CONSISTENT SYSTEMS AND POLE ASSIGNMENT PROBLEM

V. A. Zaitsev (Izhevsk, Russia)

Consider a bilinear control system

$$\dot{x} = (A(t) + u_1(t)A_1(t) + \ldots + u_r(t)A_r(t))x, \quad t \in \mathbb{R}, \quad x \in \mathbb{K}^n, \quad \mathbb{K} = \mathbb{C} \vee \mathbb{R}, \tag{1}$$

with bounded piecewise continuous functions $A(\cdot)$, $A_l(\cdot)$, $u_l(\cdot)$, $l = \overline{1,r}$. The system (1) is said to be *consistent on* $[t_0, t_1]$ [1] if for any matrix $G \in M_{n,n}(\mathbb{K})$ there exists a bounded piecewise continuous control function $\widehat{u} = (\widehat{u}_1, \ldots, \widehat{u}_r) : [t_0, t_1] \to \mathbb{K}^r$ such that the solution of the matrix initial value problem $\dot{Z} = A(t)Z + (\widehat{u}_1(t)A_1(t) + \ldots + \widehat{u}_r(t)A_r(t))X(t, t_0)$, $Z(t_0) = 0$ satisfies condition $Z(t_1) = G$; $X(t, s)$ denotes the Cauchy matrix of the system $\dot{x} = A(t)x$. Let us assume that system (1) is time-independent:

$$\dot{x} = (A + u_1A_1 + \ldots + u_rA_r)x, \quad x \in \mathbb{K}^n. \tag{2}$$

We consider pole assignment problem for system (2). We shall say that system (2) is *arbitrarily pole assignable* if for a given polynomial $p(\lambda) = \lambda^n + \gamma_1\lambda^{n-1} + \ldots + \gamma_n$ with $\gamma_i \in \mathbb{K}$ there exists a control $\widehat{u} = (\widehat{u}_1, \ldots, \widehat{u}_r) \in \mathbb{K}^r$ such that $\det(\lambda I - (A + \widehat{u}_1A_1 + \ldots + \widehat{u}_rA_r)) = p(\lambda)$. Consider a linear control system with static output feedback

$$\dot{x} = Ax + Bu, \quad y = C^*x, \quad u = Uy, \quad (x, u, y) \in \mathbb{K}^n \times \mathbb{K}^m \times \mathbb{K}^k.$$

The matrix $U \in M_{m,k}(\mathbb{K})$ is a compensator. The closed-loop system has the form

$$\dot{x} = (A + BUC^*)x, \quad x \in \mathbb{K}^n. \tag{3}$$

System (3) is a special case of system (2). Let us construct the matrix $Q = \{Q_{ij}\}$ from the system (2) as follows: $Q_{ij} = \mathrm{Sp}\,(A_jA^{i-1})$, $i = \overline{1,n}$, $j = \overline{1,r}$. Let J be first unit superdiagonal matrix that is $J = \sum_{i=1}^{n-1} e_ie_{i+1}^* \in M_{n,n}(\mathbb{K})$. Denote $\Omega_k = \{S = \{s_{ij}\}_{i,j=1}^n : s_{ij} = 0 \text{ for } i < j + k\} \in M_{n,n}(\mathbb{K})$, $k = \overline{0, n-1}$.

Theorem 1. *Suppose the matrix A of system (3) has the Hessenberg form that is $A = \{a_{ij}\}_{i,j=1}^n$; $a_{i,i+1} \neq 0$, $i = \overline{1, n-1}$; $a_{ij} = 0$, $j > i + 1$ and the first $(p-1)$ rows of matrix B and the last $(n-p)$ rows of matrix C are equal to zero ($p \in \{1, \ldots, n\}$). Then implications $1 \Longrightarrow 2 \Longleftrightarrow 3$ hold for the following assertions:*

1. *System (3) is consistent.*
2. *The matrices $C^*B, C^*AB, \ldots, C^*A^{n-1}B$ are linearly independent.*
3. *System (3) is arbitrarily pole assignable.*

Moreover, the implication $2 \Longrightarrow 1$ holds if one of the following conditions is satisfied: (a) $\mathrm{rank}\,B = n$; (b) $\mathrm{rank}\,C = n$; (c) $A = J$; (d) $\mathrm{rank}\,B + \mathrm{rank}\,C \geqslant n + 1$; ($e$) $n < 6$.

Theorem 2. *Suppose the matrix* A *of system* (2) *has the Hessenberg form and the first* $(p-1)$ *rows and the last* $(n-p)$ *columns of matrices* A_l, $l=\overline{1,r}$ *are equal to zero* $(p \in \{1,\ldots,n\})$. *Then implications* $4 \Longrightarrow 5 \Longleftrightarrow 6$ *hold for the following assertions:*

4. *System* (2) *is consistent.*

5. $\operatorname{rank} Q = n$.

6. *System* (2) *is arbitrarily pole assignable.*

Moreover, the implication $5 \Longrightarrow 4$ *holds if one of the following conditions is satisfied:* (*a*) $A=J$; (*b*) $n<3$.

Theorem 3. *Suppose the matrix* A *of system* (2) *has the Hessenberg form,* $r=n$, *and* $A_l \in \Omega_{l-1}$, $l=\overline{1,n}$. *Then implications* $4 \Longleftrightarrow 5 \Longleftrightarrow 6$ *hold.*

References. 1. Zaitsev V.A., Tonkov Ye.L. // Russian mathematics. 1999. Vol. 43, №2. P. 42–52.