PROCESSOR SHARING SYSTEMS WITH RANDOM CAPACITY DEMANDS

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We discuss processor sharing queueing systems with non-homogeneous demands. This non-homogenity means that each demand (independently of others) has some random capacity and its length (or amount of work for its service) generally depends on the capacity. In real systems, a total sum of capacities of demands presenting in the system is limited by some constant value (memory volume) $V > 0$. But we estimate loss characteristcs for such system using queueing models with unlimited memory volume.

Keywords: random capacity demand, total demands capacity, memory volume.

1. INTRODUCTION

Egalitarian processor sharing (EPS) systems are used for modeling of computer and communicating networks [1]. Presently, they are applicable to situations where a common resource is shared by a varying number of concurrent users [2] (for example, to WEB-servers modeling [3]).

We introduce the following additional assumption for the classical $M/G/1 - EPS$ system. Assume that each demand is characterized by some non-negative random capacity. This random variable can be interpreted as a part of system's memory space used by the demand during its presence in the system. A total sum of demands capacities $\sigma(t)$ in the system at arbitrary time instant t is referred as the total demands capacity. The random value $\sigma(t)$ can be limited by some constant value V $(0 < V < \infty)$, which is called the memory volume of the system. In this case we have a non-classical processor sharing system that will be notated by $M/G/1(V) - EPS$. Later on, we shall call demand length the amount of work necessary for demand's service, i.e. the service time under condition that there are no other demands in the system during its presence in it. Analogously, we shall call residual length of the demand its residual service time after some time instant under the same condition (see [2]).

The purpose of the paper is 1) to obtain the non-stationary and stationary distribution of total demands capacity in the system $M/G/1 - EPS$; 2) to determine some estimations of loss characteristics for systems $M/G/1(V) - EPS$ with limited memory volume $(V < \infty)$ based on the model with unlimited one; 3) to compare processor sharing systems $M/G/1(V) - EPS$ and $M/G/1 - EPS$ from the point of view of estimation of loss characteristics.

2. CLASSICAL PROCESSOR SHARING SYSTEM

Denote by $\eta(t)$ the number of demands present in the system at the time instant t and $\xi_i^*(t)$ be the residual length of ith demand at this instant, $i = \overline{1, \eta(t)}$. Let $F(x,t) = \mathbf{P}\{\zeta < x, \xi < t\}$ be the joint distribution function of the demand capacity ζ and its length ξ (we assume that the demand capacity and its length doesn't depend on his arrival time and on characteristics of other demands). Then $L(x) = F(x,\infty)$ and $B(t) = F(\infty,t)$ be the distribution functions of the random variables ζ and ξ consequently. Let a be an arrival rate of entrance flow of demands, $\alpha(s,q) = \int_0^\infty \int_0^\infty e^{-sx-qt} dF(x,t)$ be the double Laplace-Stieltjes transform (with respect to x and t) of the distribution function $F(x,t)$, $\varphi(s) = \alpha(s,0)$ and $\beta(q) = \alpha(0,q)$ be the Laplace-Stieltjes transform (LST) of the distribution functions $L(x)$ and $B(t)$ consequently, $D(x,t) = \mathbf{P}\{\sigma(t) < x\}$ be the distribution function of total demands capacity at the time instant t, $\delta(s,t) = \int_0^\infty e^{-sx} d_x D(x,t)$ be the LST of the function $D(x,t)$ with respect to x, $\overline{\delta}(s,q) = \int_0^\infty e^{-qt}\delta(s,t)dt$ be the Laplace transform of the function $\delta(s,t)$ with respect to t. The mixed $(i+j)$th moments of the random variables ζ and ξ (if they exist) take the form: $\alpha_{ij} = (-1)^{i+j} \frac{\partial^{i+j}}{\partial s^i \partial q^j} \alpha(s,q)\Big|_{s=0,q=0}$.

Assume that demands in the considered system at an arbitrary time t are numerated as random; i.e. if the number of demands is k, then there are $k!$ ways to enumerate them, and each enumeration can be chosen with the same probability $1/k!$.

One can easily show that the system under consideration is described by the Markov process

$$(\eta(t), \xi_i^*(t), i = \overline{1, \eta(t)}), \tag{1}$$

where components $\xi_i^*(t)$ are absent if $\eta(t) = 0$. In this case we also have $\sigma(t) = 0$.

In what follows, to simplify the notation, we denote $Y_k = (y_1, \ldots, y_k)$. We characterize the process (1) by functions with the following probabilistic sense:

$$P_0(t) = \mathbf{P}\{\eta(t) = 0\}; \tag{2}$$

$$\Theta_k(Y_k, t) = \mathbf{P}\{\eta(t) = k, \xi_j^*(t) < y_j, j = \overline{1,k}\},\ k = 1, 2, \ldots; \tag{3}$$

$$P_k(t) = \mathbf{P}\{\eta(t) = k\} = \Theta_k(\infty_k, t),\ k = 1, 2, \ldots, \tag{4}$$

where $\infty_k = (\infty, \ldots, \infty)$ is a k-component vector.

Note that the functions $\Theta_k(Y_k,t)$ are symmetric with respect to permutations of components of the vector Y_k due to our random enumeration of customers in the system.

Let us determine the function $\overline{\delta}(s,q)$ under zero initial condition $\eta(0) = \sigma(0) = 0$.

Denote by $\overline{p}_0(q) = \int_0^\infty e^{-qt} P_0(t)dt$ and $\overline{\theta}_k(Y_k,q) = \int_0^\infty e^{-qt}\Theta_k(Y_k,t)dt$ the Laplace transforms with respect to t of the functions $P_0(t)$ and $\Theta_k(Y_k,t)$ consequently. It's known (see [2]) that

$$\overline{p}_0(q) = [q + a - a\pi(q)]^{-1} \tag{5}$$

under zero initial condition, where $\pi(q)$ is the LST of the busy period distribution function for the system under consideration. Note [2] that $\pi(q)$ is a unique solution of the functional equation $\pi(q) = \beta(q + a - a\pi(q))$ such that $|\pi(q)| \le 1$.

Lemma. *Under zero initial condition, the functions $\overline{\theta}_k(Y_k, q)$, $k = 1, 2, \ldots,$ have the form $\overline{\theta}_k(Y_k, q) = \overline{p}_0(q) \prod_{i=1}^{k} \int_0^{y_i} [q + a - aB(u)]du$.*

Let $\beta_i = \mathbf{E}\xi^i = (-1)^i \beta^{(i)}(0)$ be the ith moment of customer length, $i = 1, 2, \ldots$.

Corollary 1. *If $\rho = a\beta_1 < 1$, limits $\theta_k(Y_k) = \lim_{t\to\infty} \Theta_k(Y_k, t)$, $k = 1, 2, \ldots,$ exist being independent of initial condition and have the form $\theta_k(Y_k) = (1-\rho)a^k \prod_{i=1}^{k} \int_0^{y_i} [1-$ $-B(u)]du$.*

Corollary 2. *Let $\overline{p}_k(q)$ be the Laplace transform of the function $P_k(t)$, $k = 0, 1, \ldots,$ under zero initial condition. Then we have $\overline{p}_k(q) = \frac{a^k(1-\pi(q))^k}{(q+a-a\pi(q))^{k+1}}$.*

From the corollary 1 we can obtain the known relation for the stationary distribution $\{p_k\}$ of the number of demands in the system ($\rho = a\beta_1 < 1$)[2]: $p_k = \theta_k(\infty_k) =$ $= (1-\rho)\rho^k$, $k = 0, 1, \ldots$.

Let $\chi(t)$ be the capacity of a demand being on service at the time t and $\xi^*(t)$ be the residual length of this demand at the time t. We shall use the notation $E_y(x) =$ $= \mathbf{P}\{\chi(t) < x | \xi^*(t) = y\}$. It is known [4] that the LST of the conditional distribution function $E_y(x)$ has the form:

$$e_y(s) = [1 - B(y)]^{-1} \int_{x=0}^{\infty} e^{-sx} \int_{u=y}^{\infty} dF(x, u). \tag{6}$$

Theorem 1. *For zero initial condition the function $\overline{\delta}(s, q)$ is determined by the relation $\overline{\delta}(s, q) = \{[q+a-a\pi(q)][1-I(s, q)]\}^{-1}$, where $I(s, q) = \int_0^{\infty} (q+a-aB(y))e_y(s)dy$ and $e_y(s)$ is determined by the relation (6).*

Corollary 3. *If the random variables ζ and ξ are independent, we have:*

$$\overline{\delta}(s, q) = [q + a(1 - \pi(q))(1 - \varphi(s))]^{-1}. \tag{7}$$

Corollary 4. *Under zero initial condition, the Laplace transform $g(s, q)$ with respect to t of generation function $P(z, t) = \sum_{k=0}^{\infty} P_k(t)z^k$, $|z| \le 1$, of the demands number in the system at time instant t have the following form:*

$$g(z, q) = \int_0^{\infty} e^{-qt} P(z, t)dt = [q + a(1 - z)(1 - \pi(q))]^{-1}. \tag{8}$$

Corollary 5. *Let $\rho = a\beta_1 < 1$. Then stationary mode exists. The LST $\delta(s)$ of the stationary distribution function $D(x) = \lim_{t\to\infty} D(x, t)$ of demands total capacity has the form:*

$$\delta(s) = \frac{1 - \rho}{1 + a\alpha_q'(s, q)|_{q=0}}. \tag{9}$$

Note that the relation (9) was first obtained by B. Sengupta [5].

Corollary 6. *Let $\delta_1(t)$ be the first moment of the total demands capacity $\sigma(t)$ under zero initial condition, $\overline{\delta}_1(q)$ be the Laplace transform of the function $\delta_1(t)$. Then we have:*

$$\overline{\delta}_1(q) = \frac{a\alpha_{11} + q \int_0^{\infty} \int_0^{\infty} xS(t)dF(x, t)}{[q + a - a\pi(q)] \left[1 - \rho - q \int_0^{\infty} S(t)dB(t)\right]^2},$$

where $S(t) = \int_0^t [1 - B(y)]^{-1} dy$.

Let σ be a stationary total demands capacity ($\sigma(t) \Rightarrow \sigma$ in the sense of a weak convergence). From the relation (9) the following known formulas [5] can be obtained:

$$\delta_1 = \mathbf{E}\sigma = -\delta'(0) = \frac{a\alpha_{11}}{1-\rho}, \; \delta_2 = \mathbf{E}\sigma^2 = \delta''(0) = \frac{a\alpha_{21}}{1-\rho} + 2\delta_1^2. \tag{10}$$

For some special cases we can obtain the view of the distribution function $D(x)$ from the formula (9). For example, consider the case when the demand capacity ζ and its length ξ are connected by the relation $\xi = c\zeta + \xi_1$, $c > 0$, where the random variables ζ and ξ_1 are independent (such dependence for demand capacity and its length is true for many real information systems).

Denote by $\kappa_1 = \mathbf{E}\xi_1$ the first moment of the random variable ξ_1. In this case we have $\alpha(s,q) = \varphi(s+cq)\kappa(s)$, where $\kappa(s)$ is the Laplace–Stieltjes transform of the distribution function of the random variable ξ_1. The the relation (9) takes the following form:

$$\delta(s) = \frac{1-\rho}{1 + a[c\varphi'(s) - \kappa_1\varphi(s)]}. \tag{11}$$

Assume that customer capacity ζ has an exponential distribution with the parameter $f > 0$. Then from the formula (11) we obtain: $\delta(s) = \frac{(1-\rho)(s+f)^2}{(s+f)^2 - \rho_1 f^2 - \rho_2 f(s+f)}$, where $\rho_1 = ac/f$, $\rho_2 = a\kappa_1$, so that $\rho = a\beta_1 = \rho_1 + \rho_2$.

Now we can determine the original of Laplace transform $\delta(s)/s$, where $\delta(s)$ is defined by formula (11), and obtain the view of the stationary distribution function $D(x)$:

$$D(x) = 1 - \frac{(1-\rho)e^{-fx}}{2b}\left[\frac{(\rho_2+b)^2 e^{(\rho_2+b)fx/2}}{2-\rho_2-b} - \frac{(\rho_2-b)^2 e^{(\rho_2-b)fx/2}}{2-\rho_2+b}\right], \tag{12}$$

where $b = \sqrt{\rho_2^2 + 4\rho_1}$.

3. ESTIMATION OF LOSS CHARACTERISTICS

The $M/G/1 - EPS$ is a system without losing of customers ($V = \infty$). But with the help of this model we can estimate the memory capacity V in order to guarantee inexceeding of given loss probability.

Assume that we have a stationary queueing system Q_∞ with Poisson entrance flow without losses of demands. Let Q_V be a stationary system that differs from Q_∞ only with the fact that its total capacity is limited by the constant value V. We denote by $D(x)$ the distribution function of total demands capacity for the system Q_∞ and by $D_V(x)$ the distribution function of this random value for the system Q_V.

Theorem 2. *The inequality* $D(x) \le D_V(x)$ *takes place for all* $x > 0$.

Proof of the theorem see in [4].

It follows from theorem 2 that the loss probability P for the system Q_V satisfies the following inequality [4]:

$$P = 1 - \int_0^V D_V(V-x)dL(x) \le 1 - \int_0^V D(V-x)dL(x) = P^*. \tag{13}$$

Thus, the value P^* is an upper estimation of loss probability for the system Q_V. If we choose V under condition that P^* is given so that the equality $\int_0^V D(V-x)dL(x) = 1 - P^*$ is satisfied, then the real loss probability P doesn't exceed P^*. If only very rare losses are permitted in the system under consideration, the difference between the values P and P^* is inessential.

Note that the loss probability is not exhaustive characteristic of losses, because its value shows a part of losing demands, not a part of losing capacity or, in other words, information being lost. Really, it is obvious that demands having large capacity will be lost more often. Therefore, more objective losses estimation is the value $Q = 1 - \frac{1}{\varphi_1}\int_0^V xD_V(V-x)dL(x)$.

The value Q is the probability of losing of a unit of demand capacity. The next inequality follows from theorem 2:

$$Q = 1 - \frac{1}{\varphi_1}\int_0^V xD_V(V-x)dL(x) \leq 1 - \frac{1}{\varphi_1}\int_0^V xD(V-x)dL(x) = Q^*.$$

If only very rare losses are permitted in the system under consideration, the difference between the values Q and Q^* is inessential.

For example, in the case of the distribution function (12) we obtain:

$$P^* = \left\{1 - \frac{1-\rho}{b}\left[a_1\frac{1-e^{-(1-b_1)fV}}{b+\rho_2} + a_2\frac{1-e^{-(1-b_2)fV}}{b-\rho_2}\right]\right\}e^{-fV},$$

where $a_1 = \frac{(\rho_2+b)^2}{2-\rho_2-b}$, $a_2 = \frac{(\rho_2-b)^2}{2-\rho_2+b}$, $b_1 = -1 + \frac{\rho_2+b}{2}$, $b_2 = -1 + \frac{\rho_2-b}{2}$;

$$Q^* = \left\{1 + fV - \frac{2(1-\rho)}{b}\left[\frac{(a_1+a_2)fV}{8\rho_1} + a_1\frac{1-e^{-(1-b_1)fV}}{(b+\rho_2)^2} - a_2\frac{1-e^{-(1-b_2)fV}}{(b-\rho_2)^2}\right]\right\}e^{-fV}.$$

Note that in most cases the calculation and estimation of the probability Q is very complicated. Therefore, we often must restrict ourselve to the calculation and estimationof the loss probability P.

If it is impossible to determine the view of the distribution function $D(x)$, we can estimate the value P^* by approximation of the function $\Phi(x) = \int_0^x D(x-u)dL(u)$, being the distribution function of the sum of independent random variables σ and ζ, with the distribution function of gamma distribution $\Phi^*(x) = \gamma(h, rx)/\Gamma(h)$, where $\gamma(h, rx) = \int_0^{rx} t^{h-1}e^{-t}dt$ is the incomplete gamma function, $\Gamma(h) = \gamma(h, \infty)$ is the gamma function. The parameters h and r of the approximate distribution should be chosen so that its first and second moments $f_1^* = h/r$ and $f_2^* = h(h+1)/r^2$ be equal to the first and second moments of the distribution function $\Phi(x)$ respectively. It is obvious that these moments have the form

$$f_1 = \delta_1 + \varphi_1, \ f_2 = \delta_2 + \varphi_2 + 2\delta_1\varphi_1. \tag{14}$$

Thus, the parameters of the distribution function $\Phi^*(x)$ should be chosen as follows: $h = \frac{f_1^2}{f_2 - f_1^2}$, $r = \frac{f_1}{f_2 - f_1^2}$, where f_1 and f_2 can be calculated from relations (10), (14). Hence, we have the approximate formula $P^* \cong 1 - \Phi^*(V)$.

Note that in the case of not very small permissible loss probabilities, using the estimation P^* instead of P leads to unjustifiably surplus choice of the capacity volume V. Therefore, the direct analysis of processor sharing systems with limited memory space is very important.

4. THE CASE OF LIMITED TOTAL CAPACITY

The system $M/G/1(V) - EPS$ with demands of different types was analyzed in detail in [6]. We shall concider a special case of demands of the same type. Then, for stationary probabilities of number of demands present in the system we have:

$$p_0 = \left(\sum_{k=0}^{\infty} a^k A_*^{(k)}(V) \right)^{-1}, \; p_k = p_0 a^k A_*^{(k)}(V), \; k = 1, 2, \dots,$$

where $A_*^{(k)}(x)$ is a kth order Stieltjes convolution of the function $A(x) = \int_{u=0}^{x} \int_{t=0}^{\infty} u dF(u,t)$. The loss probability has the form

$$P = 1 - p_0 \left[L(V) - \sum_{k=1}^{\infty} a^k A_*^{(k)}(V) \right].$$

Assume additionally that demand capacity has an exponential distribution with parameter f, and let the demand length be proportional to its capacity ($\xi = c\zeta$, $c > 0$). Then, we obtain after some calculation:

$$p_0 = \begin{cases} \dfrac{1-\rho}{1 - \sqrt{\rho} e^{-fV} \left[\sinh(\sqrt{\rho} fV) + \sqrt{\rho} \cosh(\sqrt{\rho} fV) \right]}, & \text{if } \rho \neq 1, \\[2ex] \dfrac{1 + e^{-2fV}}{1 + fV}, & \text{if } \rho = 1; \end{cases}$$

$$p_k = p_0 \rho^k \left[1 - e^{-fV} \sum_{i=0}^{2k-1} \frac{(fV)^i}{i!} \right], \; k = 1, 2, \dots; \; P = p_0 e^{-fV} \cosh(\sqrt{\rho} fV),$$

where $\rho = ac/f$.

Now we can compare values P^* and P or Q^* and Q using analytical results or simulation. Table 1 presents the dependence of loss characteristics upon the memory capacity V. We assume here that $\rho = 0.6$, the demand length is proportional to its capacity ($\xi = c\zeta$), where $c = 1$, and capacity ζ has an exponential distribution with parameter $f = 1$.

Values P^*, Q^*, P were obtained by calculation from above relations, and the value Q was estimated by simulation. The table shows that estimators P^*, Q^* are not very precise and we can use them for the case when the proper loss characteristics are near zero.

Table 1: Probabilities P and Q for $\rho = 0.6$

V	P^*	Q^*	P	Q
0.0	1.00000	1.00000	1.00000	1.00000
0.2	0.92721	0.99569	0.81994	0.98269
0.4	0.86622	0.98366	0.67754	0.94034
0.6	0.81392	0.96529	0.56700	0.88482
0.8	0.76815	0.94194	0.48156	0.82409
1.0	0.72735	0.91487	0.41516	0.76311
2.0	0.56855	0.75562	0.23586	0.51290
3.0	0.45178	0.60242	0.15775	0.35596
4.0	0.35651	0.47628	0.11281	0.25640
5.0	0.28750	0.37679	0.08340	0.18993
6.0	0.22947	0.29888	0.06291	0.14330
7.0	0.18316	0.23763	0.04811	0.10963
8.0	0.14620	0.18925	0.03716	0.08464
10.0	0.09314	0.12034	0.02263	0.05165
15.0	0.03018	0.03896	0.00697	0.01589
20.0	0.00978	0.01262	0.00222	0.00512
30.0	0.00103	0.00133	0.00023	0.00054
40.0	0.00011	0.00014	0.00002	0.01589
50.0	0.00001	0.00002	0.00000	0.00001

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