WEAK SOLUTION OF THE QUASI-LINEAR ONE-DIMENSIONAL PARABOLIC NEUMANN BOUNDARY-VALUE PROBLEM WITH NONLINEAR BOUNDARY CONDITIONS

I. A. Chernov (Petrozavodsk, Russia)

We consider the following parabolic single-dimensional boundary-value problem:

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\begin{aligned}
\partial_t c &= A\partial_x^2 c + B\partial_x c - \hat{B}\partial_x c - Ec + F,\ x \in (0,L),\ t \in (0,T),\\
\partial_x c(t,L) &= -G\big(t, s(t), c(t,L)\big), \quad t \in [0,T],\\
\partial_x c(t,0) &= g\big(t, s(t), c(t,0)\big), \quad t \in [0,T],\\
\dot{s}(t) &= \Gamma\big(t, s(t), c(t,0), c(t,L)\big), \quad t \in [0,T],\\
c(0,x) &= \varphi(x) \in C^2([0,L]), \quad 0 \leqslant \varphi(x) \leqslant 1, \quad s(0) = s_0 \in R^m.
\end{aligned}
$$

Here t is for time, x is the spatial variable, $s(t) \in R^m$ is called the state of the system. Coefficient $A = A(t,x,s) \geqslant \bar{A} > 0$ is continuous and bounded in $\Omega_x = [0,T] \times [0,L] \times R^m$ together with its partial derivatives. Non-negative coefficients B, $\hat{B}$, E, and F depend also on boundary values of the solution, e.g. $B = B(t,x,s,c(t,0),c(t,L))$. They are continuous and bounded together with their partial derivatives in $\Omega_c = \Omega_x \times R^2$. Besides, in Ω_c: either $E > 0$ or $E \equiv 0$, $0 \leqslant F \leqslant E$, $\partial_x B \leqslant 0$, $\partial_x \hat{B} \leqslant 0$ and does not depend on x. Neumann boundary conditions are nonlinear; the right-hand sides G and g are bounded and continuous together with their partial derivatives in $\Omega = [0,T] \times R^{m+1}$, and $\partial_c G(t,s,c) \geqslant 0$, $G(t,s,1) \geqslant 0$, $G(t,s,0) < 0$, the same for g. The right-hand side Γ of the evolution equation for the state $s(t)$ is bounded and continuous together with its partial derivatives in $\Omega \times R$.

Such problems appear as models of heat and mass transfer, e.g. models of hydride formation and decomposition (e. g. [1]) after eliminating the free boundary by a change of variables. The state $s(t)$ in this case is the position of the free boundary. Besides, «usual» state components are temperature of the sample and gas pressure. The boundary values are important due to nonlinear chemical processes on the phase boundaries, which may influence on the coefficients of the equation and the boundary conditions. The results can be generalized to a case when coefficients depend on the solution in other points (with fixed x), including «aftereffect» (e. g. dependence of $c(t-d, L)$), or the functionals on the solution, e. g. integral of $c(t,x)$ over $[0, L]$ for each t (the amount of matter).

We introduce a uniform lattice in $\Pi = [0,T] \times [0,L]$ and construct the implicit difference scheme with explicit approximation of the coefficients of the PDE. Then we prove the maximum principle to show that the lattice solution, if any exists, is bounded together with the first lattice derivatives provided that the lattice steps h and τ are sufficiently small. The assumptions about E, F and G, g are crucial for the solution to be bounded, those about $\hat{B}$ are important for boundness of the spatial lattice derivative. Finally, we show that the unique lattice solution exists provided that $\tau = o(h)$ and h is small enough. The lattice solution is obtained by a special sweeping method: $c_n^i = \alpha_n^i c_n^{i+1} + \beta_n^i + \gamma_n^i c_n^I$. Here n and i are time and spatial indices respectively, c_n^i is the lattice solution, c_n^I is its boundary value. This allows to eliminate linear equations on each time layer and to reduce the system for two nonlinear equations for c_n^0 and c_n^I for each n.

By constructing linear interpolations of lattice solutions we get the family U of continuous functions in Π. They are uniformly bounded in the Sobolev space $H_1(\Pi)$ and thus uniformly bounded and equicontinuous. The linear interpolations S of the lattice state vector s_n are equicontinuous on $[0,T]$ and uniformly bounded. Thus the family U is weakly compact in $H_1(\Pi)$ and compact in $C(\Pi)$ and S is compact in $C([0,T])$. The idea was taken from [2].

Therefore we proved the convergence of the lattice approximations. The weak solution [2,3] is the pair of continuous $c(t,x) \in H_1(\Pi)$ and $s(t) \in C([0,T])$ such that it satisfies the integral identities

$$\int_0^L v(0,x)\varphi(x)\,dx + \int_\Pi c\partial_t v\,dxdt - \int_0^T A\big(t,0,s(t)\big)v(t,0)g\big(t,s(t),c(t,0),c(t,L)\big)\,dt-$$

$$-\int_0^T A\big(t,L,s(t)\big)v(t,L)G\big(t,s(t),c(t,L),c(t,0)\big)\,dt - \int_\Pi \partial_x\big(A(t,x,s)v\big)\partial_x c\,dxdt+$$

$$+\int_\Pi (B-\hat{B})v\partial_x c\,dxdt - \int_\Pi \Big(Ec - F\Big)v\,dxdt,$$

$$s(t) = s_0 + \int_0^t \Gamma\big(\zeta, s(\zeta), c(\zeta,0), c(\zeta,L)\big)\,d\zeta$$

for any continuous $v(t,x) \in H_1(\Pi)$, $v(T,x) = 0$.

The constructed functions form the weak solutions provided that the steps tend to zero. Note that $s(t)$ has the continuous derivative. Thus we have proved the existence (but not the uniqueness) of the weak solution to the problem in a constructive way: the constructed difference scheme can be used for numerical solution of the problem.

In the report we discus the difference scheme, proofs of the properties of the lattice solutions, and the construction of the weak solution. Also a few examples of the boundary-value problems of the considered type are presented.

References

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