ON STABILITY FOR NONSTATIONARY $M/M/N/N+R$ QUEUE

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We consider nonstationary $M/M/N/N+R$ queueing model and obtain the first bounds of stability.

Keywords: Nonstationary queues; $M_t/M_t/N/N+R$ queueing model; bounds; stability.

1. INTRODUCTION

In this note we consider the estimates of stability for the simplest generalization of nonstationary Erlang queueing model. There is a number of investigations of nonstationary continuous-time Markov chains, see for instance first results in [5], and more detail studies for birth and death processes (BDPs) in [1, 6]. Now we consider nonstationary $M_t/M_t/N/N+R$ queue and obtain some simple stability bounds.

Let $X = X(t)$, $t \geq 0$ be queue-length process for $M_t/M_t/N/N+R$ queue. This is a BDP on state space $E_{N+R} = \{0, 1\ldots, N+R\}$ and birth and death rates $\lambda_n(t) = \lambda(t)$, $\mu_n(t) = \min(n, N)\,\mu(t)$ respectively. We suppose that arrival and service intensities $\lambda(t)$ and $\mu(t)$ are locally integrable on $[0, \infty)$. Let $p_i(t) = Pr\{X(t) = i\}$ be state probabilities of $X(t)$, and $\mathbf{p}(t) = (p_0(t), \ldots, p_{N+R}(t))^T$ be the respective column vector.

Then we can write the forward Kolmogorov system

$$\begin{cases} \frac{dp_0}{dt} = -\lambda(t)p_0 + \mu(t)p_1, \\ \frac{dp_k}{dt} = \lambda(t)p_{k-1} - (\lambda(t) + k\mu(t))\,p_k + (k+1)\mu(t)p_{k+1}, 1 \leq k \leq N-1, \\ \frac{dp_k}{dt} = \lambda(t)p_{k-1} - (\lambda(t) + N\mu(t))\,p_k + N\mu(t)p_{k+1}, N \leq k < N+R, \\ \frac{dp_N}{dt} = \lambda(t)p_{N-1} - N\mu(t)p_N \end{cases} \tag{1}$$

in the following form:

$$\frac{d\mathbf{p}}{dt} = A\,(t)\,\mathbf{p}, \quad t \geq 0, \tag{2}$$

where $A(t) = \{a_{ij}(t),\ t \geq 0\}$ is the transposed intensity matrix of the process, and

$$a_{ij}(t) = \begin{cases} \lambda(t), & \text{if} \quad j = i-1, \\ \min(i+1, N)\,\mu(t), & \text{if} \quad j = i+1, \\ -(\lambda(t) + \min(i, N)\,\mu(t)), & \text{if} \quad j = i, \\ 0, & \text{overwise.} \end{cases} \tag{3}$$

We denote throughout the paper by $\| \bullet \|$ the l_1-norm, i.e. $\|\mathbf{x}\| = \sum |x_i|$, for $\mathbf{x} = (x_0, ..., x_{N+R})^T$ and $\|B\| = \max_j \sum_i |b_{ij}|$ for $B = (b_{ij})_{i,j=0}^{N+R}$.

Let $\Omega = \{\mathbf{x} :\ \mathbf{x} \geq 0,\ \|\mathbf{x}\| = 1\}$ be a set of all stochastic vectors.

Let $E_k(t) = E\{X(t)\,|X(0) = k\}$ be the mean of the process at the moment t under initial condition $X(0) = k$, and $E_{\mathbf{p}}(t)$ be the mathematical expectation (the mean) at the moment t under initial probability distribution $\mathbf{p}(0) = \mathbf{p}$.

Consider also a "perturbed" queue-length process $\bar{X} = \bar{X}(t),\ t \geq 0$ with general structure of intensity matrix $\bar{A}(t)$. In general, $\bar{X}(t)$ is not BDP. Put $\hat{A}(t) = \bar{A}(t) - A(t)$. We assume that the perturbations are uniformly small, i.e. $\|\hat{A}(t)\| \leq \varepsilon$ for almost all $t \geq 0$.

2. STABILITY BOUNDS

Let $d_1, \ldots, d_{N+R}$ be positive numbers. Consider the following expression:

$$\alpha_i(t) = \lambda(t) + \min(i, N)\,\mu(t) - \frac{d_{i+1}}{d_i}\lambda(t) - \frac{d_{i-1}}{d_i}\min(i-1, N)\,\mu(t),$$
$$i = 1, 2, \ldots, N+R, \tag{4}$$

where $d_0 = d_{N+R+1} = 0$. Put $G = \sum_{i=1}^{N+R} d_i$ and $d = \min_{1 \leq i \leq N+R} d_i$.

Theorem 1. *Let there exist a positive sequence $\{d_i\}$ and a positive number θ such that*

$$\alpha_i(t) \geq \theta, \quad i = 1, 2, \ldots, N+R,\ t \geq 0. \tag{5}$$

Then the following stability bounds hold:

$$\limsup_{t \to \infty} \|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\| \leq \frac{\varepsilon\left(1 + \log\frac{4G}{d}\right)}{\theta}, \tag{6}$$

and

$$\limsup_{t \to \infty} \left|E_{\mathbf{p}}(t) - \bar{E}_{\bar{\mathbf{p}}}(t)\right| \leq \frac{(N+R)\,\varepsilon\left(1 + \log\frac{4G}{d}\right)}{\theta}, \tag{7}$$

for arbitrary initial probability distributions $\mathbf{p}(0)$ and $\bar{\mathbf{p}}(0)$ for $X(t)$ and $\bar{X}(t)$ respectively.

Proof. Firstly we find the basic estimate of the rate of convergence. The property $\sum_{i=0}^{N+R} p_i(t) = 1$ for any $t \geq s$ allows to put $p_0(t) = 1 - \sum_{i \geq 1} p_i(t)$, then we obtain the following system from (2)

$$\frac{d\mathbf{z}(t)}{dt} = B(t)\mathbf{z}(t) + \mathbf{f}(t), \tag{8}$$

where $\mathbf{z}(t) = (p_1(t), \dots, p_{N+R}(t))^T$, $\mathbf{f}(t) = (\lambda(t), 0, \dots, 0)^T$, $B(t) = (b_{ij}(t))_{i,j=1}^{N+R}$ and respective $b_{ij}(t)$, see details in [7, 8]. Consider now the triangular matrix

$$D = \begin{pmatrix} d_1 & d_1 & d_1 & \cdots & d_1 \\ 0 & d_2 & d_2 & \cdots & d_2 \\ 0 & 0 & d_3 & \cdots & d_3 \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & d_{N+R} \end{pmatrix}, \tag{9}$$

and the respective norms $\|\mathbf{x}\|_{1D} = \|D\mathbf{x}\|$, and $\|B\|_{1D} = \|DBD^{-1}\|$.

We have now the following bound of the logarithmic norm $\gamma(B(t))$ in $1D-$norm (see for instance [2, 3, 7, 9]):

$$\gamma(B)_{1D} = \max_i \left(\frac{d_{i+1}}{d_i}\lambda(t) + \frac{d_{i-1}}{d_i} \min(i-1, N)\,\mu(t) - \right. \\ \left. (\lambda(t) + \min(i, N)\,\mu(t)) \right) = \max(-\alpha_i(t)) \le -\theta, \tag{10}$$

in accordance with (5). Therefore the following inequality holds:

$$\|\mathbf{z}^*(t) - \mathbf{z}^{**}(t)\|_{1D} \le e^{-\theta(t-s)}\|\mathbf{z}^*(s) - \mathbf{z}^{**}(s)\|_{1D}, \tag{11}$$

for any initial conditions $\mathbf{z}^*(s)$, $\mathbf{z}^{**}(s)$ and any $s, t,\ 0 \le s \le t$. Then we obtain

$$\begin{aligned} \|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \le 2\|\mathbf{z}^*(t) - \mathbf{z}^{**}(t)\| = \\ 2\|D^{-1}D(\mathbf{z}^*(t) - \mathbf{z}^{**}(t))\| \le \\ \frac{4}{d}\|\mathbf{z}^*(t) - \mathbf{z}^{**}(t)\|_{1D} \le \\ \frac{4}{d}e^{-\theta(t-s)}\|\mathbf{z}^*(s) - \mathbf{z}^{**}(s)\|_{1D} \le \\ \frac{4G}{d}e^{-\theta(t-s)}\|\mathbf{z}^*(s) - \mathbf{z}^{**}(s)\| \le \\ \frac{4G}{d}e^{-\theta(t-s)}\|\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\| \le \frac{8G}{d}e^{-\theta(t-s)}, \end{aligned} \tag{12}$$

for any initial conditions $\mathbf{p}^*(s)$, $\mathbf{p}^{**}(s)$ and any $s, t,\ 0 \le s \le t$.

Consider the forward Kolmogorov system for perturbed process:

$$\frac{d\bar{\mathbf{p}}}{dt} = \bar{A}(t)\bar{\mathbf{p}}(t). \tag{13}$$

We can apply the approach of paper [4]. Put

$$\beta(t,s) = \sup_{\|\mathbf{v}\|=1, \sum v_i=0} \|U(t,s)\mathbf{v}\| = \frac{1}{2}\max_{i,j}\sum_k |p_{ik}(t,s) - p_{jk}(t,s)|, \tag{14}$$

where $U(t,s)$ is Cauchy matrix of (2), and $p_{ik}(t,s) = Pr\{X(t) = k|X(s) = i\}$. Mitrophanov in [4] proved the bound of stability, that in the nonstationary case is the following one:

$$\|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\| \le \beta(t,s)\|\mathbf{p}(s) - \bar{\mathbf{p}}(s)\| + \int_s^t \|\hat{A}(u)\|\beta(u,s)\,du. \tag{15}$$

Moreover, the following estimates hold:

$$\beta(t,s) \le 1, \quad \beta(t,s) \le \frac{ce^{-b(t-s)}}{2}, \ 0 \le s \le t, \tag{16}$$

where under our assumptions $c = \frac{8G}{d}$, $b = \theta$. Finally the following stability bound holds:

$$\|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\| \le \begin{cases} \|\mathbf{p}(s) - \bar{\mathbf{p}}(s)\| + (t-s)\varepsilon, \\ 0 < t-s < b^{-1}\log\frac{c}{2}, \\ b^{-1}(\log\frac{c}{2} + 1 - ce^{-b(t-s)})\varepsilon + \\ \frac{c}{2}e^{-b(t-s)}\|\mathbf{p}(s) - \bar{\mathbf{p}}(s)\|, \quad t-s \ge b^{-1}\log\frac{c}{2}, \end{cases} \tag{17}$$

for any initial conditions $\mathbf{p}(s)$, $\bar{\mathbf{p}}(s)$. Let $t - s \to \infty$. Then (17) implies our claim. □

Consider here the case of sufficiently large service rate, namely let there exist $d > 1$ such that the following assumption holds:

$$N\mu(t) - d\lambda(t) \ge \theta^* > 0, \tag{18}$$

for any $t \ge 0$.

Put $d_1 = 1$, $\frac{d_{k+1}}{d_k} = \delta_k = 1$, $k \le N-2$, and $\frac{d_{k+1}}{d_k} = \delta_k = d$, $k \ge N-1$.

Then

$$\alpha_k(t) = \begin{cases} \mu(t), & k < N-1; \\ \mu(t) - (d-1)\lambda(t), & k = N-1; \\ \left(1-\frac{1}{d}\right)(N\mu(t) - d\lambda(t)), & N \le k \le N+R-2; \\ N\mu(t)\left(1-\frac{1}{d}\right) - \lambda(t), & k = N+R-1. \end{cases} \tag{19}$$

Let $d \le \frac{N}{N-1}$, then we obtain

$$\theta = \inf_k \alpha_k(t) = \left(1-\frac{1}{d}\right)(N\mu(t) - d\lambda(t)) \ge \left(1-\frac{1}{d}\right)\theta^*. \tag{20}$$

Hence we obtain the following statement.

Theorem 2. *Under assumption (18) stability bounds (6) and (7) hold for* $\theta = \left(1 - \frac{1}{d}\right)\theta^*$, $d = 1$, *and* $G = N - 1 + \sum_{k=1}^{R+1} d^k$.

Let now our original process have 1-periodic intensities.

Then the following claim holds.

Theorem 3. *Let* $\lambda(t)$ *and* $\mu(t)$ *be 1-periodic. Let there exist a positive sequence* $\{d_i\}$ *and a positive number* φ^* *such that*

$$\alpha_i(t) \geq \varphi(t), \quad i = 1, 2, \ldots, N + R, 0 \leq t \leq 1, \tag{21}$$

where

$$\int_0^1 \varphi(t)\, dt \geq \varphi^*. \tag{22}$$

Let

$$K = \sup_{|t-s|\leq 1} \int_s^t \varphi(\tau)\, d\tau < \infty. \tag{23}$$

Then we have the following stability bounds:

$$\limsup_{t\to\infty} \|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\| \leq \frac{\varepsilon\left(1 + \log\frac{4Ge^K}{d}\right)}{\varphi^*}, \tag{24}$$

and

$$\limsup_{t\to\infty} \left|E_{\mathbf{p}}(t) - \bar{E}_{\bar{\mathbf{p}}}(t)\right| \leq \frac{(N+R)\,\varepsilon\left(1 + \log\frac{4Ge^K}{d}\right)}{\varphi^*}, \tag{25}$$

for arbitrary initial probability distributions $\mathbf{p}(0)$ *and* $\bar{\mathbf{p}}(0)$ *for* $X(t)$ *and* $\bar{X}(t)$ *respectively.*

Proof. The statement follows from inequality $e^{-\int_s^t \varphi(u)\,du} \leq e^K e^{-\varphi^*(t-s)}$. □

In the case of sufficiently large service rate we obtain the following claim.

Theorem 4. *Let arrival and service rates be 1-periodic, and let*

$$\int_0^1 \left(N\mu(\tau) - d\lambda(\tau)\right)\, d\tau = \psi > 0, \tag{26}$$

instead of (18). Then stability bounds (24) and (25) hold for $\varphi^* = \left(1 - \frac{1}{d}\right)\psi$, $d = 1$, *and* $G = N - 1 + \sum_{k=1}^{R+1} d^k$.

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