FLUID LIMIT FOR CUMULATIVE IDLE TIME IN MULTIPHASE QUEUES

S. Minkevičius[1,2], G. Kulvietis[2]

[1] *Institute of Mathematics and Informatics, Akademijos 4, 08663*

[2] *Vilnius Gediminas Technical University, Sauletekio 11, 10223*

[1], [2] *Vilnius, Lithuania*

[1] stst@ktl.mii.lt, [2] gk@fm.vtu.lt

The object of this research in the queueing theory is the Functional-Strong-Law-of-Large-Numbers (FSLLN) under the conditions of heavy traffic in Multiphase Queueing Systems (MQS). A FSLLN is known as fluid limit or fluid approximation. In this paper, the FSLLN is proved for values of important probabilistic characteristic of the MQS investigated as well as the cumulative idle time of a customer.

Keywords: heavy traffic, multiphase queues.

1. INTRODUCTION

Interest in the field of multiphase queueing systems was stimulated by the theoretical values of the results as well as by their possible applications in information and computing systems, communication networks, and automated technological processes (see, for example, [13]). The methods of investigation of single phase queueing systems are considered in [2], [3], etc. The asymptotic analysis of models of queueing systems in heavy traffic is of special interest (see, for example, [7], [8], [4], [5], etc.). The papers [9], [12] and others desribed the beginning of the investigation of diffusion approximation to queueing networks. Intermediate models - multiphase queueing systems - are considered rarer due to serious technical difficulties (see, for example, book [6]).

We present some definitions in the theory of metric spaces (see, for example, [1]). Let C be a metric space consisting of real continuous functions in [0, 1] with a uniform metric

$$\rho(x,y) = \sup_{0 \le t \le 1} |x(t) - y(t)|, \; x, y \in C.$$

Let D be a space of all real-valued right-continuous functions in [0,1] having left limits and endowed with the Skorokhod topology induced by the metric d (under which D is complete and separable). Also, note that $d(x,y) \le \rho(x,y)$ for $x, y \in D$.

In this paper, we will constantly use an analog of the theorem on converging together (see, for example, [1]):

Theorem 1.

$$\text{Let } \varepsilon > 0 \text{ and } \boldsymbol{X}_n,\ \boldsymbol{Y}_n,\ \boldsymbol{X} \in D. \text{ If } \boldsymbol{P}\left(\lim_{n\to\infty} d(\boldsymbol{X}_n, \boldsymbol{X}) > \varepsilon\right) = 0$$
$$\text{and } \boldsymbol{P}\left(\lim_{n\to\infty} d(\boldsymbol{X}_n, \boldsymbol{Y}_n) > \varepsilon\right) = 0, \text{ then } \boldsymbol{P}\left(\lim_{n\to\infty} d(\boldsymbol{Y}_n, \boldsymbol{X}) > \varepsilon\right) = 0. \tag{1}$$

We investigate here a k-phase multiphase queueing system (i.e., when a customer has been served at the j-th phase of the multiphase queue, he goes to the $j+1$-th phase of the multiphase queue, and after the customer has been served at the k-th phase of the multiphase queue, he leaves the multiphase queue). Let us denote by t_n the time of arrival of the n-th customer, by $S_n^{(j)}$ the service time of the n-th customer at the j-th phase of the multiphase queue, $z_n = t_{n+1} - t_n$; by $\tau_{j,n}$ the departure of the n-th customer after service at the j-th phase of the multiphase queue, $j = 1, 2, \ldots, k$.

Let interarrival times (z_n) at the multiphase queue and service times $(S_n^{(j)})$ at every phase of the multiphase queue for $j = 1, 2, \ldots, k$ be mutually independent identically distributed random variables.

Next, denote by $BI_{j,n}$ the idle time of the n-th customer at the j-th phase of the multiphase queue; $I_{j,n} = \sum_{l=1}^{n} BI_{j,l}$ stands for a cummulative idle time of the n-th customer at the j-th phase of the multiphase queue, $j = 1, 2, \ldots, k$.

Suppose that the idle time of a customer at each phase of the multiphase queue is unlimited, the service principle of customers is "first come, first served" (FCFS). All random variables are defined on the common probability space (Ω, F, P).

We form such a modified multiphase queue in which $BI_{j,n} = 0,\ j = 1, 2, \ldots, k,\ n < k$. Limit distributions for the modified multiphase queue and the usual multiphase queue working in heavy traffic conditions are coincidental (see, for example, [3]). Thus, later on we will investigate only the modified multiphase queue and admit that $n \geq k$.

When $j = 1, 2, \ldots, k$, let us define

$$\delta_{j,n} = \begin{cases} S^{(j)}_{n-(j-1)} - z_n, & \text{if } n \geq k \\ 0, & \text{if } n < k. \end{cases}$$

Denote $S_{j,n} = \sum_{l=1}^{n-1} \delta_{j,l},\ \ S_{0,n} \equiv 0,\ \ \hat{S}_{j,n} = S_{j-1,n} - S_{j,n},\ \ x_{j,n} = \tau_{j,n} - t_n,$
$x_{0,n} \equiv 0,\ \hat{x}_{j,n+1} = x_{j,n} - \delta_{j,n+1},\ \ \hat{x}_{0,n} \equiv 0,\ \ \alpha_j = M(z_n - S_n^{(j)}),\ \ \hat{y}_{j,n} = \hat{x}_{j,n} - S_{j,n},\ \ j = 1, 2, \ldots, k$. Assume $S_{j,0} = 0,\ \ j = 1, 2, \ldots, k$.

Also assume the following condition to be fulfilled:

$$\alpha_k > \alpha_{k-1} > \cdots \alpha_1 > 0. \tag{2}$$

2. MAIN RESULT

At first we present one of the main result of paper - theorem on the FSLLN for the cumulative idle time of a customer in MQS.

Theorem 2. *If conditions (2) are fulfilled, then*

$$\left(\frac{I_{1,n}}{n};\frac{I_{2,n}}{n};\ldots;\frac{I_{k,n}}{n}\right)\Rightarrow(\alpha_1;\alpha_2;\ldots;\alpha_k).$$

Proof. At first we using that for each fixed $\varepsilon>0$ (see [11])

$$P\left(\lim_{n\to\infty}\frac{|I_{j,n}-\hat{y}_{j,n}|}{\sqrt{n}}>\varepsilon\right)=0,\ j=1,2,\ldots,k,\ n\geq k. \tag{3}$$

So,

$$P\left(\lim_{n\to\infty}\frac{|I_{j,n}-\hat{y}_{j,n}|}{n}>\varepsilon\right)=0,\ j=1,2,\ldots,k,\ n\geq k. \tag{4}$$

First we prove that

$$\frac{\hat{y}_{j,n}-\sum\limits_{i=1}^{j}\hat{S}_{i,n}}{n}=\frac{\hat{y}_{j,n}-(-S_{i,n})}{n}\Rightarrow 0,\ j=1,2,\ldots,k,\ n\geq k.$$

Using relations of [11] we obtain that

$$\begin{aligned}\hat{y}_{j,n}&=\hat{x}_{j,n}-S_{j,n}=\max_{0\leq l\leq n}(\hat{x}_{j-1,l}-S_{j,l})=\max_{0\leq l\leq n}(\hat{x}_{j-1,l}-S_{j-1,l}+S_{j-1,l}-S_{j,l})\\&=\max_{0\leq l\leq n}(\hat{y}_{j-1,l}+S_{j,l}),\ j=1,2,\ldots,k,\ n\geq k.\end{aligned} \tag{5}$$

Thus,

$$\hat{y}_{j,n}=\max_{0\leq l\leq n}(\hat{y}_{j-1,l}+S_{j,l}),\ j=1,2,\ldots,k,\ \hat{y}_{0,\cdot}\equiv 0,\ \ n\geq k. \tag{6}$$

Also, we see that (see (6))

$$\begin{aligned}\hat{y}_{j,n}-\sum_{i=1}^{j}\hat{S}_{i,n}&=\max_{0\leq l\leq n}(\hat{y}_{j-1,l}+\hat{S}_{j,l})-\sum_{i=1}^{j}\hat{S}_{i,n}\geq\hat{y}_{j-1,n}+\hat{S}_{j,n}-\sum_{i=1}^{j}\hat{S}_{i,n}\\&=\hat{y}_{j-1,n}-\sum_{i=1}^{j-1}\hat{S}_{i,n}\geq\ldots\geq\hat{y}_{1,n}-\hat{S}_{1,n}=\max_{0\leq l\leq n}\hat{S}_{1,n}-\hat{S}_{1,n}\geq 0,\end{aligned} \tag{7}$$

$j=1,2,\ldots,k,\ n\geq k.$

But

$$\hat{y}_{j,n}\leq\max_{0\leq l\leq n}\hat{y}_{j-1,l}+\max_{0\leq l\leq n}\hat{S}_{j,l}=\hat{y}_{j-1,n}+\max_{0\leq j\leq l}\hat{S}_{j,l}\leq\ldots\leq\sum_{i=1}^{j}\max_{0\leq l\leq n}\hat{S}_{i,l}, \tag{8}$$

$j=1,2,\ldots,k,\ n\geq k.$

Using (7) and (8) we get that

$$0\leq\hat{y}_{j,n}-\sum_{i=1}^{j}\hat{S}_{i,n}\leq\sum_{i=1}^{j}\{\max_{0\leq l\leq n}\hat{S}_{i,l}-\hat{S}_{i,n}\}, \tag{9}$$

$j = 1, 2, \ldots, k,\ n \geq k.$

Applying (9) we achieve for each fixed $\varepsilon > 0$

$$P\left(\frac{|\hat{y}_{j,n} - \sum\limits_{i=1}^{j} \hat{S}_{i,n}|}{n} > \varepsilon\right) = P\left(\frac{\hat{y}_{j,n} - \sum\limits_{i=1}^{j} \hat{S}_{i,n}}{n} > \varepsilon\right)$$

$$\leq P\left(\frac{\sum\limits_{i=1}^{k}\{\max\limits_{0\leq l\leq n} \hat{S}_{i,l} - \hat{S}_{i,n}\}}{n} > \varepsilon\right) \leq P\left(\frac{\sum\limits_{i=1}^{j}\{\max\limits_{0\leq l\leq n} \hat{S}_{i,l} - \hat{S}_{i,n}\}}{n} > \varepsilon\right) \tag{10}$$

$$\leq \sum_{i=1}^{k} P\left(\frac{\max\limits_{0\leq l\leq n} \hat{S}_{i,l} - \hat{S}_{i,n}}{n} > \varepsilon\right) = \sum_{i=1}^{k} P\left(\frac{\max\limits_{0\leq l\leq n}(-\hat{S}_{i,n-l})}{n} > \varepsilon\right),$$

$$= \sum_{i=1}^{k} P\left(\frac{\max\limits_{0\leq l\leq n}(-\hat{S}_{i,l})}{n} > \varepsilon\right) \leq \sum_{i=1}^{k} P\left(\frac{\max\limits_{0\leq l\leq n}(-\hat{S}_{i,l})}{\sqrt{n}} > \varepsilon\right),$$

$j = 1, 2, \ldots, k,\ n \geq k.$

Thus, for each fixed $\varepsilon > 0$

$$P\left(\frac{|\hat{y}_{j,n} - \sum\limits_{i=1}^{j} \hat{S}_{i,n}|}{n} > \varepsilon\right) \leq \sum_{i=1}^{k} P\left(\frac{\max\limits_{0\leq l\leq n}(-\hat{S}_{i,l})}{\sqrt{n}} > \varepsilon\right), \tag{11}$$

$j = 1, 2, \ldots, k,\ n \geq k.$

Note (see, for example, [11]) that for each fixed $\varepsilon > 0$

$$P\left(\lim_{n\to\infty} \frac{\max\limits_{0\leq l\leq n}(-\hat{S}_{j,l})}{\sqrt{n}} > \varepsilon\right) = 0,\ j = 1, 2, \ldots, k, \tag{12}$$

if conditions (2) are fulfilled.

Using relation $\sum\limits_{i=1}^{j} \hat{S}_{i,n} = -S_{j,n},\ j = 1, 2, \ldots, k$, and (11)-(12) we obtain that for each fixed $\varepsilon > 0$,

$$P\left(\lim_{n\to\infty} \frac{|\hat{y}_{j,n} - (-S_{j,n})|}{n} > \varepsilon\right) = 0,\ j = 1, 2, \ldots, k, ,\ n \geq k. \tag{13}$$

So, we get for each fixed $\varepsilon > 0$ (see (13))

$$\begin{aligned} &P\left(\lim_{n\to\infty} \frac{|I_{j,n} - \alpha_j \cdot n|}{n} > \varepsilon\right) \leq P\left(\lim_{n\to\infty} \frac{|I_{j,n} - \hat{y}_{j,n}|}{n} > \frac{\varepsilon}{3}\right) \\ &+ P\left(\lim_{n\to\infty} \frac{|\hat{y}_{j,n} - (-S_{j,n})|}{n} > \frac{\varepsilon}{3}\right) + P\left(\lim_{n\to\infty} \frac{|(-S_{j,n}) - \alpha_j \cdot n|}{n} > \frac{\varepsilon}{3}\right) = 0, \end{aligned} \tag{14}$$

$j = 1, 2, \ldots, k,\ n \geq k.$

Thus, if conditions (2) are fulfilled, then

$$P\left(\lim_{n\to\infty} \frac{|I_{j,n} - \alpha_j \cdot n|}{n} > \varepsilon\right) = 0,\ j = 1, 2, \ldots, k,\ n \geq k. \tag{15}$$

The proof is complete. □

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