STATISTICAL ANALYSIS OF QUEUEING NETWORKS WITH INFINITE NUMBER OF SERVERS

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In this paper a problem of successive changing of individuals states or technical means from some population is considered. This changing is connected as with individuals aging so with appearance of new individuals. The problem is solved using queueing networks with infinite numbers of servers in their nodes.

Keywords: product theorem, Markov process, stationary distribution.

1. INTRODUCTION

Last years in the reliability theory a large attention is devoted to the life time models with a large number of states [1]. New applications appear in models of a dynamics of populations with multistage individuals [2]. These applications demand to develop theory of queueing networks with infinite number of servers beginning from product theorems through an aggregation of nodes to convenient statistical procedures of parameters estimates.

2. OPENED NETWORKS

Consider opened queueing network with the nodes set $S = \{0\} \bigcup I,\ I = \{1, \ldots, m\}$, the indivisible route matrix $\Theta = ||\theta(i,j)||_{i,j\in S}$ and Poisson input flow with parameter $\lambda(0) > 0$. The node $i \in I$ contains infinite number of servers with exponentially distributed service times with the parameter $\mu(i) > 0$. The node 0 is imaginary, it is a source of arrival customers and a runoff for customers leaving the network. The route matrix Θ is called indivisible if for any $i, j \in S,\ i \neq j$, there are $i_1, \ldots, i_s \in S$, so that the inequality

$$0 < \theta(i, i_1) \cdot \theta(i_1, i_2) \cdot \ldots \cdot \theta(i_{s-1}, i_s) \cdot \theta(i_s, j). \tag{1}$$

is true. From (1) we have that for fixed $\lambda(0) > 0$ the system of linear algebraic equations

$$(\lambda(0), \lambda(1), \ldots, \lambda(m)) = \Theta(\lambda(0), \lambda(1), \ldots, \lambda(m)) \tag{2}$$

[3] has single solution $(\lambda(1), \ldots, \lambda(m)),\ \lambda(i) > 0,\ i \in I$. Then the process $(n_1(t), \ldots, n_m(t))$ which describes numbers of customers in nodes of opened network [4, Proposition

1.10, Example 1.29] is ergodic and its limit distribution

$$P(n_1, \ldots, n_m) = \prod_{i=1}^{m} p_i(n_i), \; p_i(n_i) = \mathbf{e}^{(} - \rho(i)) \frac{(\rho(i))^{n_i}}{n_i!}, \; \rho(i) = \frac{\lambda(i)}{\mu(i)}, \; i \in I. \qquad (3)$$

Theorem 1. *Almost surely*

$$\lim_{T \to \infty} \frac{\int_0^T n_i(t)dt}{T} = \rho_i, \; i \in I. \qquad (4)$$

Proof. From the formula (3) the distribution $p_i(n_i)$ is Poisson and its mean coincides with $\rho(i)$. So the law of large numbers for ergodic markov processes [4, Theorem 1.2] leads to the formula (4). □

Theorem 2. *Suppose that the sets $I_1, \ldots, I_r$ create a decomposition of the set I onto nonintersected subsets. Denote $N_k(t) = \sum_{i \in I_k} n_i(t), \; k = 1, \ldots, r$, then the random process $(N_1(t), \ldots, N_r(t))$ has limit distribution $\prod_{k=1}^{r} P_k(N_k)$ where*

$$P_k(N_k) = \mathbf{e}^{(} - R_k) \frac{R_k^{N_k}}{N_k!}, \; R_k = \sum_{i \in I_k} \rho(i), \qquad (5)$$

and almost surely

$$\lim_{T \to \infty} \frac{\int_0^T N_j(t)dt}{T} = R(j), \; j = 1, \ldots, r. \qquad (6)$$

Proof. The formula (5) arises from the formula (3) and from well known fact that a sum of independent random variables x, y with Poisson distributions with the parameters a, b is a random variable with Poisson distribution with the parameter $a + b$. Indeed assume that generating functions of random variables x, y are $\varphi_x(z) = \mathbf{e}^{(} a(z-1)), \; \varphi_y(z) = \mathbf{e}^{(} b(z-1))$ then $\varphi_{x+y}(z) = \varphi_x(z)\varphi_y(z) = \mathbf{e}^{(} (a+b)(z-1))$. The formula (6) arises from (4), (5). □

Remark. Denote $N(t) = \sum_{i \in I} n_i(t)$, then from Theorem 1 the discrete random process $N(t)$ has Poisson limit distribution with the parameter $R = \sum_{i \in I} \rho(i)$. Assume that k-the input customer remains the time η_k in the network with the mean $f = M\eta_k$. The quantity f may be interpreted as mean life time of some individual. So from [5, § 31, Theorem 6] we have $R = \lambda(0)f$. Consequently if there is statistical estimate of input Poisson flow intensity $\lambda(0)$ then using Theorem 2 it is possible to estimate the parameter R and so them mean individual life time f. This statement may be spread onto nodes subset $I' \subset I$ where $\theta(i,j) = 0, \; i \in I \setminus I', \; j \in I'$ and η_k is time interval of k-the input customer stay in I'.

Example. Analogously with [2] assume that the route matrix Θ contains the following nonzero elements:

$$\theta(1,2), \theta(1,0) = 1 - \theta(1,2);\ \theta(2,3), \theta(2,0) = 1 - \theta(2,3); \ldots;$$

$$\theta(m-1,m), \theta(m-1,0) = 1 - \theta(m-1,m);\ \theta(m,0) = 1.$$

Then it is easy to obtain that

$$\lambda(1) = \lambda,\ \lambda(2) = \lambda(1)\theta(1,2),\ \lambda(3) = \lambda(2)\theta(2,3), \ldots, \lambda(m) = \lambda(m-1)\theta(m-1,m),$$

$$\rho(i) = \frac{\lambda(i)}{\mu(i)},\ i = 1, \ldots, m,$$

with the mean life time

$$M = \frac{1}{\mu(1)} + \frac{\theta(1,2)}{\mu(2)} + \frac{\theta(1,2)\theta(2,3)}{\mu(3)} + \ldots + \frac{\prod_{k=1}^{m-1} \theta(k,k+1)}{\mu(m)}.$$

3. CLOSED NETWORKS

Consider now closed queueing network with the nodes set S, the route matrix Θ, n customers circulating in the network and n (that is equivalent infinity) servers in each node with service intensity μ_i on a server of i-th node, $i \in S$. Assume that for fixed $\lambda(0) > 0$ positive numbers $\lambda(1), \ldots, \lambda(m)$ create single solution $(\lambda(1), \ldots, \lambda(m))$ of the system (2). Denote $\rho(0) = \lambda(0)/\mu(0)$, then discrete Markov process $(n_0(t), \ldots, n_m(t))$, describing number of customers in nodes of the closed network has polynomial limit distribution [4, Example 1.29]

$$P(n_0, \ldots, n_m) = n! \prod_{i=0}^{m} \frac{d_i^{n_i}}{n_i!},\ d_i = \frac{\rho(i)}{\rho(0) + \ldots + \rho(m)},\ i \in S,\ \sum_{i=0}^{m} n_i = n. \tag{7}$$

Lemma. *The random process $(n_0(t) + n_1(t), n_2(t), n_3((t), \ldots, n_m(t))$ has limit distribution*

$$P(n_0 + n_1, n_2, n_3, \ldots, n_m) = n! \frac{(d_0 + d_1)^{n_0+n_1}}{(n_0 + n_1)!} \prod_{i=2}^{m} \frac{d_i^{n_i}}{n_i!}. \tag{8}$$

Proof. Lemma statement is based on binomial theorem. □

Theorem 3. *Almost surely*

$$\lim_{T\to\infty} \frac{\int_0^T n_i(t)dt}{T} = nd_i,\ i \in I_0. \tag{9}$$

Proof. Using the formula (8) by mathematical induction it is easy to prove that the random process $(n_0(t), n_1(t)+\ldots+n_m(t))$ has limit distribution

$$P(n_0, n_1+\ldots+n_m) = \frac{n! d_0^{n_0}(d_1+\ldots+d_m)^{n-n_0}}{n_0!(n-n_0)!}.$$

Consequently the random process $n_0(t)$ has limit Bernoulli distribution with the mean nd_0. Analogously it is possible to prove that the random process $n_i(t)$ has limit Bernoulli distribution with the mean $nd_i,\ i=1,\ldots,m$. So from the law of large numbers for ergodic discrete Markov process [4, Theorem 1.2] we obtain the formula (9). □

Theorem 4. *Suppose that $I_1,\ldots,I_r$ create a decomposition of the set S onto non-intersected subsets. Denote $N_k(t)=\sum_{i\in I_k} n_i(t),\ k=1,\ldots,r$, then the random process $(N_1(t),\ldots,N_r(t))$ has polynomial limit distribution*

$$P(N_1,\ldots,N_r) = n!\prod_{k=1}^{r}\frac{D_k^{N_k}}{N_k!},\ D_k=\sum_{i\in I_k} d_i,\ k=1,\ldots,r, \tag{10}$$

and almost surely

$$\lim_{T\to\infty}\frac{\int_0^T N_k(t)dt}{T} = nD_k,\ k=1,\ldots,r. \tag{11}$$

Proof. The formula (10) may be obtained from the formula (8) by mathematical induction. The formula (11) arises from the formula (10) and Theorem 3. □

4. CONCLUSION

Theorems 1 - 4 allow to aggregate nodes of initial network and so to simplify statistical estimate of its parameters. As a result random process, which describes customers motion along nodes-states and which is defined by the route matrix Θ with the dimension $m\times m$ and by service intensities $\mu(i),\ i=1,\ldots,m$, is replaced by statistical model of customers distribution in aggregated nodes. It allows not only to get rid of complicated solution of the system (2) but to simplify a procedure of observation and a collection of necessary information. It is well known that to define a state i is more difficult than to define its belonging to the subset $I_j\subseteq I$. More over it is possible to construct sufficiently simple estimate of mean life time also. This quantity is important in analysis of efficiency indexes.

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