ON APPROXIMATION OF RETRIAL QUEUES WITH VARYING SERVICE RATE

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The paper deals with Markov model of retrial queueing system in which service rate depends on queue length. Investigation method is based on an approximation of the input system by the system with truncated state space. The accuracy of such approximation is also discussed.

Keywords: Stochastic system, retrials, steady state.

1. INTRODUCTION

The research of wide class of stochastic systems with repeated calls faces the problem of calculating characteristics of the system with the poisson incoming flow. Markov process that describes the behavior of such system has an infinite state space and the transaction matrix usually does not have special properties that simplify the process of finding Kolmogorov set of equations explicit solution. Described features lead to the fact that only few simple models were examined in details [1].

Usually the problem of computing stationary probabilities for such systems is solved by computation algorithms or recurrent schemas [2]. To apply them system with finite state space is taken into account. Usually it is considered that the queue length does not exceed the value of M. If the request comes to the system when there is no free server and there are M sources of repeated calls already it is lost. It is intuitively obvious that by choosing M big enough we can approximate characteristics of the input system with the predefined accuracy.

In this paper we present the described above technic for the retrial system with Poisson input flow and varying service rate. Its integral characteristics are approximated by characteristics of the system with truncated space state which can be explicitly written in terms of system's parameters. The error of such approximation for different service rate switching policies is also discussed.

2. MARKOV MODELS OF THE INPUT AND TRUNCATED SYSTEMS

Consider continuous time Markov chain $X(t) = (C(t); N(t))$, $C(t) \in \{0, 1, \dots, c\}$, $N(t) \in \{0, 1, \dots\}$, which is defined by its infinitesimal characteristics $a_{(i,j)(i',j')}$, (i, j), $(i', j') \in S(X) = \{0, 1, \dots, c\} \times \{0, 1, \dots\}$:

1) if $i = \{0, 1, \dots, c-1\}$, then

$$a_{(i,j)(i',j')} = \begin{cases} \lambda, & (i',j') = (i+1,j); \\ j\mu, & (i',j') = (i+1,j-1); \\ i\nu_j, & (i',j') = (i-1,j); \\ -[\lambda + j\mu + i\nu_j], & (i',j') = (i,j); \\ 0, & \text{otherwise.} \end{cases}$$

2) if $i = c$, then

$$a_{(c,j)(i',j')} = \begin{cases} \lambda, & (i',j') = (c,j+1); \\ c\nu_j, & (i',j') = (c-1,j); \\ -[\lambda + c\nu_j], & (i',j') = (c,j); \\ 0, & \text{otherwise.} \end{cases}$$

Process $X(t)$ describes the behavior of the following system. The incoming flow of events is Poisson with rate λ. There are c identical servers. If there is any free server the request is served immediately. Service time is exponentially distributed random variable with parameter ν_j that depends on the current queue of repeated calls length. If request finds all servers busy it tries to get service in a random period of time that has exponential distribution with parameter μ. The number of busy servers at any time t is defined by the first component of $X(t)$ and the number of retrials Ц by the second.

Let us find up the condition of $X(t), t > 0$ stationary mode existence.

Lemma 1. *Let* $\nu = \varliminf_{j\to\infty} \nu_j$. *Then if* $\frac{\lambda}{c\nu} < 1$ *process* $X(t)$ *– is ergodic and its boundary distribution* $\pi_{ij}, (i,j) \in S(X)$ *coincides with a single stationary.*

Proof. Consider the following Lyapunov functions $\varphi(i,j) = \alpha i + j, (i,j) \in S(X)$, where parameter α will be determined later on. Then the mean drifts

$$y_{ij} = \sum_{(i',j') \neq (i,j)} a_{(i,j)(i',j')}(\varphi(i',j') - \varphi(i,j)$$

are given by

$$y_{ij} = \begin{cases} \lambda\alpha - i\nu_j\alpha + j\mu(\alpha - 1), & 0 \le i \le c-1, \\ \lambda - c\nu_j\alpha, & i = c. \end{cases}$$

When $\frac{\lambda}{c\nu} < 1$ for any value of $\alpha \in (\frac{\lambda}{c\nu}, 1)$ there exists such $\varepsilon > 0$ that $y_{ij} < -\varepsilon$ for all $(i,j) \in S(X)$ excluding finite number of states (i,j). So the assumptions of Tweedie's theorem[[?], p.97] are hold for functions $\varphi(i,j) = \alpha i + j, \alpha \in (\frac{\lambda}{c\nu}, 1)$. □

Consider truncated system. It functions in the same way as an input system but has the limitation on maximum number of retrials. This means that incoming calls are lost when all servers are busy and there are M sources of repeated calls already. Formally the system is described by the Markov chain $X(t,M) = (C(t,M); N(t,M))$, where $C(t, M) \in \{0, 1, \ldots, c\}$, $N(t, M) \in \{0, 1, \ldots, M\}$ with infinitesimal characteristics $a^{(M)}_{(i,j)(i',j')}, (i, j), (i', j') \in S(X, M) = \{0, 1, \ldots, c\} \times \{0, 1, \ldots, M\}$:

1) if $i = \{0, 1, \ldots, c-1\}, j = \{0, 1, \ldots, M\}$, then

$$a^{(M)}_{(i,j)(i',j')} = \begin{cases} \lambda, & (i', j') = (i+1, j); \\ j\mu, & (i', j') = (i+1, j-1); \\ i\nu_j, & (i', j') = (i-1, j); \\ -[\lambda + j\mu + i\nu_j, & (i', j') = (i, j); \\ 0, & \text{otherwise.} \end{cases}$$

2) if $i = c, j = \{0, 1, \ldots, M-1\}$, then

$$a^{(M)}_{(c,j)(i',j')} = \begin{cases} \lambda, & (i', j') = (c, j+1); \\ c\nu_j, & (i', j') = (c-1, j); \\ -[\lambda + c\nu_j], & (i', j') = (c, j); \\ 0, & \text{otherwise.} \end{cases}$$

3) if $i = c, j = M$, then

$$a^{(M)}_{(c,M)(i',j')} = \begin{cases} c\nu_M, & (i', j') = (c-1, M); \\ -c\nu_M, & (i', j') = (c, M); \\ 0, & \text{otherwise.} \end{cases}$$

The state space $S(X, M)$ of process $X(t, M)$ is finite thus the stationary mode always exists and by $\pi_{ij}(M), (i, j) \in S(X, M)$ we define its stationary probabilities.

Let us consider the service process of the truncated system in more detail.

3. STATIONARY PROBABILITIES OF THE TRUNCATED SYSTEM

For the given system stationary probabilities satisfy the following set of Kolmogorov equations and normalizing condition:

$$[\lambda + j\mu + i\nu_j]\pi_{ij}(M) = (j+1)\mu\pi_{i-1j+1}(M) + \lambda\pi_{i-1j}(M) + (i+1)\nu_j\pi_{i+1j}(M), \quad (1)$$

$$j = 0, \ldots, M-1, i = 0, ..., c-1,$$

$$[\lambda + M\mu + i\nu_M]\pi_{iM}(M) = \lambda\pi_{i-1M}(M) + (i+1)\nu_M\pi_{i+1M}(M), i = 0, ..., c-1, \quad (2)$$

$$[\lambda + c\nu_j]\pi_{cj}(M) = (j+1)\mu\pi_{c-1j+1}(M) + \lambda\pi_{c-1j}(M) + \lambda\pi_{cj-1}(M), j = 0, \ldots, M-1,$$

$$c\nu_M\pi_{cM}(M) = \lambda\pi_{c-1M}(M) + \lambda\pi_{cM-1}(M), \quad (3)$$

$$\sum_{i=0}^{c}\sum_{j=0}^{M}\pi_{ij}(M) = 1.$$

Consider the following definitions:

$$e_i(n) = (\delta_{i0},\ \delta_{i1},\ \ldots,\ \delta_{in-1})^T, \quad \delta_{ij} = \begin{cases} 1, i = j; \\ 0, i \neq j; \end{cases}$$

$1(c)$ - vector of length that consists from 1,

$$A_j = \|a^j_{ik}\|^{c-1}_{i,k=0}, \quad a^j_{ik} = \begin{cases} -\lambda, & k = i-1, \\ \lambda + j\mu + i\nu_j, & k = i, \\ -(i+1)\nu_j, & k = i+1, \\ 0, & \text{otherwise}; \end{cases}$$

if $i \neq 0, c-1$. In case $i = 0$

$$a^j_{0k} = \begin{cases} \lambda + j\mu, & k = 0, \\ -\nu_j, & k = 1, \\ 0, & \text{otherwise}; \end{cases}$$

and if $i = c-1$

$$a^j_{c-1k} = \begin{cases} -\lambda, & k = c-2, \\ \lambda + j\mu + (c-1)\nu_j, & k = c-1, \\ 0, & \text{otherwise}; \end{cases}$$

$$B_j = \|b^j_{ik}\|^{c-1}_{i,k=0}, \quad b^j_{ik} = \begin{cases} (j+1)\mu, & k = i-1, \\ 0, & \text{otherwise}; \end{cases}$$

if $i \neq 0, c-1$. In case $i = 0, b^j_{0k} = 0, k = 0, 1, \ldots, c-1$, and if $i = c-1$

$$b^j_{c-1k} = \begin{cases} \frac{c(j+1)\mu\nu_j}{\lambda}, & k \neq c-2, \\ \frac{(j+1)\mu[\lambda + c\nu_j]}{\lambda}, & k = c-2; \end{cases}$$

$$C = \|c_{ik}\|^{c-1}_{i,k=0}, \quad c_{ik} = \begin{cases} 1, & k = 0, i = 0, \\ a^M_{i-1k}, & \text{otherwise}; \end{cases}$$

$$\Phi_j = \left(\prod_{i=j}^{M-1} A_i^{-1} B_i \right) C^{-1} e_0(c).$$

Vector Φ_j is defined correctly by the last equation. As $|C| = (-1)^{c-1}(c-1)\nu_M^{c-1} \neq 0$ so C^{-1} always exists. Matrices $A_j, j = 0, 1, \ldots, M$ are not singular because they satisfy the Adamar column condition ([3] p. 406).

Probabilities $\pi_{ij}(M), (i,j) \in S(X, M)$ can be explicitly expressed in terms of the system'Ÿs parameters.

Theorem 1. *Stationary probabilities of the system are defined by the following set of equations:*

$$\pi_j(M) = \Phi_j \pi_{0M}(M), j = 0, \ldots, M,$$

$$\pi_{cj}(M) = \frac{(j+1)\mu}{\lambda} 1(c)^T \Phi_{j+1} \pi_{0M}(M), j = 0, \ldots, M-1,$$

$$\pi_{cM}(M) = \frac{\lambda e_{c-1}^T(c) + M\mu 1(c)^T}{c\nu_M} C^{-1} e_0(c) \pi_{0M}(M),$$

where $\pi_j(M) = (\pi_{0j}(M),\ \pi_{1j}(M),\ \ldots,\ \pi_{c-1j}(M))^T$,

$$\pi_{0M}(M) = \left\{ \sum_{j=0}^{M} \left(1 + \frac{j\mu}{\lambda}\right) 1(c)^T \Phi_j + \frac{\lambda e_{c-1}^T(c) + M\mu 1(c)^T}{c\nu_M} \Phi_M \right\}^{-1}.$$

It should be noticed that in case of $c = 1, 2$ the above formulas turn into equations of the scalar type [4].

Next we will show that the system with truncated state space approximates the input system.

4. APPROXIMATION RESEARCH

To proof that characteristics of the finite system approximate characteristics of the input system let us use stochastic order concept [5].

Lemma 2. *If conditions of lemma 1 are true and :*

1) if $X(0,M) \le_{st} X(0)$, *then* $X(t,M) \le_{st} X(t)$ *for all* $t \ge 0$ *and* $X(M) \le_{st} X$, *where* $X = (C,N)$, $X(M) = (C(M), N(M))$ *– random vectors distributed as* $\pi_{ij}, (i,j) \in S(X)$ *and* $\pi_{ij}(M), (i,j) \in S(X,M)$ *correspondingly;*

2) if $X(0,M) \le_{st} X(0, M+1)$, *then* $X(t,M) \le_{st} X(t, M+1)$ *for all* $t \ge 0$ *and* $X(M) \le_{st} X(M+1)$.

Lemma 2 appears from the results of stochastic order of migration processes. It leades to the following theorem.

Theorem 2. *If conditions of lemma 1 take place, then for any* $(i,j) \in S(X)$ $\pi_{ij} = \lim_{M\to\infty} \pi_{ij}(M)$.

We have already shown that the probabilities $\pi_{ij}(M)$ tend to π_{ij} as the value of M increases. It is interesting to find the value of difference between them. Let us examine it for some pretty general types of policies.

Theorem 3. *If conditions of lemma 1 take place and the sequence* ν_j *is strictly monotone, then for* $(i,j) \in \{1, \ldots, c\} \times \{0, 1, \ldots\}$

$$0 \le \bar{\pi}_{ij} - \bar{\pi}_{ij}(M) \le \lambda\left(\nu_j - \nu_{j-1}\right)^{-1} \frac{\frac{1}{M!}(\frac{\lambda}{c\mu})^M \prod\limits_{\alpha=0}^{M} \frac{\lambda+\alpha\mu}{\nu_\alpha}}{\sum\limits_{\beta=0}^{M} \frac{1}{\beta!}(\frac{\lambda}{c\mu})^\beta [c + \frac{\lambda+\beta\mu}{\nu_\beta}] \prod\limits_{\alpha=0}^{\beta-1} \frac{\lambda+\alpha\mu}{\nu_\alpha}},$$

where $\bar{\pi}_{ij} = \sum\limits_{\alpha \geq i, \beta \geq j} \pi_{\alpha\beta} = P(C \geq i, N \geq j)$, $\bar{\pi}_{ij}(M) = \sum\limits_{\alpha \geq i, \beta \geq j} \pi_{\alpha\beta}(M) = P(C(M) \geq i, N(M) \geq j)$.

Theorem 4. *If conditions of lemma 1 take place and* $\nu_j \leq \nu_{j+1}, j = 0, 1, \ldots, \lambda < c\nu_0$ *then*

$$\sup_{(i,j) \in S(X)} (\bar{\pi}_{ij} - \bar{\pi}_{ij}(M)) \leq \frac{\frac{c[(\lambda + c\mu)\lambda + (M+c)\nu_0]}{M!\nu_0} (\frac{\lambda}{c\mu})^{M+1} \prod\limits_{\alpha=0}^{M} \frac{\lambda + \alpha\mu}{\nu_\alpha}}{(c\nu_0 - \lambda) \sum\limits_{\beta=0}^{M} \frac{1}{\beta!} (\frac{\lambda}{c\mu})^\beta [c + \frac{\lambda + \beta\mu}{\nu_\beta}] \prod\limits_{\alpha=0}^{\beta-1} \frac{\lambda + \alpha\mu}{\nu_\alpha}},$$

In monograph of Failn and Tempelton the differences between some major integral functionals of the input and the truncated system were estimated in case of uncontrolled retrial queues [5]. In case of controlled retrial queues similar results can be found but such an estimate strongly relates on the control policy properties. We have provided such estimates for few of them.

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