Classical solution of problem of control boundary conditions in case of the first mixed problem for one-dimensional wave equation

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Abstract. *Boundary conditions ensuring the value of the solution and its derivative with respect to the time variable at the preassigned time are written in the form of analytical expressions in terms of given functions included in the initial and boundary conditions at the other time intervals.*

1 Introduction

A number of papers by Il'in and Moiseev [1–20] are devoted to problems of control solutions of mixed problems for one-dimensional equation of string vibration by boundary conditions. Because these problems for sufficiently large time variable have nonunique solution, so optimization of functions included in the boundary conditions is considered in addition. Weak solutions satisfying corresponding integral relations are considered here.

Basing on some results of the paper [22], functions included in the boundary conditions that ensure the values of required solution and its derivative with respect to the time variable at the given time are written in the paper in the form of analytical relations in terms of already defined functions. The solution of the problem is classical and under some restrictions on given functions smoothness is continuous and has all continuous derivatives included in the equation of the problem.

2 Statement of problem

With respect to the required function $u\colon \mathbb{R}^2 \supset Q \ni (t,x) \to u(t,x) \in \mathbb{R}$ of independent variables $(t,x) \in \mathbb{R}^2$ one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(t,x),\ \ (t,x) \in Q, \tag{1}$$

is considered, where $\mathbb{R}^2$ and $\mathbb{R}$ are two-dimensional and one-dimensional Euclidean spaces, $Q = (0, \infty) \times (0, l)$, $(0, l)$ is interval, $(0, \infty) = \{t \mid 0 < t < \infty\}$. On the lower base of Q initial conditions

$$u|_{t=0} = \varphi(x), \ \left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi(x), \ x \in (0, l), \tag{2}$$

are given.

On the lateral rays of the boundary ∂Q the Dirichlet boundary conditions

$$u|_{x=0} = \mu^{(1)}(t), \ u|_{x=l} = \mu^{(2)}(t), \ t \in (0, \infty), \tag{3}$$

are given.

Furthermore, we require desired function and its derivative satisfy conditions

$$u|_{t=\frac{kl+\tau}{a}} = \tilde{\varphi}(x), \ \left.\frac{\partial u}{\partial t}\right|_{t=\frac{kl+\tau}{a}} = \tilde{\psi}(x), \ x \in (0, l), \tag{4}$$

at time $t = kl/a + \tau/a$ in terms of given on the interval $(0, l)$ functions $\tilde{\varphi}$ and $\tilde{\psi}$, where $k \in \{0, 1, 2, 3, \ldots\}$, τ is a number from the segment $[0, l] \subset \mathbb{R}$.

Problem (1)–(4) is overdetermined. In order that it has solution, conditions on given functions are required. These conditions we impose on the functions $\mu^{(j)}$, $j = 1, 2$. Such problem we call problem of control with the help of boundary conditions $\mu^{(j)}$ ($j = 1, 2$), in order the required solution u and its derivative $\partial u/\partial t$ get the given value at $t = (kl + \tau)/a$. The word "control" is connected with the physical meaning of such problems.

For finding solution u of problem (1)–(4) with control we use the classical solution of problem (1)–(3) [22]. In order to find solution of stated problem (1)–(4) and its controlling functions, we find classical solution of problem (1), (2), (4).

3 Problem (1), (2), (4)

We divide the rectangle $\left(0, \frac{kl+\tau}{a}\right) \times (0, l)$ into rectangles $\widetilde{Q}^{(m)}$, starting from the top, $\widetilde{Q}^{(m)} = \left(\frac{(m-1)l+\tau}{a}, \frac{ml+\tau}{a}\right) \times (0, l)$, $m = k, k-1, \ldots, 2, 1$, and $\widetilde{Q}^{(0)} = \left(0, \frac{\tau}{a}\right) \times (0, l)$. In $\widetilde{Q}^{(m)}$ we consider mixed problem

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0, \ (t, x) \in \widetilde{Q}^{(m)}, \tag{5}$$

$$u|_{t=\frac{ml+\tau}{a}} = \tilde{\varphi}^{(m)}(x), \ \left.\frac{\partial u}{\partial t}\right|_{t=\frac{ml+\tau}{a}} = \tilde{\psi}^{(m)}(x), \ x \in (0, l), \tag{6}$$

$$u|_{x=0} = \mu^{(1)}(t), \ u|_{x=l} = \mu^{(2)}(t), \ t \in \left(\frac{(m-1)l+\tau}{a}, \frac{ml+\tau}{a}\right). \tag{7}$$

Here $\tilde{\varphi}^{(k)}(x) = \tilde{\varphi}(x)$, $\tilde{\psi}^{(k)}(x) = \tilde{\psi}(x)$. For the other m $\tilde{\varphi}^{(m)}$ and $\tilde{\psi}^{(m)}$ determined as a result of solving mixed problem (5)–(7).

We use common solution (5) [22] of equation (5), i. e.

$$u(t,x) = r(at - x - (m-1)l - \tau) + g(x + at - ml - \tau). \tag{8}$$

the characteristics $(m-1)l + \tau + x - at = 0$ and $ml + \tau - x - at = 0$ partition $\widetilde{Q}^{(m)}$ into four subdomains $\widetilde{Q}_j^{(m)}$, $j = 1, 2, 3, 4$, presented on Fig. 1.

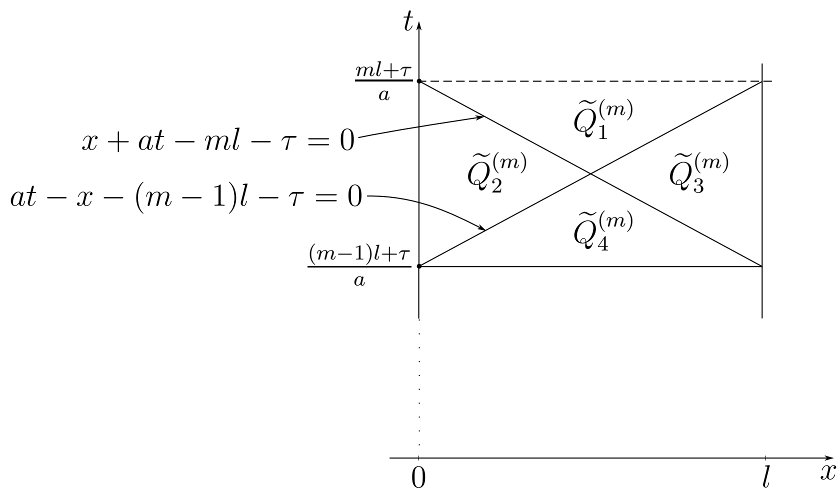

Figure 1: Partition of the domain $\widetilde{Q}^{(m)}$.

From common solution (8) we select solutions satisfying conditions (6) and (7). As a result we get

$$u(t,x) = \frac{1}{2}[\tilde{\varphi}^{(m)}(x - at + ml + \tau) + \tilde{\varphi}^{(m)}(x + at - ml - \tau)] +$$

$$+ \frac{1}{2a} \int\limits_{x-at+ml+\tau}^{x+at-ml-\tau} \tilde{\psi}^{(m)}(\xi)\mathrm{d}\xi, \ (t,x) \in \widetilde{Q}_1^{(m)}, \tag{9}$$

$$u(t,x) = \frac{1}{2}[\tilde{\varphi}^{(m)}(x - at + ml + \tau) - \tilde{\varphi}^{(m)}(ml + \tau - x - at)] +$$

$$+ \frac{1}{2a} \int\limits_{x-at+ml+\tau}^{ml+\tau-x-at} \tilde{\psi}^{(m)}(\xi)\mathrm{d}\xi + \mu^{(1)}\left(t + \frac{x}{a}\right), \ (t,x) \in \widetilde{Q}_2^{(m)}, \tag{10}$$

$$u(t,x) = \frac{1}{2}[\tilde{\varphi}^{(m)}(x + at - ml - \tau) - \tilde{\varphi}^{(m)}(2l - ml - \tau - x + at)] +$$

$$+ \frac{1}{2a} \int\limits_{2l-ml-\tau-x+at}^{x+at-ml-\tau} \tilde{\psi}^{(m)}(\xi)\mathrm{d}\xi + \mu^{(2)}\left(t + \frac{l-x}{a}\right), \ (t,x) \in \widetilde{Q}_3^{(m)}, \tag{11}$$

$$u(t,x) = \mu^{(1)}\left(t + \frac{x}{a}\right) + \mu^{(2)}\left(t + \frac{l-x}{a}\right) - \frac{1}{2}[\tilde{\varphi}^{(m)}(2l - ml - \tau - x + at) +$$

$$+\tilde{\varphi}^{(m)}(ml+\tau-x-at)]+\frac{1}{2a}\int\limits_{2l-ml-\tau-x+at}^{ml+\tau-x-at}\tilde{\psi}^{(m)}(\xi)\mathrm{d}\xi,\ (t,x)\in\widetilde{Q}_4^{(m)}, \qquad (12)$$

In order function u represented by (9)–(12) in the subdomains $\widetilde{Q}_j^{(m)}$ $(j=1,2,3,4)$ is solution of problem (5)–(7) in the domain $\widetilde{Q}^{(m)}$ from the class of twice continuously differentiable functions, $\tilde{\varphi}^{(m)}\in C^2[0,l]$, $\tilde{\psi}^{(m)}\in C^1[0,l]$, $\mu^{(j)}\in C^2\big[\frac{(m-1)l+\tau}{a},\frac{ml+\tau}{a}\big]$, where $C^s[c,d]$ is a set of continuously differentiable up to the order s on the segment $[c,d]$ functions. One can see this from (9)–(12). Furthermore, the functions from (9)–(12) together with its derivatives of the first and second order must be equal on the characteristics $x+at-ml-\tau=0$ and $at-x-(m-1)l-\tau=0$, considering them at the points of these characteristics as limit values from the corresponding subdomains $\widetilde{Q}_j^{(m)}$, $j=1,2,3,4$. Demanding this, we get the following agreement conditions:

$$\begin{gathered}\tilde{\varphi}^{(m)}(0)=\mu^{(1)}\left(\frac{ml+\tau}{a}\right),\ \tilde{\varphi}^{(m)}(l)=\mu^{(2)}\left(\frac{ml+\tau}{a}\right),\\ \tilde{\psi}^{(m)}(0)=\mu^{(1)\prime}\left(\frac{ml+\tau}{a}\right),\ \tilde{\psi}^{(m)}(l)=\mu^{(2)\prime}\left(\frac{ml+\tau}{a}\right),\\ \mu^{(1)\prime\prime}\left(\frac{ml+\tau}{a}\right)=a^2\tilde{\varphi}^{(m)\prime\prime}(0),\ \mu^{(2)\prime\prime}\left(\frac{ml+\tau}{a}\right)=a^2\tilde{\varphi}^{(m)\prime\prime}(l),\end{gathered} \qquad (13)$$

where $\mu^{(j)\prime}(y),\mu^{(j)\prime\prime}(y),\tilde{\varphi}^{(m)\prime\prime}(y)$ are the derivatives of the first and second order of the functions $\mu^{(j)}$ and $\tilde{\varphi}^{(j)}$, $j=1,2$.

Considering the limit values of the function u from (12) and its derivative with respect to t at $t=\frac{(m-1)l+\tau}{a}$, we get

$$\begin{aligned}\tilde{\varphi}^{(m-1)}(x)&=\mu^{(1)}\left(\frac{x-l+ml+\tau}{a}\right)+\mu^{(2)}\left(\frac{ml+\tau-x}{a}\right)-\\&\qquad-\tilde{\varphi}^{(m)}(l-x),\ x\in[0,l],\\ \tilde{\psi}^{(m-1)}(x)&=\mu^{(1)\prime}\left(\frac{x-l+ml+\tau}{a}\right)+\mu^{(2)\prime}\left(\frac{ml+\tau-x}{a}\right)-\\&\qquad-\tilde{\psi}^{(m)}(l-x),\ x\in[0,l].\end{aligned} \qquad (14)$$

Repeating this process further, we find solutions of problem (5)–(7) for $m=1,\dots,k$ and in $\widetilde{Q}^{(0)}$ by (9)–(12).

Using (14), we can determine $\tilde{\varphi}^{(m)}$ and $\tilde{\psi}^{(m)}$ in terms of $\tilde{\varphi}=\tilde{\varphi}^{(k)}$ and $\tilde{\psi}=\tilde{\psi}^{(k)}$ for all $m=1,\dots,k-1$. Express consequently in terms of each other, we obtain

$$\tilde{\varphi}^{(m)}(x)=(-1)^{k-m}\tilde{\varphi}(s(k-m)l+(-1)^{k-m}x)+$$

$$+\sum_{j=1}^{k-m}(-1)^{j-1}\left[\mu^{(1)}\left(\frac{[m+j-1-s(j-1)]l+\tau+(-1)^{j-1}x}{a}\right)+\right.$$

$$+ \mu^{(2)}\left(\frac{[m+j-s(j-1)]l+\tau+(-1)^j x}{a}\right)\Bigg], \tag{15}$$

$$\tilde{\psi}^{(m)}(x) = (-1)^{k-m}\tilde{\psi}(s(k-m)l+(-1)^{k-m}x) +$$

$$+\sum_{j=1}^{k-m}(-1)^{j-1}\left[\mu^{(1)\prime}\left(\frac{[m+j-s(j)]l+\tau+(-1)^{j-1}x}{a}\right)+\right.$$

$$\left.+ \mu^{(2)\prime}\left(\frac{[m+j-s(j-1)]l+\tau+(-1)^j x}{a}\right)\right], \tag{16}$$

where function s define on the set of integers and $s(n)=1$, if n is odd, and $s(n)=0$, if n – even.

Lemma 1. *If $\tilde{\varphi}\in C^2[0,l]$, $\tilde{\psi}\in C^1[0,l]$, $\mu^{(j)}\in C^2[0,(kl+\tau)/a]$ $(j=1,2)$ and agreement conditions*

$$\tilde{\varphi}(0)=\mu^{(1)}\left(\frac{kl+\tau}{a}\right),\ \tilde{\varphi}(l)=\mu^{(2)}\left(\frac{kl+\tau}{a}\right),$$

$$\tilde{\psi}(0)=\mu^{(1)\prime}\left(\frac{kl+\tau}{a}\right),\ \tilde{\psi}(l)=\mu^{(2)\prime}\left(\frac{kl+\tau}{a}\right), \tag{17}$$

$$\mu^{(1)\prime\prime}\left(\frac{kl+\tau}{a}\right)=a^2\tilde{\varphi}''(0),\ a^2\tilde{\varphi}''(l)=\mu^{(2)\prime\prime}\left(\frac{kl+\tau}{a}\right),$$

are fulfilled, then $\tilde{\varphi}^{(m)}\in C^2[0,l]$, $\tilde{\psi}^{(m)}\in C^1[0,l]$ and agreement conditions (13) *for $m=1,2,\ldots,k$ are fulfilled.*

P r o o f. $\tilde{\varphi}^{(m)}\in C^2[0,l]$, $\tilde{\psi}^{(m)}\in C^1[0,l]$, follow from Lemma's conditions and (14).

Let conditions (17) be fulfilled. We suppose $m=k$ in (14)and check the agreement conditions (13) for $k-1$. Really,

$$\tilde{\varphi}^{(k-1)}(0)=\mu^{(1)}\left(\frac{(k-1)l+\tau}{a}\right)+\mu^{(2)}\left(\frac{kl+\tau}{a}\right)-\tilde{\varphi}(l)=\mu^{(1)}\left(\frac{(k-1)l+\tau}{a}\right),$$

$$\tilde{\varphi}^{(k-1)}(l)=\mu^{(1)}\left(\frac{kl+\tau}{a}\right)+\mu^{(2)}\left(\frac{(k-1)l+\tau}{a}\right)-\tilde{\varphi}(0)=\mu^{(2)}\left(\frac{(k-1)l+\tau}{a}\right).$$

Analogously, using (14), the other agreement conditions (13) for $m=k-1$ are proved.

Further, based on the proved agreement conditions (13) for $m=k-1$, using (14), agreement conditions (13) for $m=k-2$ are checked.

Moving in such way by chain further, we prove the fulfilment of agreement conditions (13) and for $m=k-3,\ldots,1$. ⊠

We write homogeneous equation (1) in the domain Q, i. e.

$$\frac{\partial^2 u}{\partial t^2}-a^2\frac{\partial^2 u}{\partial x^2}=0,\ (t,x)\in Q. \tag{18}$$

Theorem 1. *If conditions of Lemma 1, then a unique classical solution of problem* (18), (2), (4) *from the class $C^2(\overline{Q})$ exists, $\overline{Q}$ is closure of th domain Q.*

P r o o f actually follows from the previous arguments. Solution is represented by (9)–(12). It follows from here, that $u \in C^2(\overline{Q^{(m)}})$ for all $m = 0, \ldots, k$. Furthermore, from construction of a solution $u \in C^1(\overline{Q})$. for finishing the proving of the theorem it remains to show the continuity of the second derivatives with respect to t of the solution u represented by (9)–(12) at joining points of the domains $\widetilde{Q}^{(m)}$.

Let $t = (ml + \tau)/a$, $m \in \{1, \ldots, k-1\}$. From (12), calculating the derivative $\partial^2 u/\partial t^2$, we get

$$\left.\frac{\partial^2 u}{\partial t^2}\right|_{t=\frac{ml+\tau}{a}} = \mu^{(1)\prime\prime}\left(\frac{ml+\tau+x}{a}\right) + \mu^{(2)\prime\prime}\left(\frac{(m+1)l+\tau-x}{a}\right) - a^2\tilde{\varphi}^{(m+1)\prime\prime}(l-x). \tag{19}$$

Analogously, calculating the derivative of (9), we have

$$\left.\frac{\partial^2 u}{\partial t^2}\right|_{t=\frac{ml+\tau}{a}} = a^2\tilde{\varphi}^{(m)\prime\prime}(x). \tag{20}$$

According to the first relation from (14), the right parts of (19) and (20) are equal. ⊠

Now we consider the case of inhomogeneous equation (1), i. e. problem (1), (2), (4). We represent its solution in the form of sum of two functions $u = \tilde{u} + v$, where $\tilde{u}$ is solution of problem (18), (2), (4), and v is solution of equation (1), fulfilled conditions

$$v|_{t=\frac{kl+\tau}{a}} = \left.\frac{\partial v}{\partial t}\right|_{t=\frac{kl+\tau}{a}} = 0,\ x \in [0, l], \tag{21}$$

$$v|_{x=0} = v|_{x=l} = 0,\ t \in \left[0, \frac{kl+\tau}{a}\right]. \tag{22}$$

Function v is determined in terms of another function w as follows:

$$v(t, x) = \int\limits_{\frac{kl+\tau}{a}}^{t} w\left(\frac{kl+\tau}{a} + t - \xi, \xi, x\right) \mathrm{d}\xi, \tag{23}$$

where $w\colon [0, (kl+\tau)/a] \times [0, (kl+\tau)/a] \times [0, l] \ni (t, \xi, x) \to w(t, \xi, x) \in \mathbb{R}$ is the solution of equation (5) with respect to the independent variables t and x, satisfying conditions

$$\begin{gathered} w|_{t=\frac{kl+\tau}{a}} = 0,\ \left.\frac{\partial w}{\partial t}\right|_{t=\frac{kl+\tau}{a}} = f(\xi, x),\ \xi \in \left[0, \frac{kl+\tau}{a}\right],\ x \in [0, l]. \\ w|_{x=0} = w|_{x=l} = 0, \end{gathered} \tag{24}$$

Solution w of problem (5), (24) is determined by (9)–(12). Furthermore, function w must satisfy agreement conditions (17). These conditions for w is fulfilled, if the function f satisfies conditions

$$f(0, t) = f(t, l) = 0,\ t \in \left[0, \frac{kl+\tau}{a}\right]. \tag{25}$$

We formulate the result of arguments in the form of theorem.

Theorem 2. *Suppose that conditions of Lemma 1 are fulfilled, function f is from the class $C^2(\overline{Q})$ and satisfies conditions* (25)*. Then in the class of functions $C^2(\overline{Q})$ a unique classical solution $u = \tilde{u} + v$ of problem* (1), (2), (4) *exists, where $\tilde{u}$ is solution of problem* (18), (2), (4)*, and function v is determined by* (23) *in terms of w, w is solution of problem* (18), (24)*.*

4 Problem of control by boundary conditions

We consider problem (18), (2), (3). In [22] classical solution for this problem is determined with the help of (13), (17), (20), (21). Let

$$u\big|_{t=\frac{ml}{a}} = \varphi^{(m)}(x), \ \left.\frac{\partial u}{\partial t}\right|_{t=\frac{ml}{a}} = \psi^{(m)}(x), \ x \in [0,l],$$

where u is the solution of problem (1)–(3), $m = 0, 1, \ldots, k$, $\varphi^{(0)}(x) = \varphi(x)$, $\psi^{(0)}(x) = \psi(x)$. For the functions $\varphi^{(m)}$ and $\psi^{(m)}$ the relations

$$\varphi^{(m)}(x) = \mu^{(1)}\left(\frac{ml-x}{a}\right) + \mu^{(2)}\left(\frac{(m-1)l+x}{a}\right) - \varphi^{(m-1)}(l-x), \ x \in [0,l], \tag{26}$$

$$\psi^{(m)}(x) = \mu^{(1)\prime}\left(\frac{ml-x}{a}\right) + \mu^{(2)\prime}\left(\frac{(m-1)l+x}{a}\right) - \psi^{(m-1)}(l-x), \ x \in [0,l], \tag{27}$$

$$\begin{gathered}\varphi^{(m)}(x) = (-1)^m\varphi(s(m)l + (-1)^m x) + \\ + \sum_{j=1}^{m}(-1)^{m+j}\left[\mu^{(1)}\left(\frac{[j-s(m+j)]l + (-1)^{m+j-1}x}{a}\right) + \right. \\ \left. + \mu^{(2)}\left(\frac{[j-s(m+j-1)]l + (-1)^{m+j}x}{a}\right)\right], \ m = 1, 2, \ldots, k,\end{gathered} \tag{28}$$

$$\begin{gathered}\psi^{(m)}(x) = (-1)^m\psi(s(m)l + (-1)^m x) + \\ + \sum_{j=1}^{m}(-1)^{m+j}\left[\mu^{(1)\prime}\left(\frac{[j-s(m+j)]l + (-1)^{m+j-1}x}{a}\right) + \right. \\ \left. + \mu^{(2)\prime}\left(\frac{[j-s(m+j-1)]l + (-1)^{m+j}x}{a}\right)\right], \ m = 1, \ldots, k,\end{gathered} \tag{29}$$

are valid.

We suppose, that equation (1) is homogeneous, i. e. consider equation (18). According to Theorem 2, solution $u\colon \overline{Q} \ni (t,x) \to u(t,x) \in \mathbb{R}$ of problem (18), (2), (4) from the class $C^2(\overline{Q})$ is determined in $\overline{Q}$. Let

$$u\big|_{t=\frac{kl}{a}} = p(x), \ \left.\frac{\partial u}{\partial t}\right|_{t=\frac{kl}{a}} = q(x), \ x \in [0,l], \tag{30}$$

for the solution u of this problem.

By virtue of (26) and (27)

$$\begin{aligned} p(x) &= \mu^{(1)}\left(\frac{kl-x}{a}\right) + \mu^{(2)}\left(\frac{(k-1)l+x}{a}\right) - \varphi^{(k-1)}(l-x),\ x \in [0,l], \\ q(x) &= \mu^{(1)\prime}\left(\frac{kl-x}{a}\right) + \mu^{(2)\prime}\left(\frac{(k-1)l+x}{a}\right) - \psi^{(k-1)}(l-x),\ x \in [0,l]. \end{aligned} \tag{31}$$

We consider system (31) with respect to $\mu^{(1)}$ and $\mu^{(2)}$. Really,

$$\begin{gathered} \mu^{(1)}\left(\frac{kl-x}{a}\right) + \mu^{(2)}\left(\frac{(m-1)l+x}{a}\right) = p(x) + \varphi^{(k-1)}(l-x),\ x \in [0,l], \\ -\mu^{(1)}\left(\frac{kl-x}{a}\right) + \mu^{(2)}\left(\frac{(m-1)l+x}{a}\right) = \\ = \int\limits_0^x [q(y) + \psi^{(k-1)}(l-y)]\mathrm{d}y + C,\ x \in [0,l]. \end{gathered} \tag{32}$$

From here the functions $\mu^{(1)}$ and $\mu^{(2)}$ are determined uniquely on the segment $\left[\frac{(k-1)l}{a}, \frac{kl}{a}\right]$ in terms of p, q, $\varphi^{(k-1)}$, $\psi^{(k-1)}$. Here p and q are determined by $\tilde{\varphi}$ and $\tilde{\psi}$ by Cauchy conditions (4), and the functions $\varphi^{(k-1)}$, $\psi^{(k-1)}$ in terms of φ and ψ of initial conditions Cauchy (2), and the values of the functions $\mu^{(j)}$ $(j = 1,2)$, defined on the segments $\left[0, \frac{(k-1)l}{a}\right] \cup \left[\frac{kl}{a}, \frac{kl+\tau}{a}\right]$, too. Thus, give the values of the functions $\mu^{(j)}$ $(j = 1,2)$ on the segment $\left[\frac{(k-1)l}{a}, \frac{kl}{a}\right]$ in corresponding way, according to system (31), we get unique solution of problem (18), (2)–(4).

If instead of equation (18) in Q one consider inhomogeneous equation (1), then in this case in (30) the functions p and q change in corresponding way, and in (31) terms because of the function v adds to the equations. In case of equation (1) arguments don't change in principle.

Note that the controlling functions $\mu^{(j)}$ can be found not only on the segments $\left[\frac{(k-1)l}{a}, \frac{kl}{a}\right]$, but on the other $\left[\frac{(m-1)l}{a}, \frac{ml}{a}\right]$, too, where $m \in \{1, 2, \ldots, k-1\}$, moving on the segment $\left[\frac{(m-1)l}{a}, \frac{ml}{a}\right]$ from above from the functions $\tilde{\varphi}$ and $\tilde{\psi}$ and from below from the functions φ and ψ.

If the functions φ, ψ and $\mu^{(j)}$ $(j = 1,2)$ satisfy agreement conditions

$$\begin{gathered} \varphi(0) = \mu^{(1)}(0),\ \varphi(l) = \mu^{(2)}(0), \\ \psi(0) = \mu^{(1)\prime}(0),\ \psi(l) = \mu^{(2)\prime}(0), \\ \mu^{(1)\prime\prime}(0) = a^2\varphi''(0),\ \mu^{(2)\prime\prime}(0) = a^2\varphi''(l), \end{gathered} \tag{33}$$

then for $t = (k-1)l/a$ agreement conditions

$$\mu^{(1)}\left(\frac{(k-1)l}{a}\right) = \varphi^{(k-1)}(0),\ \mu^{(2)}\left(\frac{(k-1)l}{a}\right) = \varphi^{(k-1)}(l),$$

$$\mu^{(1)\prime}\left(\frac{(k-1)l}{a}\right)=\psi^{(k-1)}(0),\ \mu^{(2)\prime}\left(\frac{(k-1)l}{a}\right)=\psi^{(k-1)}(l), \tag{34}$$
$$\mu^{(1)\prime\prime}\left(\frac{(k-1)l}{a}\right)=a^2\varphi^{(k-1)\prime\prime}(0),\ \mu^{(2)\prime\prime}\left(\frac{(k-1)l}{a}\right)=a^2\varphi^{(k-1)\prime\prime}(l)$$

are fulfilled. It follows from (28) and (29). From (31) by strength of (34) agreement conditions

$$p(0)=\mu^{(1)}\left(\frac{kl}{a}\right),\ p(l)=\mu^{(2)}\left(\frac{kl}{a}\right),$$
$$q(0)=\mu^{(1)\prime}\left(\frac{kl}{a}\right)),\ q(l)=\mu^{(2)\prime}\left(\frac{kl}{a}\right), \tag{35}$$
$$\mu^{(1)\prime\prime}\left(\frac{kl}{a}\right)=a^2p''(0),\ \mu^{(2)\prime\prime}\left(\frac{kl}{a}\right)=a^2p''(l)$$

follow. By strength of agreement conditions (34) and (35) and the controlling functions $\mu^{(j)}$ $(j=1,2)$ of system (32) it follows that on the whole the functions $\mu^{(j)}\in C^2\left[0,\frac{kl+\tau}{a}\right]$, $j=1,2$. The last statement can be proved for $m\in\{1,2,\ldots,k-1\}$, too.

Thus, the following theorem is valid.

Theorem 3. *If $\varphi,\tilde{\varphi}\in C^2[0,l]$, $\psi,\tilde{\psi}\in C^1[0,l]$, $\mu^{(j)}\in C^2\left(\left[0,\frac{(m-1)l}{a}\right]\cup\left[\frac{ml}{a},\frac{kl+\tau}{a}\right]\right)$ $(j=1,2)$, $f\in C^1(\overline{Q})$ and agreement conditions* (17), (25) *and* (33) *are fulfilled, then such functions $\mu^{(j)}$ (controlling functions) exist on the segment $\left[\frac{(m-1)l}{a},\frac{ml}{a}\right]$, $j=1,2$, that*

- *the functions $\mu^{(j)}$ $(j=1,2)$ defined already on the segment $\left[0,\frac{kl+\tau}{a}\right]$ belong to the class $C^2\left[0,\frac{kl+\tau}{a}\right]$;*

•• *for problem* (1)–(4) *a unique classical solution u from the class $C^2(\overline{Q})$ determined by* (9)–(12), (23) *exists, where it takes in* (23) *$m\in\{1,2,\ldots,k\}$ instead k and 0 instead $(kl+\tau)/a$.*

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