|g_i\|_{t_1} := \int_{t_0}^{t_1} g_i(s) ds \leq \int_{c_{i-1}}^{\infty} \frac{dv}{\phi_i(v)}, \quad c_0 = c, \quad c_1 = \Phi_1^{-1}(\|g_1\|_{t_1}). \quad (1.9)$$

Corollary 1.1. Let $u(t)$, $f(t)$, $g(t)$ be continuous and nonnegative functions on $[t_0, T]$, and $c > 0$ a constant. Moreover, suppose that

$$u(t) \leq c + \int_{t_0}^t f(s) ds + \int_{t_0}^t g(s) u^2(s) ds, \quad t \geq t_0,$$

and t_1 is the largest number such that $t_1 \geq t_0$ and

$$\int_{t_0}^{t_1} g(s) ds \leq \frac{1}{c + \int_{t_0}^{t_1} f(s) ds}.$$

Then for $t \in [t_0, t_1]$ the following inequality holds:

$$u(t) \leq \frac{c + \int_{t_0}^t f(s) ds}{1 - (c + \int_{t_0}^t f(s) ds) \int_{t_0}^t g(s) ds}. \quad (1.10)$$

Proof. To prove this statement, we put

$$\phi_1(u(s)) \equiv 1, \quad \phi_2(u(s)) = u^2(s), \quad g_1(t) = f(t), \quad g_2(t) = g(t)$$

in the conditions of Lemma 1.1 Then

$$\Phi_1(u) = u - c, \quad \Phi_1^{-1}(v) = v + c; \quad \Phi_2(u) = \frac{1}{u_0} - \frac{1}{u}, \quad \Phi_2^{-1}(v) = \frac{u_0}{1 - u_0 v}, \quad u_0 > 0,$$

$$c_1 = c + \int_{t_0}^{t_1} f(s) ds.$$

Using these relations and Lemma 1.1, we obtain inequality (1.10). $\square$

Remark 1.1. Note that the well-known Gronwall inequality follows from Lemma 1.1 with $g_1(t) \equiv 0$, $\phi_2(u) \equiv 1$.

Applying Corollary 1.1 to the solution of problem (1.3), (1.4) we get

$$\left(\|v_0\|_{C_\gamma} + \int_0^t \|F(\xi)\|_C d\xi \right) t = 1. \quad (1.11)$$

For the given $F(t)$ the solution of equation (1.11) gives a time-point t^* to which the derivative of the solution of problem (1.1), (1.2) is always bounded, i.e., the estimate

$$\|v(t)\|_{\bar{C}} \leq \frac{\|v_0\|_{C_\gamma} + \int_0^t \|F(\xi)\|_C d\xi}{1 - t \left(\|v_0\|_{C_\gamma} + \int_0^t \|F(\xi)\|_C d\xi \right)}, \quad t - t^* \geq \delta_1 > 0$$

holds. Since $\partial u / \partial x \rightarrow \infty$ as $t \rightarrow t^*$, consequently, shock wave can be generated. However, in this case, the moment of discontinuity generation cannot be determined exactly. The time-point t_s of shock wave generation belongs to some interval $t_s \in [t^*, \bar{t}^*]$. The above approach allows to determine only the lower bound t^* of this interval. To set the upper bound of this $\bar{t}^*$, we need to obtain the lower bound of the function $v(t)$.

1.2. Multidimensional scalar conservation law. Now let us study how the initial data influence the generation of discontinuity of the solution in multidimensional case. Namely, we study the behavior of the solution gradient depending on the initial data.

Consider the Cauchy problem for the homogeneous quasi-linear transport equation in the space $\mathbb{R}^d$, $d \geq 2$.

$$\frac{\partial u}{\partial t} + \sum_{i=1}^d \alpha_i u \frac{\partial u}{\partial x_i} = 0, \quad \mathbf{x} \in \mathbb{R}^d, \quad t \in (0, +\infty), \quad (1.12)$$

$$u(0, \mathbf{x}) = u_0(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d. \quad (1.13)$$

Here $0 \leq u_0(\mathbf{x}) \leq U = \text{const}$ for all $\mathbf{x} \in \mathbb{R}^d$ and α_i ($i = 1, 2, \dots, d$) is a positive constant.

Denote by

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \sum_{i=1}^d \alpha_i u p_i,$$

the derivative along the direction given by the system of equations

$$\frac{dx_i}{dt} = \alpha_i u(t, \mathbf{x}), \quad i = 1, 2, \dots, d,$$

where $p_i = \frac{\partial u}{\partial x_i}$, $i = 1, 2, \dots, d$. Then we can rewrite equation (1.12) in the following form:

$$\frac{du}{dt} = 0. \quad (1.14)$$

Differentiating equation (1.14) with respect to x_k ($k = 1, 2, \dots, d$), we get

$$\frac{dp_k}{dt} = - \sum_{i=1}^d \alpha_i p_k p_i, \quad k = 1, 2, \dots, d.$$

Multiplying both sides of the last relation by α_k and summing the results in k from 1 to d , we obtain the Riccati equation with respect to the function $v(t) = \sum_{i=1}^d \alpha_i p_i(t, \mathbf{x}(t))$:

$$\frac{dv}{dt} = -v^2. \quad (1.15)$$

From (1.13) we obtain the initial conditions

$$v(0) = v_0 = \sum_{i=1}^d \alpha_i \frac{\partial u_0(\mathbf{x})}{\partial x_i} = \vec{\alpha} \cdot \nabla u_0(\mathbf{x}), \quad (1.16)$$

where $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_d)$ is an element of the space $\mathbb{R}^d$, and $\cdot$ denotes the inner product in $\mathbb{R}^d$.

The solution of problem (1.15), (1.16) is determined by (1.5). Hence, as for the one-dimensional case, we conclude that a shock wave is generated if $\vec{\alpha} \cdot \nabla u_0(\mathbf{x}) < 0$ at least at one point of $\mathbb{R}^d$. For finding the moment of shock wave generation we have

$$t^* = \frac{1}{\max_{\mathbf{x} \in \mathcal{N}} |\vec{\alpha} \cdot \nabla u_0(\mathbf{x})|},$$

where $\mathcal{N} = \{\mathbf{x} \in \mathbb{R}^d : \vec{\alpha} \cdot \nabla u_0(\mathbf{x}) < 0\}$.

2. Preliminary results

In this section, we discuss some concepts and results necessary for the further investigation of the difference schemes.

2.1. Stability. In general, the stability of a difference scheme means that its solution continuously depends on the input data [25]. In other words, the computational process is called stable if small errors of input data involve small errors of the solution (increase of errors is limited).

Approximation of problems of mathematical physics yields a family of difference schemes, which can be interpreted as an operator equation with operators acting in linear normalized finite-dimensional spaces [25].

For formal description of the difference scheme it is natural to introduce the families of linear spaces $\mathcal{B}_h^{(1)}$ and $\mathcal{B}_h^{(2)}$, *spaces of grid functions* given on grid $\bar{\omega}_h$ with vector parameter h provided with norm $|h| > 0$. Suppose that $\mathcal{B}_h^{(1)}$, $\mathcal{B}_h^{(2)}$ are finite-dimensional spaces whose dimension depends on h and can tend to infinity as $|h| \rightarrow 0$. We say that the family of operator equations

$$\mathcal{A}_h(y_h) = \varphi_h, \quad y_h \in \mathcal{B}_h^{(1)}, \quad \varphi_h \in \mathcal{B}_h^{(2)}, \quad (2.1)$$

is a *difference scheme*. Here $\mathcal{A}_h(y_h)$ is the nonlinear operator from $\mathcal{D}(\mathcal{A}_h) = \mathcal{B}_h^{(1)}$ to $\mathcal{R}(\mathcal{A}_h) \subseteq \mathcal{B}_h^{(2)}$. The operator $\mathcal{A}_h$ defines the structure of a concrete difference scheme, y_h the solution of the difference scheme, and φ_h the input data of the problem. For example, for the time-dependent problems we have

$$\varphi_h = \begin{cases} f_h & \text{is the right-hand side at inner grid points,} \\ \mu_h & \text{is the boundary conditions at boundary grid points,} \\ u_{0h} & \text{is the initial condition at } t = 0. \end{cases}$$

Perturbing the input data $\tilde{\varphi}_h \in \mathcal{B}_h^{(2)}$ in (2.1) we get the equation with respect to the perturbed solution $\tilde{y}_h$:

$$\mathcal{A}_h(\tilde{y}_h) = \tilde{\varphi}_h. \quad (2.2)$$

The solution y_h of the difference scheme (2.1) is called *stable* if this solution depends on the input data φ_h continuously and uniformly with respect to h . In other words, there exists a positive constant M independent of the choice of h and φ_h , $\tilde{\varphi}_h$ such that for sufficiently small $|h| < h_0$ and for any $\varphi_h, \tilde{\varphi}_h \in \mathcal{B}_h^{(2)}$ the following inequality holds:

$$\|\tilde{y}_h - y_h\|_{(1_h)} \leq M \|\tilde{\varphi}_h - \varphi_h\|_{(2_h)}. \quad (2.3)$$

Here $\|\cdot\|_{(1_h)}, \|\cdot\|_{(2_h)}$ are the norms of the spaces $\mathcal{B}_h^{(1)}$ and $\mathcal{B}_h^{(2)}$ respectively.

To derive a stability estimate of the form of (2.3), we need to obtain a problem for the perturbation of the difference solution $\delta y_h = \tilde{y}_h - y_h$. Subtracting equation (2.1) from (2.2), we get

$$\mathcal{P}_h(\delta y_h, \tilde{y}_h, y_h) = \delta \varphi_h, \quad (2.4)$$

where, in general, the grid operator $\mathcal{P}_h$ depends on δy_h , the solutions $\tilde{y}_h, y_h$, and their difference derivatives [16]. Thus, to investigate problem (2.4), it is necessary to obtain the corresponding *a priori* estimates for the solutions of problems (2.1) and (2.2) in the strong norms. R. Courant has stressed this property of investigating the solutions of the nonlinear mathematical physics problems in his book [5]. Practically, this means that we have to prove the existence of a solution.

As is well known [25], the concept of stability is a part of the concept of *well-posedness* of the difference scheme (2.1): the difference scheme (2.1) is *well-posed* if for a sufficiently small parameter h and for all input data from a certain admissible family there exists a unique solution y_h and this solution is stable.

We stress that from estimate (2.3) it follows that the solution is unique. Suppose $\tilde{\varphi}_h = \varphi_h$, $y_h = y_1$, $\tilde{y}_h = y_2$ (y_1 and y_2 are two different solutions for the same input data of problems (2.1) and (2.2)); then from inequality (2.3) we get $\|y_1 - y_2\|_{(1_h)} \leq M \|0\|_{(2_h)} = 0$; whence $y_1 = y_2$. So, in the nonlinear case (as well as in the linear case), the requirement of uniqueness of the solution can be omitted in the definition of the well-posedness.

2.2. Difference approximation and convergence. We say that the difference scheme (2.1) approximates the operator equation

$$\mathcal{A}(u) = f, \quad u \in \mathcal{B}^{(1)}, \quad f \in \mathcal{B}^{(2)}, \quad \mathcal{A} : \mathcal{B}^{(1)} \rightarrow \mathcal{B}^{(2)}, \quad (2.5)$$

with order $n > 0$ if

$$\|\varphi_h - \mathcal{P}_h^{(2)}(f)\|_{(2_h)} = \|\mathcal{A}_h(u_h) - \mathcal{P}_h^{(2)}(\mathcal{A}(u))\|_{(2_h)} = \|\mathcal{A}_h(\mathcal{P}_h^{(1)}(u)) - \mathcal{P}_h^{(2)}(\mathcal{A}(u))\|_{(2_h)} = O(|h|^n).$$

Here the projection operators $\mathcal{P}_h^{(1)} : \mathcal{B}^{(1)} \rightarrow \mathcal{B}_h^{(1)}$ and $\mathcal{P}_h^{(2)} : \mathcal{B}^{(2)} \rightarrow \mathcal{B}_h^{(2)}$ such that

$$\begin{aligned} \mathcal{P}_h^{(1)}(u) &= u_h \in \mathcal{B}_h^{(1)}, & \text{if } u \in \mathcal{B}^{(1)}, \\ \mathcal{P}_h^{(2)}(f) &= f_h \in \mathcal{B}_h^{(2)}, & \text{if } f \in \mathcal{B}^{(2)}, \end{aligned}$$

and

$$\lim_{|h| \rightarrow 0} \|\mathcal{P}_h^{(1)}(u)\|_{(1_h)} = \|u\|_{(1)}, \quad \lim_{|h| \rightarrow 0} \|\mathcal{P}_h^{(2)}(f)\|_{(2_h)} = \|f\|_{(2)},$$

where $\|\cdot\|_{(k)}$ is the norm in the space $\mathcal{B}^{(k)}$, $k = 1, 2$.

To prove the convergence of the solution y_h of the difference scheme (2.1) to the solution of the operator equation (2.5), i.e. to obtain the estimate

$$\|y_h - u_h\|_{(1_h)} \leq M|h|^n, \quad u_h = \mathcal{P}_h^{(1)}(u),$$

with a constant $M > 0$ independent of h , we can use the following

Theorem 2.1. *If scheme (2.1) is stable and approximates problem (2.5), then the solution y_h of problem (2.1) converges to the solution $u \in \mathcal{B}^{(1)}$ of equation (2.5) as $|h| \rightarrow 0$. Moreover, the order of convergence of scheme (2.1) is equal to the order of the difference approximation.*

The proof for linear and nonlinear operator $\mathcal{A}_h$ is found in [25].

2.3. Monotonicity of the difference schemes. In the theory of computational methods the concept of monotonicity is very important because monotonicity ensures the absence of nonphysical oscillations in numerical computations. Keeping the property of monotonicity of the difference scheme is very important for problems with highly varying input data [9].

In the linear case, the monotonicity of the difference scheme follows from the positiveness of the coefficients of the difference scheme [9] or from the grid maximum principle [25, 27, 28]. The maximum principle is also fulfilled under positive coefficients of difference scheme (2.1). In the nonlinear case, we need a more precise definition [15]. The most correct and natural definition is given in the monograph [8]. For the abstract problem (2.1) this definition is formulated as follows.

Definition 2.1. The difference scheme (2.1) is called *monotone* if from the condition $\tilde{\varphi}_h \geq \varphi_h$ ($\tilde{\varphi}_h \leq \varphi_h$) it follows that $\tilde{y}_h \geq y_h$ ($\tilde{y}_h \leq y_h$) at all grid points $\bar{\omega}_h$.

In other words, the difference scheme is monotone if from the condition $\delta\varphi \geq 0$ ($\delta\varphi \leq 0$) it follows that $\delta y \geq 0$ ($\delta y \leq 0$). This implies that in the nonlinear case problem (2.4) for the investigation of stability is the same as for the investigation of monotonicity. Certainly, the conditions of positiveness of the difference operator coefficients or the fulfillment of the grid maximum principle used in the linear case can also be used for the nonlinear problems. But these conditions have to be applied to the nonlinear problem for perturbation (2.4) mismatched with problem (2.1) or (2.2). Note that in this case the requirement of positiveness of the coefficients leads to the requirement of monotone behavior for the difference derivatives [16, 21, 22].

2.4. Grid maximum principle. Let $\bar{\omega}_h$ be a finite set of points (grid) in a certain bounded domain of the Euclidian space of dimension d . The grid point $\mathbf{x} \in \bar{\omega}_h$ is called the *boundary point* of the grid $\bar{\omega}_h$ if at this grid point the value $y(\mathbf{x}) = \mu(\mathbf{x})$ of the function $y(\mathbf{x})$ is given. The set of all boundary grid points is said to be the *boundary* of the grid and is denoted by $\partial\omega_h$. Other grid points of $\bar{\omega}_h$ are called *interior grid points* and the set of interior grid points is denoted by ω_h .

On the grid $\bar{\omega}_h$ we consider the difference scheme

$$\alpha(\mathbf{x})y(\mathbf{x}) = \sum_{\mathbf{s} \in \mathcal{M}(\mathbf{x})} \beta(\mathbf{x}, \mathbf{s})y(\mathbf{s}) + \varphi(\mathbf{x}), \quad \mathbf{x} \in \omega_h, \quad y|_{\mathbf{x} \in \partial\omega_h} = \mu(\mathbf{x}), \quad \mathbf{x} \in \partial\omega_h. \quad (2.6)$$

Scheme (2.6) is called the *canonical form* of difference equations [25]. Here $\alpha(\mathbf{x})$, $\beta(\mathbf{x}, \mathbf{s})$, $\varphi(\mathbf{x})$ are the grid functions defined for all $\mathbf{x}, \mathbf{s} \in \omega_h$. Summation is performed over a certain subset $\mathcal{M}(\mathbf{x})$ of the grid $\bar{\omega}_h$, $\mathcal{M}(\mathbf{x})$ does not contain the grid point $\mathbf{x}$, and $\mathcal{T}(\mathbf{x}) = \mathcal{M}(\mathbf{x}) \cup \{\mathbf{x}\}$ is the *stencil* of the difference scheme. The subset $\mathcal{M}(\mathbf{x})$ is called the *neighborhood of the grid point $\mathbf{x}$* . Suppose that the grid $\bar{\omega}_h$ is *connected*, i.e., for any two grid points $\bar{\mathbf{x}} \in \bar{\omega}_h$ and $\bar{\bar{\mathbf{x}}} \in \bar{\omega}_h$ there exist grid points $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m \in \bar{\omega}_h$ such that

$$\bar{\mathbf{x}} \in \mathcal{M}(\mathbf{x}_1), \quad \mathbf{x}_1 \in \mathcal{M}(\mathbf{x}_2), \dots, \mathbf{x}_{m-1} \in \mathcal{M}(\mathbf{x}_m), \quad \mathbf{x}_m \in \mathcal{M}(\bar{\bar{\mathbf{x}}}).$$

Let us estimate the solution $y(\mathbf{x})$ of problem (2.6) in the $C(\bar{\omega}_h)$ -norm. The following notation is needed for the sequel

$$\|y\|_{C(\bar{\omega}_h)} = \max_{\mathbf{x} \in \bar{\omega}_h} |y(\mathbf{x})|, \quad \|y\|_{C(\omega_h)} = \max_{\mathbf{x} \in \omega_h} |y(\mathbf{x})|$$

Lemma 2.1. *Suppose the following conditions hold:*

$$\begin{aligned} \alpha(\mathbf{x}) &\geq 0, \quad \varphi(\mathbf{x}) \geq f > 0, \quad \text{for all } \mathbf{x} \in \omega_h, \\ \beta(\mathbf{x}, \mathbf{s}) &\geq 0, \quad \text{for all } \mathbf{x}, \mathbf{s} \in \omega_h, \\ 0 < \gamma(\mathbf{x}) = \alpha(\mathbf{x}) - \sum_{\mathbf{s} \in \mathcal{M}} \beta(\mathbf{x}, \mathbf{s}) &\leq d_2, \quad \text{for all } \mathbf{x} \in \omega_h, \\ \mu(\mathbf{x}) &\geq m_1 > 0, \quad \text{for all } \mathbf{x} \in \partial\omega_h. \end{aligned} \quad (2.7)$$

Then

$$y(\mathbf{x}) \geq c_1 > 0 \quad \text{for all } \mathbf{x} \in \bar{\omega}_h \quad c_1 = \min \left\{ m_1, \frac{f}{d_2} \right\}. \quad (2.8)$$

Proof. If the function $y(\mathbf{x})$ reaches its minimum on the boundary of the grid $\partial\omega_h$, then

$$y(\mathbf{x}) \geq m_1 > 0, \quad \mathbf{x} \in \bar{\omega}_h. \quad (2.9)$$

If the function $y(\mathbf{x})$ reaches its minimum at some interior grid point $\mathbf{x} = \mathbf{x}_* \in \omega_h$, then from (2.6) we have

$$\alpha(\mathbf{x}_*)y(\mathbf{x}_*) = \sum_{\mathbf{s} \in \mathcal{M}} \beta(\mathbf{x}_*, \mathbf{s})y(\mathbf{s}) + \varphi(\mathbf{x}_*) \geq \sum_{\mathbf{s} \in \mathcal{M}(\mathbf{x}_*)} \beta(\mathbf{x}_*, \mathbf{s})y(\mathbf{x}_*) + \varphi(\mathbf{x}_*).$$

Whence,

$$\left(\alpha(\mathbf{x}_*) - \sum_{\mathbf{s} \in \mathcal{M}(\mathbf{x}_*)} \beta(\mathbf{x}_*, \mathbf{s}) \right) y(\mathbf{x}_*) \geq \varphi(\mathbf{x}_*).$$

Taking into account conditions (2.7) from the last relation, we get

$$y(\mathbf{x}_*) \geq f/d_2 > 0. \quad (2.10)$$

Combining (2.9) and (2.10), we obtain inequality (2.8). $\square$

Lemma 2.2. *Suppose that for all $\mathbf{x}, \mathbf{s} \in \omega_h$ the following conditions are fulfilled:*

$$\alpha(\mathbf{x}) \geq 0, \quad \beta(\mathbf{x}, \mathbf{s}) \geq 0, \quad \gamma(\mathbf{x}) = \alpha(\mathbf{x}) - \sum_{\mathbf{s} \in \mathcal{M}(\mathbf{x})} \beta(\mathbf{x}, \mathbf{s}) > 0. \quad (2.11)$$

Then

$$\|y(\mathbf{x})\|_{C(\bar{\omega}_h)} \leq \max \left\{ \|\mu\|_{C(\partial\omega_h)}, \left\| \frac{\varphi}{\gamma} \right\|_{C(\omega_h)} \right\}. \quad (2.12)$$

Proof. If the grid function $y(\mathbf{x})$ reaches its maximum on the boundary of the grid $\partial\omega_h$, then $\|y\|_{C(\bar{\omega}_h)} = \|\mu\|_{C(\partial\omega_h)}$. But if the grid function y reaches its maximum at an interior grid-point $\mathbf{x}^* \in \omega_h$, then from (2.6) we get

$$\alpha(\mathbf{x}^*)\|y\|_{C(\bar{\omega}_h)} \leq \sum_{\mathbf{s} \in \mathcal{M}(\mathbf{x}^*)} \beta(\mathbf{x}^*, \mathbf{s})\|y\|_{C(\bar{\omega}_h)} + |\varphi(\mathbf{x}^*)|.$$

Now using (2.11), from the last inequality we get the estimate

$$\|y\|_{\bar{C}} \leq \|\varphi/\gamma\|_{C(\omega_h)}.$$

Thus we obtain estimate (2.12). $\square$

Lemma 2.3. *Suppose the following recurrence relation holds:*

$$\|y^{n+1}\|_{C(\bar{\omega}_h)} \leq \max \{ \mu^{n+1}, \|y^n\|_{C(\bar{\omega}_h)} \} + \tau \|f^{n+1}\|_{C(\omega_h)}. \quad (2.13)$$

Then

$$\begin{aligned} \|y^{n+1}\|_{C(\bar{\omega}_h)} &\leq c_{2n}, \quad n = 0, 1, \dots, \\ c_{2n} &= \max \left\{ \max_{0 \leq k \leq n} \mu^{k+1}, \|y^0\|_{C(\bar{\omega}_h)} \right\} + \sum_{k=0}^n \tau \|f^{k+1}\|_{C(\omega_h)}. \end{aligned} \quad (2.14)$$

This lemma can be proved by induction (see [20]).

2.5. Discrete Bihari-type inequality. In section 1, in the general case, we use an integral Bihari-type inequality to obtain estimates of the derivative of the solution. Here we prove a discrete Bihari-type inequality.

Lemma 2.4 (discrete analogue of Corollary 1.1). *Let $\mathcal{E}_n$, $\mathcal{G}_n$, g_n and f_n be non-negative functions defined on the grid $\omega_\tau = \{t_n = \tau n, n = 0, 1, \dots\}$. Besides, suppose that $\mathcal{G}_{n+1} \geq \mathcal{G}_n$ and $\mathcal{G}_n \geq \mathcal{E}_0$ for all $n = 1, 2, \dots$. Then, if the inequalities*

$$\mathcal{E}_{n+1} \leq \mathcal{G}_{n+1} + \sum_{k=0}^n \tau g_k \mathcal{E}_{k+1} \mathcal{E}_k, \quad (2.15)$$

$$\sum_{k=0}^n \tau g_k < \frac{1}{\mathcal{G}_{n+1}}, \quad n = 0, 1, \dots, \quad (2.16)$$

hold, then

$$\mathcal{E}_{n+1} \leq \frac{\mathcal{G}_{n+1}}{1 - \mathcal{G}_{n+1} \sum_{k=0}^n \tau g_k}. \quad (2.17)$$

Proof. The proof is by induction on n . For $n = 0$, taking into account the properties of the grid function $\mathcal{G}_n$, from inequalities (2.15) and (2.16) we have

$$\mathcal{E}_1 \leq \mathcal{G}_1 + \tau g_0 \mathcal{E}_1 \mathcal{E}_0 \leq \mathcal{G}_1 + \tau g_0 \mathcal{E}_1 \mathcal{G}_1, \quad \tau g_0 < 1/\mathcal{G}_1.$$

Hence,

$$\mathcal{E}_1 \leq \frac{\mathcal{G}_1}{1 - \mathcal{G}_1 \tau g_0}.$$

Consequently, for $n = 0$ estimate (2.17) holds. Now assume that inequality (2.17) holds for $n = p \geq 1$. Then, taking into account the properties of the function $\mathcal{G}_n$ and inequalities (2.15) and (2.16), we have

$$\mathcal{E}_{p+2} \leq \mathcal{G}_{p+2} + \sum_{k=0}^{p+1} \tau g_k \mathcal{E}_{k+1} \mathcal{E}_k \leq \mathcal{G}_{p+2} + \frac{\tau g_{p+1} \mathcal{E}_{p+2} \mathcal{G}_{p+1} + \mathcal{G}_{p+1} \mathcal{G}_p \sum_{k=0}^p \tau g_k}{1 - \mathcal{G}_{p+1} \sum_{k=0}^p \tau g_k}.$$

Hence,

$$\left(1 - \mathcal{G}_{p+1} \sum_{k=0}^{p+1} \tau g_k \right) \mathcal{E}_{p+2} \leq \mathcal{G}_{p+2} - \mathcal{G}_{p+1} (\mathcal{G}_{p+2} - \mathcal{G}_p) \sum_{k=0}^p \tau g_k. \quad (2.18)$$

On the assumption of lemma $\mathcal{G}_{p+2} \geq \mathcal{G}_{p+1} \geq \mathcal{G}_p$. Thus from (2.18) it follows that inequality (2.17) also holds for $n = p + 1$. This completes the proof. $\square$

Below Lemma 2.4 is used to obtain *a priori* estimates of the difference derivative of the solution of the implicit difference scheme approximating the nonlinear transport equation.

3. Difference schemes for the nonlinear scalar conservation law

Let us study the properties of the difference algorithms approximating problems for nonlinear equations.

In the domain $\bar{Q} = \{(t, x) : 0 \leq t \leq T, 0 \leq x \leq l\}$, consider the problem

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = f(t, x), \quad 0 < t \leq T, \quad 0 < x \leq l, \quad (3.1)$$

$$u(t, 0) = \mu(t), \quad 0 < t \leq T, \quad u(0, x) = u_0(x), \quad 0 \leq x \leq l. \quad (3.2)$$

Suppose that the input data satisfy the following conditions:

$$\begin{aligned} u_0(x) &\in C^1[0, l], \quad \mu(t) \in C^1[0, T], \\ \mu(t) &\geq m_1 > 0, \quad u_0(x) \geq m_2 > 0, \quad f(t, x) \geq 0, \quad 0 \leq t \leq T, \quad 0 \leq x \leq l. \end{aligned} \quad (3.3)$$

In the domain $\bar{Q}$, let us introduce the grids

$$\begin{aligned} \bar{\omega} &= \bar{\omega}_h \times \bar{\omega}_\tau, \quad \bar{\omega}_h = \{x_i = ih : i = 0, 1, 2, \dots, N_x\} = \omega_h \cup \{x_0 = 0\}, \\ \bar{\omega}_\tau &= \{t_n = n\tau : n = 0, 1, \dots, N\} = \omega_\tau \cup \{T = N\tau\}. \end{aligned}$$

3.1. Implicit difference scheme. On the grid $\bar{\omega}$ we approximate problem (3.1), (3.2) by the linearized implicit difference scheme

$$\begin{aligned} y_t + y \hat{y}_{\bar{x}} &= \varphi, \quad \varphi(t, x) = f(t + \tau, x) \quad x \in \omega_h, \quad t \in \omega_\tau, \\ \hat{y}(t, 0) &= \hat{\mu}(t), \quad t \in \omega_\tau, \quad y(0, x) = u_0, \quad x \in \bar{\omega}_h. \end{aligned} \quad (3.4)$$

Here we use the following notation of the theory of difference schemes [25]:

$$y = y_i^n = y(t_n, x_i), \quad \hat{y} = y_i^{n+1}, \quad y_t = \frac{\hat{y} - y}{\tau}, \quad y_{\bar{x}} = \frac{y_i^n - y_{i-1}^n}{h}.$$

Rewrite problem (3.4) in the form

$$\begin{aligned} (1 + \gamma y_i^n) y_i^{n+1} &= \gamma y_i^n y_{i-1}^{n+1} + y_i^n + \tau \varphi_i^n, \quad i = 1, 2, \dots, N_x, \quad n = 1, 2, \dots, N-1, \\ y_0^{n+1} &= \mu^{n+1}, \quad n = 0, 1, \dots, N-1, \quad y_i^0 = u_{0,i}, \quad i = 0, 1, \dots, N_x, \quad \gamma = \frac{\tau}{h}. \end{aligned} \quad (3.5)$$

Using Lemmas 2.1–2.3, conditions (3.3), and relations (3.5), it is easy to obtain (by induction on n) the estimate

$$0 < c_1 \leq y_i^{n+1} \leq \|y^{n+1}\|_{C(\bar{\omega}_h)} \leq \max \left\{ \max_{1 \leq k \leq n+1} \mu^k, \|u_0\|_{C(\bar{\omega}_h)} \right\} + \sum_{k=0}^N \tau \|\varphi^k\|_{C(\omega_h)} = c_2, \quad (3.6)$$

where $c_1 = \min\{m_1, m_2\}$.

Consider the difference scheme

$$\begin{aligned} \tilde{y}_t + \tilde{y} \hat{\tilde{y}}_{\bar{x}} &= \tilde{\varphi}, \quad \tilde{\varphi}(t, x) = \tilde{f}(t + \tau, x), \quad x \in \omega_h, \quad t \in \omega_\tau, \\ \hat{\tilde{y}}(t, 0) &= \hat{\mu}(t), \quad t \in \omega_\tau, \quad \tilde{y}(0, x) = \tilde{u}_0, \quad x \in \bar{\omega}_h, \end{aligned} \quad (3.7)$$

approximating the differential problem

$$\frac{\partial \tilde{u}}{\partial t} + \tilde{u} \frac{\partial \tilde{u}}{\partial x} = \tilde{f}(t, x), \quad 0 < t \leq T, \quad 0 < x \leq l, \quad (3.8)$$

$$\tilde{u}(t, 0) = \tilde{\mu}(t), \quad 0 < t \leq T, \quad \tilde{u}(0, x) = \tilde{u}_0(x), \quad 0 \leq x \leq l, \quad (3.9)$$

with perturbed input data

$$\begin{aligned} \tilde{u}_0(x) &\in C^1[0, l], \quad \tilde{\mu}(t) \in C^1[0, T], \\ \tilde{\mu}(t) &\geq \tilde{m}_1 > 0, \quad \tilde{u}_0(x) \geq \tilde{m}_2 > 0, \quad \tilde{f}(t, x) \geq 0, \quad 0 \leq t \leq T, \quad 0 \leq x \leq l, \end{aligned} \quad (3.10)$$

As above, for the solution $\tilde{y}$ of the difference scheme (3.7) the following estimate holds:

$$0 < \tilde{c}_1 \leq \tilde{y}_i^{n+1} \leq \|\tilde{y}^{n+1}\|_{C(\bar{\omega}_h)} \leq \max \left\{ \max_{1 \leq k \leq n+1} \tilde{\mu}^k, \|\tilde{u}_0\|_{C(\bar{\omega}_h)} \right\} + \sum_{k=0}^N \tau \|\tilde{\varphi}^k\|_{C(\omega_h)} = \tilde{c}_2, \quad (3.11)$$

where $\tilde{c}_1 = \min\{\tilde{m}_1, \tilde{m}_2\}$.

To investigate the stability and monotonicity of the difference scheme (3.4), we write the problem for perturbation $\delta y = \tilde{y} - y$

$$\begin{aligned} (\delta y)_t + y(\delta \hat{y})_{\bar{x}} + \delta y \hat{y}_{\bar{x}} &= \delta \varphi, \quad x \in \omega_h, \quad t \in \omega_\tau, \\ \delta y(t, 0) &= \delta \mu(t), \quad t \in \omega_\tau, \quad \delta y(0, x) = \delta u_0, \quad x \in \bar{\omega}_h. \end{aligned} \quad (3.12)$$

Problem (3.12) can be rewritten in the following form:

$$\begin{aligned} (1 + \gamma y_i^n) \delta y_i^{n+1} &= \gamma y_i^n \delta y_{i-1}^{n+1} + (1 - \tau \tilde{y}_{\bar{x}, i}^{n+1}) \delta y_i^n + \tau \delta \varphi_i^n, \quad i = 1, 2, \dots, N_x, \quad n = 0, 1, \dots, N-1, \\ \delta y_0^{n+1} &= \delta \mu^{n+1}, \quad n = 0, 1, \dots, N-1, \quad \delta y_i^0 = \delta u_{0i}, \quad i = 0, 1, \dots, N_x. \end{aligned} \quad (3.13)$$

From the structure of (3.13) we see that for investigating the stability and monotonicity of the difference scheme (3.4) we need information about the first difference derivative $\tilde{y}_{\bar{x}, i}^{n+1}$. This is one of distinguishing features of investigating the stability and monotonicity in the nonlinear case.

Since the difference equation (3.4) must be satisfied at the grid point $x_0 = 0$, then, using the boundary conditions, we determine $\tilde{y}(t, -h)$ and $\tilde{u}_0(-h)$ as follows:

$$\tilde{y}(t, -h) = \tilde{\mu}(t) + h \frac{\tilde{\mu}_t(t) - \tilde{\varphi}(t, 0)}{\tilde{\mu}(t)}, \quad \tilde{u}_0(-h) = \tilde{\mu}(0) + h \frac{\tilde{\mu}_t(0) - \tilde{\varphi}(0, 0)}{\tilde{\mu}(0)}.$$

Now by the difference equations (3.7) we obtain the problem for the grid function $w = \tilde{y}_{\bar{x}}$

$$\begin{aligned} w_t + (\tilde{y} \hat{w})_{\bar{x}} &= \tilde{\varphi}_{\bar{x}}, \quad t \in \omega_\tau, \quad x \in \omega_h, \\ \hat{w}(t, 0) &= \hat{\mu}_1(t) = \frac{\hat{\varphi}(t, 0) - \hat{\mu}_t(t)}{\hat{\mu}(t)}, \quad t \in \omega_\tau, \quad w(0, x) = \tilde{u}_{0\bar{x}}(x), \quad x \in \bar{\omega}_h. \end{aligned}$$

Moreover, the boundary and initial conditions are consistent, i.e., $\tilde{u}_0(0) = \tilde{\mu}_1(0)$. The last problem can be rewritten in the following form:

$$(1 + \gamma \tilde{y}_i^n) w_i^{n+1} = \gamma \tilde{y}_{i-1}^n w_{i-1}^{n+1} + w_i^n + \tau \tilde{\varphi}_{\bar{x}, i}^n, \quad i = 1, 2, \dots, N_x, \quad n = 0, 1, \dots, N-1, \quad (3.14)$$

$$w_0^{n+1} = \tilde{\mu}_1^{n+1}, \quad n = 0, 1, \dots, N-1, \quad w_i^0 = \tilde{u}_{0\bar{x}, i}, \quad i = 0, 1, \dots, N_x. \quad (3.15)$$

3.1.1. Investigation of stability in the case where a shock wave is not generated. In this case, the input data of problem (3.1), (3.2) and the corresponding perturbed problem (3.8), (3.9) satisfy, in addition, the conditions

$$u'_0(x), \tilde{u}'_0(x) \geq 0, \quad \frac{\partial f(t, x)}{\partial x}, \frac{\partial \tilde{f}(t, x)}{\partial x} \geq 0, \quad \mu'(t), \tilde{\mu}'(t) \leq 0, \quad (t, x) \in \bar{Q}. \quad (3.16)$$

Under conditions (3.3), (3.10), (3.16) it can be proved by induction on n that all coefficients of problem (3.14), (3.15) are positive and

$$w \geq 0, \quad t \in \bar{\omega}_\tau, \quad x \in \omega_h. \quad (3.17)$$

Now we obtain the estimate of the grid function w . Suppose that w reaches its maximum at the grid point $x_0 = 0$; then $\|w^{n+1}\|_{C(\bar{\omega}_h)} = \mu_1^{n+1}$. If w reaches its maximum at a certain grid point x_{i^*} of the grid ω_h , i.e., $\|w^{n+1}\|_{C(\bar{\omega}_h)} = w_{i^*}^{n+1}$; then from (3.14) we obtain

$$(1 + \tau w_{i^*}^n) w_{i^*}^{n+1} = w_{i^*}^n + \tau \tilde{\varphi}_{\bar{x}, i^*}^n.$$

Taking into account the nonnegativity of w , we get

$$\|w^{n+1}\|_{C(\bar{\omega}_h)} \leq \|w^n\|_{C(\bar{\omega}_h)} + \tau \|\tilde{\varphi}_{\bar{x}}^n\|_{C(\omega_h)}.$$

Hence, for all $x \in \bar{\omega}_h$ the following recurrent inequality holds:

$$\|w^{n+1}\|_{C(\bar{\omega}_h)} \leq \max \{ \mu_1^{n+1}, \|w^n\|_{C(\bar{\omega}_h)} \} + \tau \|\tilde{\varphi}_{\bar{x}}^n\|_{C(\omega_h)},$$

Using Lemma 2.3, from the last inequality we get the following estimate:

$$\|w^{n+1}\|_{C(\bar{\omega}_h)} \leq \max \left\{ \max_{1 \leq k \leq N} \mu_1^k, \|w^0\|_{C(\bar{\omega}_h)} \right\} + \sum_{k=0}^N \tau \|\tilde{\varphi}_{\bar{x}}^k\|_{C(\omega_h)} = \tilde{c}_3. \quad (3.18)$$

Now we estimate the perturbation δy . Since

$$1 - \tau \tilde{y}_{\bar{x}}^{n+1} \geq 1 - \tau \tilde{c}_3 \geq 0, \quad \text{if } \tau \leq \tau_0 = 1/\tilde{c}_3, \quad (3.19)$$

it follows that the coefficients of problem (3.13) are positive for sufficiently small $\tau \leq \tau_0$. Using Lemma 2.2 and the nonnegativity of $\tilde{y}_{\bar{x}}$, for sufficiently small $\tau \leq \tau_0$, we obtain the inequality

$$\begin{aligned} \|\delta y^{n+1}\|_{C(\bar{\omega}_h)} &\leq \max \{ |\delta \mu^{n+1}|, (1 - \tau \tilde{y}_{\bar{x}, i^*}^{n+1}) \|\delta y^n\|_{C(\bar{\omega}_h)} \} + \tau \|\delta \varphi^n\|_{C(\omega_h)} \leq \\ &\max \{ |\delta \mu^{n+1}|, \|\delta y^n\|_{C(\bar{\omega}_h)} \} + \tau \|\delta \varphi^n\|_{C(\omega_h)}. \end{aligned}$$

Here i^* is the number of the grid point where $\|\delta y^{n+1}\|_{C(\omega_h)} = |\delta y_{i^*}^{n+1}|$. Hence, using Lemma 2.3, we get the following estimate:

$$\|\tilde{y}^{n+1} - y^{n+1}\|_{C(\bar{\omega}_h)} \leq \max \left\{ \max_{1 \leq k \leq n+1} |\tilde{\mu}^k - \mu^k|, \|\tilde{u}_0 - u_0\|_{C(\bar{\omega}_h)} \right\} + \sum_{k=0}^n \tau \|\tilde{\varphi}^k - \varphi^k\|_{C(\omega_h)}, \quad (3.20)$$

expressing the stability with respect to the input data. Moreover, the above conditions ensure the unconditional monotonicity of the difference scheme for sufficiently small $\tau \leq \tau_0$.

3.1.2. *Investigation of stability in the case where a shock wave is generated.* Now let the following conditions hold:

$$u'_0(x), \tilde{u}'_0(x) \leq 0, \quad \frac{\partial f(t, x)}{\partial x}, \frac{\partial \tilde{f}(t, x)}{\partial x} \leq 0, \quad \mu'(t), \tilde{\mu}'(t) \geq 0, \quad (t, x) \in \bar{Q}, \quad (3.21)$$

and the strict inequalities are satisfied at least at one point.

Let us obtain the estimate of w . Let the absolute value of the function w_i^{n+1} reach its maximum at a certain grid point $x_{i^*} > 0$ of the grid $\bar{\omega}_h$, i.e., $\|w^{n+1}\|_{C(\bar{\omega}_h)} = |w_{i^*}^{n+1}|$. Then from (3.14) it follows that

$$(1 + \tau w_{i^*}^n) \|w^{n+1}\|_{C(\bar{\omega}_h)} \leq \|w^n\|_{C(\bar{\omega}_h)} + \tau \|\tilde{\varphi}_{\bar{x}}^n\|_{C(\omega_h)}$$

and, taking into account $1 + \tau w_{i^*}^n \geq 1 - \tau \|w^n\|_{C(\bar{\omega}_h)}$, we obtain the following nonlinear relation:

$$(1 - \tau \|w^n\|_{C(\bar{\omega}_h)}) \|w^{n+1}\|_{C(\bar{\omega}_h)} \leq \|w^n\|_{C(\bar{\omega}_h)} + \tau \|\tilde{\varphi}_{\bar{x}}^n\|_{C(\omega_h)}.$$

If the absolute value of the grid function w_i^{n+1} reaches its maximum at the grid point $x_0 = 0$, then $\|w^{n+1}\|_{C(\bar{\omega}_h)} = |\mu_1^{n+1}|$. Thus, we get the nonlinear recurrent inequality

$$\|w^{n+1}\|_{C(\bar{\omega}_h)} \leq \max \{ |\mu_1^{n+1}|, \|w^n\|_{C(\bar{\omega}_h)} \} + \tau \|w^n\|_{C(\bar{\omega}_h)} \|w^{n+1}\|_{C(\bar{\omega}_h)} + \tau \|\tilde{\varphi}_{\bar{x}}^n\|_{C(\omega_h)}.$$

By analogy with the proof of Lemma 2.3, from the last inequality we get the relation

$$\|w^{n+1}\|_{C(\bar{\omega}_h)} \leq \max \left\{ \max_{1 \leq k \leq n} |\mu_1^{k+1}|, \|w^0\|_{C(\bar{\omega}_h)} \right\} + \sum_{k=0}^n \tau \|w^k\|_{C(\bar{\omega}_h)} \|w^{k+1}\|_{C(\bar{\omega}_h)} + \sum_{k=0}^n \tau \|\tilde{\varphi}_{\bar{x}}^k\|_{C(\omega_h)}. \quad (3.22)$$

To obtain the estimate with respect to only the initial and boundary conditions and the right-hand side, we apply Lemma 2.4. Let

$$\mathcal{E}_n = \|w^n\|_{C(\bar{\omega}_h)}, \quad \mathcal{G}_{n+1} = \max \left\{ \max_{1 \leq k \leq n} |\mu_1^{k+1}|, \|w^0\|_{C(\bar{\omega}_h)} \right\} + \sum_{k=0}^n \tau \|\tilde{\varphi}_{\bar{x}}^k\|_{C(\omega_h)},$$

$$g_n = 1, \quad \text{for all } n = 0, 1, 2, \dots, N.$$

Then condition (2.16) yields

$$\left(\max \left\{ \max_{1 \leq k \leq n} |\mu_1^{k+1}|, \|w^0\|_{C(\bar{\omega}_h)} \right\} + \sum_{k=0}^n \tau \|\tilde{\varphi}_{\bar{x}}^k\|_{C(\omega_h)} \right) t_n < 1.$$

If the stronger condition

$$\left(\max \left\{ \max_{1 \leq k \leq n} |\mu_1^{k+1}|, \|w^0\|_{C(\bar{\omega}_h)} \right\} + \sum_{k=0}^n \tau \|\tilde{\varphi}_{\bar{x}}^k\|_{C(\omega_h)} \right) t_n < 1 - \varepsilon, \quad 0 < \varepsilon < 1, \quad (3.23)$$

holds, then for $\|w^{n+1}\|_{\bar{C}}$ the estimate

$$\|w^{n+1}\|_{C(\bar{\omega}_h)} \leq \frac{1}{\varepsilon} \left(\max \left\{ \max_{1 \leq k \leq n^*} |\mu_1^{k+1}|, \|w^0\|_{C(\bar{\omega}_h)} \right\} + \sum_{k=0}^{n^*} \tau \|\tilde{\varphi}_{\bar{x}}^k\|_{C(\omega_h)} \right) = \tilde{c}_4 \quad (3.24)$$

holds on the interval $[0, t_{n^*}]$, where t_{n^*} is determined by inequality (3.23)

Now we obtain the estimate of the perturbation δy of the difference solution. Taking into account inequalities (3.6), (3.11) and (3.24), from (3.13) we get

$$\|\delta y^{n+1}\|_{C(\bar{\omega}_h)} \leq (1 + \tilde{c}_4 \tau) \max\{|\delta \mu^{n+1}|, \|\delta y^n\|_{C(\bar{\omega}_h)}\} + \tau \|\delta \varphi^n\|_{C(\omega_h)}, \quad n = 0, 1, \dots, n^* - 1.$$

Therefore, using the discrete Gronwall inequality [25], we obtain the estimate

$$\begin{aligned} \|\tilde{y}^{n+1} - y^{n+1}\|_{C(\bar{\omega}_h)} &\leq e^{\tilde{c}_4 t_{n+1}} \max \left\{ \max_{1 \leq k \leq n+1} |\tilde{\mu}^k - \mu^k|, \|\tilde{u}_0 - u_0\|_{C(\bar{\omega}_h)} \right\} + \\ &\sum_{k=0}^n \tau e^{\tilde{c}_4(t_n - t_k)} \|\tilde{\varphi}^k - \varphi^k\|_{C(\omega_h)}, \quad n = 0, 1, \dots, n^* - 1, \end{aligned} \quad (3.25)$$

expressing the ρ -stability of scheme (3.4).

Thus we have shown that scheme (3.4) is stable on the interval $[0, t_{n^*}]$, where t_{n^*} is determined by inequality (3.23).

3.2. Conservative, exact and stable schemes on the shock wave. Consider the Riemann problem

$$\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0, \quad 0 < t \leq T, \quad 0 < x \leq l, \quad F'(u) > 0, \quad F''(u) > 0, \quad (3.26)$$

$$u(0, x) = u_0(x) = \begin{cases} u^L & \text{for } 0 < x < \xi, \\ u^R & \text{for } \xi < x \leq l, \end{cases} \quad u^L > u^R \geq 0. \quad (3.27)$$

$$u(t, 0) = u^L, \quad 0 < t \leq T, \quad (3.28)$$

where u^L and u^R are constants.

We are interested in the weak solution. The weak solution is defined in the class of piecewise continuous functions satisfying (for any closed piecewise-smooth contour ∂V) the following integral equation [9]

$$\oint_{\partial V} u dx - F(u) dt = 0 \quad (3.29)$$

and conditions (3.27) and (3.28).

Let the solution $u(t, x)$ be discontinuous along the line $x(t) = \xi + X(t)$. In the case of the considered Riemann problem (3.26)–(3.28), the velocity $D(t)$ of the discontinuity is determined by the Rankine—Hugoniot condition [8]:

$$D = \frac{F(u^R) - F(u^L)}{u^R - u^L} = \frac{dX}{dt}. \quad (3.30)$$

If $D \equiv \text{const}$, then $D = dx/dt = h/\tau$. Thus, for the constant velocity D it follows from (3.30) that

$$\frac{F(u^R) - F(u^L)}{u^R - u^L} = \frac{h}{\tau}.$$

Hence, assuming

$$\frac{\tau D}{h} = 1, \quad (3.31)$$

we obtain the identity

$$\frac{u^R - u^L}{\tau} - \frac{F(u^R) - F(u^L)}{h} = 0. \quad (3.32)$$

For u^R , from (3.32) we have

$$u^R = u^L + \frac{\tau}{h}(F(u^R) - F(u^L)).$$

To approximate problem (3.26)–(3.28), we take the grid $\bar{\omega}_h$, as follows: the point ξ (where the initial function u_0 is discontinuous) is not a grid-point. Let $\xi \in (x_{i_0-1}, x_{i_0})$ (see figure).

On the grid $\bar{\omega}$, consider the difference scheme

$$\frac{u_i^{n+1} - u_i^n}{\tau} + \frac{F(u_i^n) - F(u_{i-1}^n)}{h} = 0, \quad t \in \omega_\tau, \quad x \in \omega_h, \quad (3.33)$$

$$u(0, x) = u_0(x), \quad x \in \bar{\omega}_h, \quad u(t, 0) = u^L, \quad t \in \omega_\tau. \quad (3.34)$$

From equation (3.33) we have

$$u_i^{n+1} = u_i^n - \frac{\tau}{h}(F(u_i^n) - F(u_{i-1}^n)).$$

Thus, since $u_{i_0-1}^n = u^L$, $u_{i_0}^n = u^R$ near the discontinuity, it follows that

$$u_{i_0}^{n+1} = u^R - \frac{\tau}{h}(F(u^R) - F(u^L)).$$

Combining this with (3.32), we get the identity

$$u_{i_0}^{n+1} = u^L = u_{i_0-1}^n,$$

and, consequently, the difference equation (3.33) approximates the Rankine—Hugoniot conditions (3.32) exactly near the discontinuity. It follows that on the shock wave under assumptions (3.31) the difference scheme (3.33) approximates problem (3.26)–(3.28) exactly.

Consider the problem with perturbed input data

$$\frac{\partial \tilde{u}}{\partial t} + \frac{\partial F(\tilde{u})}{\partial x} = 0, \quad 0 < t \leq T, \quad 0 < x \leq l, \quad F'(\tilde{u}) > 0, \quad F''(\tilde{u}) > 0, \quad (3.35)$$

$$\tilde{u}(0, x) = \tilde{u}_0(x) = \begin{cases} \tilde{u}^L & \text{for } 0 < x < \xi, \\ \tilde{u}^R & \text{for } \xi < x \leq l, \end{cases} \quad \tilde{u}^L > \tilde{u}^R \geq 0. \quad (3.36)$$

$$\tilde{u}(t, 0) = \tilde{u}^L, \quad 0 < t \leq T, \quad (3.37)$$

where $\tilde{u}^L$ and $\tilde{u}^R$ are constants. On the grid $\bar{\omega}$ we approximate this problem by the following difference scheme:

$$\frac{\tilde{u}_i^{n+1} - \tilde{u}_i^n}{\tau} + \frac{F(\tilde{u}_i^n) - F(\tilde{u}_{i-1}^n)}{h} = 0, \quad t \in \omega_\tau, \quad x \in \omega_h, \quad (3.38)$$

$$\tilde{u}(0, x) = \tilde{u}_0(x), \quad x \in \bar{\omega}_h, \quad \tilde{u}(t, 0) = \tilde{u}^L, \quad t \in \omega_\tau. \quad (3.39)$$

The difference scheme (3.38), (3.39) approximates problem (3.35), (3.36) exactly on the same grid $\bar{\omega}$ with

$$\tilde{D} = \frac{F(\tilde{u}^R) - F(\tilde{u}^L)}{\tilde{u}^R - \tilde{u}^L} = \frac{h}{\tau} = D. \quad (3.40)$$

Under conditions (3.31), (3.40) we have

$$u_i^{n+1} = u_{i-1}^n, \quad i = 1, 2, \dots, N_x, \quad n = 0, 1, \dots, N-1,$$

$$\tilde{u}_i^{n+1} = \tilde{u}_{i-1}^n, \quad i = 1, 2, \dots, N_x, \quad n = 0, 1, \dots, N-1.$$

Therefore,

$$\delta u_i^{n+1} = \delta u_{i-1}^n, \quad i = 1, 2, \dots, N_x, \quad n = 0, 1, \dots, N-1,$$

Consequently,

$$\|\delta u^{n+1}\|_{C(\bar{\omega}_h)} \leq \|\delta u_0\|_{C(\bar{\omega}_h)}.$$

Thus, we have proved that the simple explicit conservative difference scheme (3.33), (3.34), approximating the Riemann problem (3.26)–(3.28), is stable and exact for $\tau = hD > \tau_K$ and $\tilde{D} = D$. We stress that this value of τ is greater than the value τ_K admissible by the Courant—Friedrichs—Lewy condition.

3.3. Difference schemes for multidimensional problems. In the domain $\bar{Q}^{d+1} = \bar{\Omega}^d \times [0, T]$, $\bar{\Omega}^d = \{\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d : 0 \leq x_k \leq l_k, k = 1, 2, \dots, d\} = \Omega^d \cup \partial\Omega^d$, $\partial\Omega^d = \{\mathbf{x} \in \bar{\Omega}^d : x_k = 0, k = 1, 2, \dots, d\}$, consider the multi-dimensional scalar conservation law

$$\frac{\partial u}{\partial t} + \sum_{k=1}^d \alpha_k \frac{\partial(u^2)}{\partial x_k} = 0, \quad (t, \mathbf{x}) \in Q^{d+1} = \Omega^d \times (0, T], \quad (3.41)$$

$$u(t, \mathbf{x})|_{\partial\Omega^d} = \mu(t, \mathbf{x}), \quad t \in (0, T] \quad u(0, \mathbf{x}) = u_0(\mathbf{x}), \quad \mathbf{x} \in \bar{\Omega}^d. \quad (3.42)$$

Suppose that $\alpha_k = \text{const} > 0$ and the functions defining the input data satisfy the following conditions:

$$0 < m_1 \leq u_0(\mathbf{x}) \leq m_1^*, \quad \mu(t, \mathbf{x}) \geq m_2 > 0, \quad \sum_{k=1}^d \alpha_k \frac{\partial u_0^2}{\partial x_k} \geq m_3 > 0, \quad \frac{\partial \mu}{\partial t} \leq 0, \quad (3.43)$$

$$u_0(\mathbf{x}) = \mu(0, \mathbf{x}), \quad \sum_{k=1}^d \alpha_k \frac{\partial u_0(\mathbf{x})}{\partial x_k} = -\frac{\partial \mu(0, \mathbf{x})}{\partial t} \mu^{-1}(0, \mathbf{x}) \quad \mathbf{x} \in \partial\Omega^d, \quad (3.44)$$

$$\mu(t, \mathbf{x}) \in C^1([0, T] \times \partial\Omega^d), \quad u_0(\mathbf{x}) \in C^1(\bar{\Omega}^d).$$

Under conditions (3.43) the shock wave is not generated because in this case we have

$$0 < m_3 \leq \sum_{k=1}^d \alpha_k \frac{\partial u_0^2}{\partial x_k} = 2u_0 \sum_{k=1}^d \alpha_k \frac{\partial u_0}{\partial x_k} \leq 2m_1^* \sum_{k=1}^d \alpha_k \frac{\partial u_0}{\partial x_k}$$

and, consequently,

$$\sum_{k=1}^d \alpha_k \frac{\partial u_0}{\partial x_k} \geq m_4 = \frac{m_3}{2m_1^*} > 0. \quad (3.45)$$

In $\bar{Q}^{d+1}$, let us introduce the uniform grids

$$\begin{aligned} \bar{\omega}_h^d &= \{\mathbf{x} \in \bar{\Omega}^d : x_k = i_k h_k, \ i_k = 0, 1, \dots, N_{x_k}, \ h_k N_{x_k} = l_k, \ k = 1, 2, \dots, d\} = \omega_h^d \cup \partial\omega_h^d, \\ \omega_h^d &= \bar{\omega}_h^d \cap \Omega^d, \quad \partial\omega_h^d = \bar{\omega}_h^d \cap \partial\Omega^d, \quad \bar{\omega}^{d+1} = \bar{\omega}_h^d \times \bar{\omega}_\tau, \\ \partial\omega^{d+1} &= \partial\omega_h^d \times \omega_\tau \cup \{(0, \mathbf{x}) : \mathbf{x} \in \bar{\omega}_h^d\}, \quad h = \max(h_1, h_2, \dots, h_d). \end{aligned}$$

On the grid $\bar{\omega}^{d+1}$ we approximate problem (3.41), (3.42) by the following explicit conservative difference scheme:

$$y_t + \sum_{k=1}^d \alpha_k (y^2)_{\bar{x}_k} = 0, \quad (t, \mathbf{x}) \in \omega^{d+1} = \omega_h^d \cup \omega_\tau, \quad (3.46)$$

$$y|_{\partial\omega_h^d} = \mu(t, \mathbf{x}), \quad t \in \omega_\tau, \quad y(0, \mathbf{x}) = u_0(\mathbf{x}), \quad \mathbf{x} \in \bar{\omega}_h^d. \quad (3.47)$$

Here and below we use the following notation:

$$\begin{aligned} y &= y(t, \mathbf{x}), \quad \hat{y} = y(t + \tau, \mathbf{x}), \quad \check{y} = y(t - \tau, \mathbf{x}), \\ y_{(\pm 1_k)} &= y(t, \mathbf{x}_{(\pm 1_k)}) = y(t, x_1, \dots, x_k \pm h_k, \dots, x_d), \quad (t, \mathbf{x}) \in \bar{\omega}^{d+1}, \\ y^{(0.5)} &= 0.5(y + \check{y}), \quad y_t = \frac{\hat{y} - y}{\tau}, \quad y_{\bar{t}} = \frac{y - \check{y}}{\tau}, \quad y_{\bar{x}_k} = \frac{y - y_{(-i_k)}}{h_k}. \end{aligned}$$

Moreover, by definition, put

$$\|v\|_{C(\bar{\omega}_h^d)} = \max_{\mathbf{x} \in \bar{\omega}_h^d} |v(\mathbf{x})|, \quad \|v\|_{C(\omega_h^d)} = \max_{\mathbf{x} \in \omega_h^d} |v(\mathbf{x})|, \quad \|v\|_{C(\partial\omega^{d+1})} = \max_{\mathbf{x} \in \partial\omega^{d+1}} |v(\mathbf{x})|.$$

The discrete analogues of condition (3.43) are needed for the sequel. Using expansion in Taylor series, we get

$$\sum_{k=1}^d \alpha_k (u_0^2)_{\bar{x}_k} = \sum_{k=1}^d \alpha_k \frac{\partial u_0^2}{\partial x_k^2} - \sum_{k=1}^d \alpha_k \frac{h_k}{2} \frac{\partial^2 \bar{u}_k^2}{\partial x_k^2} \geq m_3 - hm_5 \geq 0, \quad \text{for } h \leq h^{(1)} = \frac{m_3}{m_5}, \quad (3.48)$$

where

$$m_5 = \frac{1}{2} \max_{\mathbf{x} \in \bar{\Omega}^d} \left| \sum_{k=1}^d \alpha_k \frac{\partial^2 u_0^2}{\partial x_k^2} \right|, \quad \bar{u}_k = u_0(x_1, \dots, \bar{x}_k, \dots, x_d), \quad \bar{x}_k \in (x_k - h_k, x_k + h_k).$$

In the same way, it can be proved that

$$\sum_{k=1}^d \alpha_k u_{0\bar{x}_k} \geq 0 \quad \text{for } h \leq h^{(2)} = \frac{m_4}{m_6}, \quad m_6 = \frac{1}{2} \max_{\mathbf{x} \in \bar{\Omega}^d} \left| \sum_{k=1}^d \alpha_k \frac{\partial^2 u_0}{\partial x_k^2} \right| \quad (3.49)$$

Let us study the properties of the solution of the difference problem (3.46), (3.47). From equation (3.46) we obtain the solution on the upper level $\hat{y}$ as follows

$$\hat{y} = \left(1 - \tau y \sum_{k=1}^d \frac{\alpha_k}{h_k} \right) y + \tau \sum_{k=1}^d \frac{\alpha_k}{h_k} y_{(-1_k)}^2, \quad t \in \omega_\tau, \quad \mathbf{x} \in \omega_h^d. \quad (3.50)$$

Lemma 3.1. *Let on the n th time level $t = t_n$ the following conditions be fulfilled:*

$$0 \leq y(t, \mathbf{x}) \leq c_2, \quad \sum_{k=1}^d \alpha_k (y^2)_{\bar{x}_k}(t, \mathbf{x}) \geq 0, \quad \mathbf{x} \in \bar{\omega}_h, \quad c_2 = \max\{\|u_0\|_{C(\bar{\omega}_h^d)}, \|\mu\|_{C(\partial\omega_h^d)}\}, \quad (3.51)$$

$$1 - 2\tau c_2 \sum_{k=1}^d \frac{\alpha_k}{h_k} \geq 0, \quad h \leq h^{(0)} = \min\{h^{(1)}, h^{(2)}\}. \quad (3.52)$$

Then on the next time level $t + \tau$ the following inequalities hold:

$$0 \leq y(t + \tau, \mathbf{x}) \leq c_2, \quad \sum_{k=1}^d \alpha_k (y^2)_{\bar{x}_k}(t + \tau, \mathbf{x}) \geq 0, \quad \mathbf{x} \in \bar{\omega}_h. \quad (3.53)$$

Proof. From (3.43) we see that under conditions (3.52) inequalities (3.51) hold true on the initial time level $t = 0$. Now assume that inequalities (3.51) are fulfilled at any instant of time t . Then from (3.50) it follows that $y(t + \tau, \mathbf{x}) \geq 0$, $\mathbf{x} \in \bar{\omega}_h$.

To obtain the upper estimate

$$\|y(t + \tau)\|_{C(\bar{\omega}_h^d)} \leq c_2, \quad (3.54)$$

we use the maximum principle technique. The absolute value of the function $\hat{y}$ can reach its maximum either on the boundary $\partial\omega_h^d$, i.e.,

$$\|\hat{y}\|_{C(\bar{\omega}_h^d)} = \|\mu\|_{C(\partial\omega_h^d)}, \quad (3.55)$$

or at a certain grid point $\mathbf{x}^* \in \omega_h^d$, i.e. $\|\hat{y}\|_{C(\bar{\omega}_h^d)} = \hat{y}(\mathbf{x}^*)$. So, from (3.50), we obtain

$$\begin{aligned} \|\hat{y}\|_{C(\bar{\omega}_h^d)} &\leq \left(1 - \tau y(\mathbf{x}^*) \sum_{k=1}^d \frac{\alpha_k}{h_k}\right) \|y\|_{C(\bar{\omega}_h^d)} + \tau \sum_{k=1}^d \frac{\alpha_k}{h_k} y_{(-1_k)}(\mathbf{x}^*) \|y\|_{C(\bar{\omega}_h^d)} \leq \\ &\left(1 - \tau \sum_{k=1}^d \alpha_k y_{\bar{x}_k}(\mathbf{x}^*)\right) \|y\|_{C(\bar{\omega}_h^d)}. \end{aligned} \quad (3.56)$$

By the induction hypothesis,

$$0 \leq \sum_{k=1}^d \alpha_k (y^2)_{\bar{x}_k}(t, \mathbf{x}) = y(t, \mathbf{x}) \sum_{k=1}^d \alpha_k y_{\bar{x}_k}(t, \mathbf{x}) - \sum_{k=1}^d \alpha_k h_k (y_{\bar{x}_k}(t, \mathbf{x}))^2 \leq y(t, \mathbf{x}) \sum_{k=1}^d \alpha_k y_{\bar{x}_k}(t, \mathbf{x}).$$

Therefore, since $\sum_{k=1}^d \alpha_k y_{\bar{x}_k}(t, \mathbf{x}) \geq 0$ it follows from (3.56) that

$$\|y(t + \tau)\|_{C(\bar{\omega}_h^d)} \leq \|y(t)\|_{C(\bar{\omega}_h^d)}. \quad (3.57)$$

Combining (3.55) and (3.57), we get estimate (3.54).

Now we must prove that $\sum_{k=1}^d \alpha_k y_{\bar{x}_k}(t + \tau, \mathbf{x}) \geq 0$, $\mathbf{x} \in \omega_h^d$. Subtracting equation (3.46) at the instant of time t from the same equation at the instant of time $t + \tau$ and denoting $y(t + \tau, \mathbf{x})$ by $v = v(t, \mathbf{x})$, we obtain

$$v + 2 \sum_{k=1}^d \alpha_k \tau (y^{(0.5)} \tilde{v})_{\bar{x}_k} = \tilde{v}, \quad (t, \mathbf{x}) \in \omega^d, \quad (3.58)$$

or

$$v = \left(1 - 2\tau y^{(0.5)} \sum_{k=1}^d \frac{\alpha_k}{h_k}\right) \check{v} + 2\tau \sum_{k=1}^d \frac{\alpha_k}{h_k} y_{(-1k)}^{(0.5)} \check{v}_{(-1k)}. \quad (3.59)$$

For the boundary conditions we have

$$v|_{\partial\omega_h^d} = \mu_t, \quad t \in \omega_\tau. \quad (3.60)$$

The initial conditions for the function v can be obtained from equation (3.46) at the time $t = 0$ and from the initial condition (3.47):

$$v(0, \mathbf{x}) = v_0(\mathbf{x}) = - \sum_{k=1}^d \alpha_k (u_0^2)_{\bar{x}_k}, \quad \mathbf{x} \in \bar{\omega}_h^d. \quad (3.61)$$

By the inductive assumption,

$$\check{v} = - \sum_{k=1}^d \alpha_k (y^2)_{\bar{x}_k} \leq 0.$$

Therefore, using the already proved estimate (3.54) and conditions (3.52), we have

$$1 - 2\tau y^{(0.5)} \sum_{k=1}^d \frac{\alpha_k}{h_k} \geq 1 - 2\tau c_2 \sum_{k=1}^d \frac{\alpha_k}{h_k} \geq 0.$$

Moreover, $y(t, \mathbf{x}), y(t + \tau, \mathbf{x}) \geq 0$ for all $\mathbf{x} \in \bar{\omega}_h$. Thus, from equation (3.59) it follows that $y_t(t + \tau, \mathbf{x}) \leq 0$ for all $\mathbf{x} \in \bar{\omega}_h$. Using equation (3.46) at $t + \tau$ by direct calculation we have $\sum_{k=1}^d \alpha_k (y^2)_{\bar{x}_k}(t + \tau, \mathbf{x}) \leq 0$ for all $\mathbf{x} \in \bar{\omega}_h$. $\square$

To investigate the stability, we consider the differential problem

$$\frac{\partial \tilde{u}}{\partial t} + \sum_{k=1}^d \alpha_k \frac{\partial(\tilde{u}^2)}{\partial x_k} = 0, \quad (t, \mathbf{x}) \in Q^{d+1} = \Omega^d \times (0, T], \quad (3.62)$$

$$\tilde{u}(t, \mathbf{x})|_{\partial\Omega^d} = \tilde{\mu}(t, \mathbf{x}), \quad \tilde{u}(0, \mathbf{x}) = \tilde{u}_0(\mathbf{x}), \quad t > 0, \quad \mathbf{x} \in \bar{\Omega}^d, \quad (3.63)$$

with perturbed input data

$$0 < \tilde{m}_1 \leq \tilde{u}_0(\mathbf{x}) \leq \tilde{m}_1^*, \quad \tilde{\mu}(t, \mathbf{x}) \geq \tilde{m}_2 > 0, \quad \sum_{k=1}^d \alpha_k \frac{\partial \tilde{u}_0^2}{\partial x_k} \geq \tilde{m}_3 > 0, \quad \frac{\partial \tilde{\mu}}{\partial t} \leq 0. \quad (3.64)$$

As above, we also have

$$\sum_{k=1}^d \alpha_k \frac{\partial \tilde{u}_0}{\partial x_k} \geq \tilde{m}_4 = \frac{\tilde{m}_3}{2\tilde{m}_1^*} > 0. \quad (3.65)$$

In addition, suppose that

$$\begin{aligned} \tilde{u}_0(\mathbf{x}) &= \tilde{\mu}(0, \mathbf{x}), \quad \sum_{k=1}^2 \alpha_k \frac{\partial \tilde{u}_0(\mathbf{x})}{\partial x_k} = - \frac{\partial \tilde{\mu}(0, \mathbf{x})}{\partial t} \tilde{\mu}^{-1}(0, \mathbf{x}) \quad \mathbf{x} \in \bar{\Omega}^d, \\ \tilde{\mu}(t, \mathbf{x}) &\in C^1([0, T] \times \partial\Omega^d), \quad \tilde{u}_0(\mathbf{x}) \in C^1(\bar{\Omega}^d). \end{aligned} \quad (3.66)$$

We approximate problem (3.62), (3.63) by the explicit difference scheme

$$\tilde{y}_t + \sum_{k=1}^d \alpha_k (\tilde{y}^2)_{\bar{x}_k} = 0, \quad (t, \mathbf{x}) \in \omega^{d+1} = \omega_h^d \cup \omega_\tau, \quad (3.67)$$

$$\tilde{y}|_{\partial\omega_h^d} = \tilde{\mu}(t, \mathbf{x}), \quad \tilde{y}(0, \mathbf{x}) = \tilde{u}_0(\mathbf{x}), \quad t \in \omega_\tau, \quad \mathbf{x} \in \bar{\omega}_h^d. \quad (3.68)$$

As for the unperturbed problem, it can be shown that

$$\sum_{k=1}^d \alpha_k (\tilde{u}_0^2)_{\bar{x}_k} \geq 0 \quad \text{for } h \leq \tilde{h}^{(1)} = \frac{\tilde{m}_3}{\tilde{m}_5}, \quad \tilde{m}_5 = \frac{1}{2} \max_{\mathbf{x} \in \bar{\Omega}^d} \left| \sum_{k=1}^d \alpha_k \frac{\partial^2 \tilde{u}_0^2}{\partial x_k^2} \right|, \quad (3.69)$$

and

$$\sum_{k=1}^d \alpha_k \tilde{u}_{0\bar{x}_k} \geq 0 \quad \text{for } h \leq \tilde{h}^{(2)} = \frac{\tilde{m}_4}{\tilde{m}_6}, \quad \tilde{m}_6 = \frac{1}{2} \max_{\mathbf{x} \in \bar{\Omega}^d} \left| \sum_{k=1}^d \alpha_k \frac{\partial^2 \tilde{u}_0}{\partial x_k^2} \right| \quad (3.70)$$

Similarly to Lemma 3.1 the following statement can be proved.

Lemma 3.2. *Let on the n th time-level $t = t_n$ the following conditions be fulfilled:*

$$0 \leq \tilde{y}(t, \mathbf{x}) \leq \tilde{c}_2, \quad \sum_{k=1}^d \alpha_k (\tilde{y}^2)_{\bar{x}_k}(t, \mathbf{x}) \geq 0, \quad \mathbf{x} \in \bar{\omega}_h, \quad \tilde{c}_2 = \max\{\|\tilde{u}_0\|_{C(\bar{\omega}_h^d)}, \|\tilde{\mu}\|_{C(\partial\omega_h^d)}\}, \quad (3.71)$$

$$1 - 2\tau\tilde{c}_2 \sum_{k=1}^d \frac{\alpha_k}{h_k} \geq 0, \quad h \leq \tilde{h}^{(0)} = \min\{\tilde{h}^{(1)}, \tilde{h}^{(2)}\}. \quad (3.72)$$

Then on the next time-level $t + \tau$ the following inequalities hold:

$$0 \leq \tilde{y}(t + \tau, \mathbf{x}) \leq \tilde{c}_2, \quad \sum_{k=1}^d \alpha_k (\tilde{y}^2)_{\bar{x}_k}(t + \tau, \mathbf{x}) \geq 0, \quad \mathbf{x} \in \bar{\omega}_h. \quad (3.73)$$

To investigate the stability of the considered difference scheme, we obtain the problem for the perturbation $\delta y = \tilde{y} - y$:

$$(\delta y)_t + \sum_{k=1}^d \alpha_k ((\tilde{y} + y)\delta y)_{\bar{x}_k} = 0, \quad (t, \mathbf{x}) \in \omega^{d+1}, \quad (3.74)$$

$$\delta y|_{\partial\omega_h^d} = 0, \quad t \in \omega_\tau, \quad \delta y(0, \mathbf{x}) = \delta u_0(\mathbf{x}), \quad \mathbf{x} \in \bar{\omega}_h^d. \quad (3.75)$$

Expressing $\delta \hat{y} = \hat{\tilde{y}} - \hat{y}$ from equation (3.74), we get

$$\delta \hat{y} = \left(1 - \tau(\tilde{y} + y) \sum_{k=1}^d \frac{\alpha_k}{h_k} \right) \delta y + \tau \sum_{k=1}^d \frac{\alpha_k}{h_k} (\tilde{y}_{(-1_k)} + y_{(-1_k)}) \delta y_{(-1_k)}. \quad (3.76)$$

Now we prove the following

Theorem 3.1. *Let the conditions of Lemmas 3.1, 3.2 hold and*

$$\tau \leq \frac{h}{2c \sum_{k=1}^d \alpha_k}, \quad c = \max\{c_2, \tilde{c}_2\}, \quad h \leq h_0 = \min\{h^{(0)}, \tilde{h}^{(0)}\}. \quad (3.77)$$

Then the difference scheme (3.46), (3.47) is stable with respect to small perturbations of the initial and boundary conditions and for any $t \in \omega_\tau$ the following estimate holds:

$$\|\tilde{y}(t) - y(t)\|_{C(\bar{\omega}_h)} \leq \max \left\{ \max_{t \in \omega_\tau} \|\tilde{\mu}(t) - \mu(t)\|_{C(\partial\omega_h^d)}, \|\tilde{u}_0 - u_0\|_{C(\bar{\omega}_h^d)} \right\}. \quad (3.78)$$

Proof. Since all coefficients on the right-hand side of equation (3.76) are positive, it follows that the difference scheme is monotone. Moreover, using the maximum principle technique, it can be shown that

$$\|\delta y(t)\|_{C(\bar{\omega}_h^d)} \leq \max \left\{ \|\mu(t)\|_{C(\partial\omega_h^d)} \left(1 - \tau \sum_{k=1}^d \alpha_k (\tilde{y}(t, \mathbf{x}^*) + y(t, \mathbf{x}^*))_{\bar{x}_k} \right) \|\delta y(t-\tau)\|_{C(\bar{\omega}_h^d)} \right\}. \quad (3.79)$$

Taking into account conditions (3.77) and Lemmas 3.1, 3.2, we have

$$1 - \tau \sum_{k=1}^d \alpha_k (\tilde{y}(t, \mathbf{x}^*) + y(t, \mathbf{x}^*))_{\bar{x}_k} \geq 0, \quad \sum_{k=1}^d \alpha_k y_{\bar{x}_k}(t - \tau, \mathbf{x}^*), \quad \sum_{k=1}^d \alpha_k \tilde{y}_{\bar{x}_k}(t - \tau, \mathbf{x}^*) \leq 0.$$

Consequently, from (3.79) we obtain the required estimate (3.78) expressing the stability of the difference scheme (3.46), (3.47) with respect to small perturbations of the initial and boundary conditions. $\square$

4. Difference schemes for parabolic equations

In this section, we study the stability and monotonicity of the explicit difference scheme approximating the initial-boundary value problem for the porous media equation given in the domain $\bar{Q}^{d+1} = \bar{\Omega}^d \times [0, T]$, $\bar{\Omega}^d = \{\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d : 0 \leq x_k \leq l_k, k = 1, 2, \dots, d\} = \Omega^d \cup \partial\Omega^d$, $\Omega^d = \{\mathbf{x} \in \mathbb{R}^d : 0 < x_k < l_k, k = 1, 2, \dots, d\}$:

$$\frac{\partial u}{\partial t} = \Delta u^m + f, \quad 0 < t \leq T, \quad \mathbf{x} \in \Omega^d, \quad m > 1, \quad (4.1)$$

$$u(t, \mathbf{x})|_{\partial\Omega^d} = \mu(t, \mathbf{x}), \quad 0 < t \leq T, \quad u(0, \mathbf{x}) = u_0(\mathbf{x}), \quad \mathbf{x} \in \bar{\Omega}^d. \quad (4.2)$$

Here Δ is the Laplacian $\Delta u = \sum_{k=1}^d \frac{\partial^2 u}{\partial x_k^2}$.

Suppose that the following conditions are satisfied:

- 1°) $0 < \kappa_1 \leq u_0(\mathbf{x}) \leq \kappa_2$, $\mathbf{x} \in \bar{\Omega}^d$, $u_0(\mathbf{x}) \in C^4(\bar{\Omega}^d)$, $\mu(t) \in C^2(0, T]$;
- 2°) $\partial\mu(t, \mathbf{x})/\partial t \leq 0$, $f(t, \mathbf{x}) \geq 0$, $\partial f(t, \mathbf{x})/\partial t \leq 0$, $0 < t \leq T$, $\mathbf{x} \in \bar{\Omega}^d$;
- 3°) $\Delta u_0^m + f(0, \mathbf{x}) \leq -\varepsilon$, $\varepsilon = \text{const} > 0$.

From 1°)–3°), in particular, it follows that

$$\Delta u_0(\mathbf{x}) \leq -\varepsilon_1, \quad \varepsilon_1 = \frac{\varepsilon}{m\kappa_1^{m-1}} > 0. \quad (4.3)$$

Indeed, by direct calculation we get

$$-\varepsilon \geq \Delta u_0^m + f(0, \mathbf{x}) \geq m u_0^{m-1} \Delta u_0 + f(0, \mathbf{x}) \geq m u_0^{m-1} \Delta u_0 \geq m \kappa_1^{m-1} \Delta u_0.$$

4.1. Difference scheme. Without loss of generality it can be assumed that $m = 2$. On the grid

$$\bar{\omega}^{d+1} = \bar{\omega}_h^d \times \bar{\omega}_\tau,$$

$$\begin{aligned}\bar{\omega}_h^d &= \{\mathbf{x} \in \bar{\Omega}^d : x_k = i_k h_k, i_k = 0, 1, \dots, N_k, N_k h_k = l_k, k = 1, 2, \dots, d\} = \omega_h^d \cup \partial\omega_h^d, \\ \omega_h^d &= \bar{\omega}_h^d \cap \Omega^d, \quad \partial\omega_h^d = \bar{\omega}_h^d \cap \partial\Omega^d,\end{aligned}$$

we approximate problem (4.1)–(4.2) by the following difference scheme:

$$y_t = \Delta_h y^2 + f, \quad (t, \mathbf{x}) \in \omega^{d+1}, \quad \Delta_h y = \sum_{k=1}^d y_{\bar{x}_k x_k}, \quad (4.4)$$

$$\hat{y}|_{\partial\omega_h^d} = \hat{\mu}, \quad t \in \omega_\tau; \quad y(0, \mathbf{x}) = u_0(\mathbf{x}), \quad \mathbf{x} \in \bar{\omega}_h^d. \quad (4.5)$$

Equation (4.4) can be rewritten in the form

$$\hat{y} = \left(1 - 2\tau \sum_{k=1}^d \frac{1}{h_k^2}\right) y + \sum_{k=1}^d \frac{\tau(y_{(-1_k)}^2 + y_{(+1_k)}^2)}{h_k^2} + \tau f. \quad (4.6)$$

Here

$$y_{\bar{x}_k x_k} = \frac{y_{x_k} - y_{\bar{x}_k}}{h_k}, \quad y_{x_k} = \frac{y_{(+1_k)} - y}{h_k}, \quad y_{\bar{x}_k} = \frac{y - y_{(-1_k)}}{h_k}.$$

Let us prove some properties of the grid operators.

Lemma 4.1. *For any grid function given on the grid $\bar{\omega}_h$ the following inequality holds:*

$$(y^2)_{\bar{x}_k x_k} \geq 2yy_{\bar{x}_k x_k}. \quad (4.7)$$

Proof. Using the formula of the difference differentiation, we get

$$(y^2)_{\bar{x}_k x_k} = ((y_{(+1_k)} + y)y_{\bar{x}_k})_{x_k} = (y_{(+1_k)} + y)y_{\bar{x}_k x_k} + (y_{\bar{x}_k} + y_{x_k})y_{\bar{x}_k},$$

$$(y^2)_{x_k \bar{x}_k} = ((y_{(+1_k)} + y)y_{x_k})_{\bar{x}_k} = (y_{(-1_k)} + y)y_{\bar{x}_k x_k} + (y_{\bar{x}_k} + y_{x_k})y_{x_k},$$

Summing the last identities, we obtain

$$(y^2)_{\bar{x}_k x_k} = \frac{y_{(+1_k)} + 2y + y_{(-1_k)}}{2} y_{\bar{x}_k x_k} + 2y_{\bar{x}_k}^2 \geq 2yy_{\bar{x}_k x_k} + 0.5h_k^2 y_{\bar{x}_k x_k}^2 \geq 2yy_{\bar{x}_k x_k}.$$

This completes the proof. $\square$

In particular, from (4.7) it follows that

$$y_t = \Delta_h y^2 + f \geq 2y\Delta_h y. \quad (4.8)$$

The discrete analogue of property 3°) is needed for the sequel. Using the Taylor series expansion of the grid function, we obtain

$$\Delta_h u_0^2 + f(0, \mathbf{x}) = \Delta u_0^2 + f(0, \mathbf{x}) + \frac{1}{12} \sum_{k=1}^d h_k^2 \frac{\partial^4 \bar{u}_k^2}{\partial x_k^4} \leq -\varepsilon + \frac{h^2 d \kappa_3}{12} \leq 0 \quad (4.9)$$

for sufficiently small $h \leq h_0^{(1)} = \sqrt{12\varepsilon/d\kappa_3}$, where

$$\kappa_3 = \max_{1 \leq k \leq d} \left\| \frac{\partial^4 u_0^2}{\partial x_k^4} \right\|_{C(\bar{\Omega}^d)}, \quad \bar{u}_k = u_0(x_1, x_2, \dots, \bar{x}_k, \dots, x_d), \quad \bar{x}_k \in (x_k - h_k, x_k + h_k).$$

Similarly, it can be proved that the discrete analogue of inequality (4.3)

$$\Delta_h u_0 \leq \varepsilon_1 < 0 \quad (4.10)$$

is satisfied for

$$h \leq h_0^{(2)} = \sqrt{12\varepsilon_1/d\kappa_4}, \quad \kappa_4 = \max_{1 \leq k \leq d} \left\| \frac{\partial^4 u_0}{\partial x_k^4} \right\|_{C(\bar{\Omega}^d)}. \quad (4.11)$$

For the solution of problem (4.4)–(4.5) the following theorem holds.

Theorem 4.1. *Suppose conditions 1°)–3°) are satisfied and*

$$h \leq h^{(0)} = \sqrt{12M/d\kappa}, \quad M = \max\{\varepsilon, \varepsilon_1\}, \quad \kappa = \max\{\kappa_3, \kappa_4\}, \quad (4.12)$$

$$0 \leq y \leq c_2, \quad \Delta_h y \leq 0, \quad 1 - 4\tau c_2 \sum_{k=1}^d \frac{1}{h_k^2} \geq 0, \quad (4.13)$$

$$c_2 = \max \left\{ \max_{1 \leq k \leq N} \|\mu^k\|_{C(\partial\omega_h^d)}, \|u_0\|_{C(\omega_h^d)} \right\} + \sum_{k=1}^N \tau \|f^k\|_{C(\omega_h^d)}.$$

Then

$$0 \leq \hat{y} \leq c_2, \quad \Delta_h \hat{y} \leq 0. \quad (4.14)$$

Proof. The nonnegativity of the grid function $\hat{y}$ follows from the structure of the difference scheme (4.6), (4.5) and from the assumptions. To prove the boundedness of the solution $\hat{y} \leq c_1$, we use the maximum principle technique. The grid function $\hat{y}$ can reach its maximum either on the boundary

$$\|\hat{y}\|_{C(\bar{\omega}_h^d)} = \|\hat{\mu}\|_{C(\partial\omega_h^d)}, \quad (4.15)$$

or at a certain interior point $\mathbf{x}^* \in \omega_h^d$

$$\hat{y}^* = \max_{\mathbf{x} \in \omega_h^d} \hat{y}(t, \mathbf{x}) = y(t, \mathbf{x}^*).$$

But then from the difference equation (4.6) it follows that

$$\|\hat{y}\|_{C(\bar{\omega}_h^d)} \leq (1 + \tau \Delta_h y^*) \|y\|_{C(\omega_h^d)} + \tau \|f\|_{C(\omega_h^d)}. \quad (4.16)$$

Since $\Delta_h y \leq 0$, combining (4.15) and (4.16), we get

$$\|\hat{y}\|_{C(\bar{\omega}_h^d)} \leq \max \left\{ \|\hat{\mu}\|_{C(\partial\omega_h^d)}, \|y\|_{C(\omega_h^d)} \right\} + \tau \|f\|_{C(\omega_h^d)}. \quad (4.17)$$

The application of Lemma 2.3 yields $\hat{y} \leq c_1$. To prove the second condition of (4.14), we formulate the problem for the difference derivative $v = y_t$ in t . Using simple identity $\varphi = \tau\varphi_{\bar{t}} + \check{\varphi}$ and the representation of the difference scheme (4.4) in the form

$$v = \Delta_h y^2 + f,$$

from (4.4), (4.5) we obtain

$$v = \check{v} + \tau \Delta_h ((y + \hat{y})v) + \tau f_{\bar{t}}, \quad (4.18)$$

$$v(0, \mathbf{x}) = \Delta_h u_0^2 + f(0, \mathbf{x}) \leq 0, \quad v|_{\partial\omega_h^d} = \mu_t \leq 0. \quad (4.19)$$

Let us write the difference equation (4.18) in a more convenient form:

$$v = \left(1 - 2\tau(y + \tilde{y}) \sum_{k=1}^d \frac{1}{h_k}\right) \tilde{v} + \tau \sum_{k=1}^d \frac{y_{(-1_k)} + \tilde{y}_{(-1_k)}}{h_k^2} \tilde{v}_{(-1_k)} + \tau \sum_{k=1}^d \frac{y_{(+1_k)} + \tilde{y}_{(+1_k)}}{h_k^2} \tilde{v}_{(+1_k)} + \tau f_{\tilde{t}}. \quad (4.20)$$

Since the Courant — Friedrichs — Lewy condition (4.13) is satisfied, it follows that all coefficients of the difference equation (4.20) are positive; $v^0 \leq 0$ by (4.14). Consequently, by induction it follows from (4.20) that $v^k \leq v^n$ for all $k \leq n$. Since we have proved that $0 \leq \hat{y} \leq c_1$, it follows from (4.20) that $v^{n+1} \leq 0$ for $k = n + 1$. Hence, by (4.8) it follows that $\Delta_h \hat{y} \leq 0$. $\square$

The properties of the difference scheme (4.4), (4.5) are determined by the properties of the differential problem and by conditions 1°)–3°). So, we believe that to formulate the problem for the perturbation $\delta y = \tilde{y} - y$ and to study the stability of the difference scheme, it is more natural to perturb the input data of the differential problem and, subsequently, to approximate the perturbed differential problem. As a result, we construct the corresponding differential and difference problems

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial t} &= \Delta \tilde{u}^2 + \tilde{f}, \quad 0 < t \leq T, \quad \mathbf{x} \in \Omega^d, \\ \tilde{u}(t, \mathbf{x})|_{\partial \Omega^d} &= \tilde{\mu}(t, \mathbf{x}), \quad 0 < t \leq T; \quad \tilde{u}(0, \mathbf{x}) = \tilde{u}_0(\mathbf{x}), \quad \mathbf{x} \in \bar{\Omega}^d, \\ \tilde{y}_t &= \Delta_h \tilde{y}^2 + \tilde{f}, \quad (t, \mathbf{x}) \in \omega^{d+1}, \\ \tilde{y}|_{\partial \omega_h^d} &= \tilde{\mu}, \quad t \in \omega_\tau; \quad \tilde{y}(0, \mathbf{x}) = \tilde{u}_0(\mathbf{x}), \quad \mathbf{x} \in \bar{\omega}_h^d. \end{aligned}$$

Suppose that the solution $\tilde{u}(t, \mathbf{x})$ and the input data satisfy conditions similar to 1°)–3°). For the solution $\tilde{y}$ a theorem similar to Theorem 4.1 holds.

The problem for the perturbation δy is of the form

$$\delta \hat{y} = \left(1 - 2\tau(y + \tilde{y}) \sum_{k=1}^d \frac{1}{h_k}\right) \delta y + \tau \sum_{k=1}^d \frac{(y_{(-1_k)} + \tilde{y}_{(-1_k)}) \delta y_{(-1_k)} + (y_{(+1_k)} + \tilde{y}_{(+1_k)}) \delta y_{(+1_k)}}{h_k^2} + \tau \delta f, \quad (4.21)$$

$$\delta y^0 = \delta u_0 = \tilde{u}_0 - u_0, \quad \delta y|_{\partial \omega_h^d} = \delta \mu = \tilde{\mu} - \mu. \quad (4.22)$$

Let us note that under the conditions of Theorem 4.1 the coefficients of the right-hand side of (4.21) are positive and for $\delta \hat{y}$ the following estimate holds:

$$\|\delta \hat{y}\|_{C(\bar{\omega}_h^d)} \leq \max \left\{ \max_{1 \leq k \leq n} |\delta \mu^{k+1}|, \|\delta u_0\|_{C(\omega_h^d)} \right\} + \sum_{k=1}^n \tau \|\delta f^k\|_{C(\omega_h^d)}. \quad (4.23)$$

The last inequality expresses the stability of the difference scheme with respect to a small perturbation of the problem input data. Note that estimate (4.23) is obtained for sufficiently small $h \leq h_0 = \min\{h^{(0)}, \tilde{h}^{(0)}\}$ and under the Courant — Friedrichs — Lewy condition

$$\begin{aligned} 1 - 2\tau(c_2 + \tilde{c}_2) \sum_{k=1}^d \frac{1}{h_k^2} &\geq 0, \\ \tilde{c}_2 &= \max \left\{ \max_{1 \leq k \leq N} \|\tilde{\mu}^k\|_{C(\partial \omega_h^d)}, \|\tilde{u}_0\|_{C(\omega_h^d)} \right\} + \sum_{k=1}^N \tau \|\tilde{f}^k\|_{C(\omega_h^d)}. \end{aligned} \quad (4.24)$$

Under the conditions of Theorem 4.1 all coefficients of equation (4.21) are nonnegative. Hence, the considered difference scheme is monotone.

Let us study the convergence of the solution of the difference scheme (4.4), (4.5) to the solution of the differential problem (4.1), (4.2). Consider the following difference scheme with respect to the exact solution $u = u(t, \mathbf{x})$:

$$u_t = \Delta_h u^2 + f + \psi, \quad (t, \mathbf{x}) \in \omega^d, \quad (4.25)$$

$$\hat{u}|_{\partial\omega_h^d} = \hat{\mu}, \quad t \in \omega_\tau, \quad u(0, \mathbf{x}) = u_0(\mathbf{x}), \quad \mathbf{x} \in \bar{\omega}_h^d. \quad (4.26)$$

Here, for sufficiently smooth solution u , $\|\psi^n\|_{C(\omega_h^d)} \leq M(h^2 + \tau)$ ($M = \text{const} > 0$) is the truncation error of the difference scheme. The problem for the error $e = y - u$ is of the form

$$\begin{aligned} \hat{e} &= \left(1 - 2\tau(y + u) \sum_{k=1}^d \frac{1}{h_k^2}\right) e + \\ &\tau \sum_{k=1}^d \frac{(y_{(-1_k)} + u_{(-1_k)})e_{(-1_k)} + (y_{(+1_k)} + u_{(+1_k)})e_{(+1_k)}}{h_k^2} + \tau\psi, \quad (t, \mathbf{x}) \in \omega^{d+1}, \end{aligned} \quad (4.27)$$

$$e|_{\partial\omega_h^d} = 0, \quad t \in \omega_\tau, \quad e(0, \mathbf{x}) = 0, \quad \mathbf{x} \in \bar{\omega}_h^d. \quad (4.28)$$

The application of the *a priori* estimate (4.23) to the solution of problem (4.27), (4.28) yields

$$\|e\|_{C(\bar{\omega}_h^d)} \leq Mt(h^2 + \tau), \quad t \in \bar{\omega}_\tau. \quad (4.29)$$

From (4.29) it follows that the difference scheme has the order of convergence $\mathcal{O}(h^2 + \tau)$ provided that there exists a sufficiently smooth solution of the differential problem.

Remark 4.1. Also, similar results can be obtained for explicit difference schemes approximating initial-boundary value problems for the more general equation

$$\frac{\partial u}{\partial t} = \Delta F(u), \quad F'(u) \geq \kappa_0 > 0,$$

$$F''(u) \geq 0, \quad \text{for } u \in \mathcal{D}_u = \{u(t, \mathbf{x}) : 0 < u_1 \leq u(t, \mathbf{x}) \leq u_2, (t, \mathbf{x}) \in \bar{Q}^{d+1}\}.$$

In this case, the following identity is useful:

$$\begin{aligned} (\varphi(y))_{\bar{x}x} &= \varphi_{\bar{y}y} (y_{\bar{x}})^2 + \frac{1}{2} (\varphi_y + \varphi_{\bar{y}}) y_{\bar{x}x}, \\ \varphi_{\bar{y}y} &= \frac{1}{0.5(y_{i+1} - y_{i-1})} \left(\frac{\varphi(y_{i+1}) - \varphi(y_i)}{y_{i+1} - y_i} - \frac{\varphi(y_i) - \varphi(y_{i-1})}{y_i - y_{i-1}} \right). \end{aligned}$$

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