INVESTIGATION OF TRAFFIC FLOWS CHARACTERISTICS IN CASE OF THE SMALL DENSITY

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A random number of vehicles crossed a transverse line of a motorway during an arbitrary interval of time and a random number of vehicles situated on an arbitrary part of the motorway at a fixed instant of time form a complicated stochastic dependence. This is indicated by, for example, a functional dependence between the intensity of the traffic and its density[1]. First mathematical model of traffic flow of non-homogeneous vehicles in case of bad weather and bad road conditions is constructed taking into account both spatial and temporal processes.

Keywords: traffic flow, Poisson flow, finite system of Kolmogorov differential equations, limiting probability distribution.

1. INTRODUCTION

There are usually considered such traffic flows in the classical queueing theory, random distances between the neighbouring vehicles in which are independent and all of them have exponential probability distributions. However in practice we often encounter with a situation, when the intervals between moments of crossing a virtual transverse line of the motorway by the consecutive vehicles are dependent and have different probability distributions. In this case constructing and investigation of the models of spacial and temporal traffic flows characteristics offer some significant difficulties. This paper proposes simple mechanism of so-called traffic batch formation. It makes constructing and researching of vehicles spacial location on the motorway possible. Assuming it as a basis we managed to find and ground simple local description of temporal characteristic of stationary traffic flow of non-homogeneous vehicles. By this token it is proved that we can use non-ordinary Poisson process to describe adequately the traffic on the motorway taking into account both its spatial and temporal characteristics. Meanwhile a number of arrivals received in a random manner at every calling instant of crossing a transverse line of the motorway by the vehicles is restricted.

2. MECHANISM OF TRAFFIC BATCH FORMATION AND ITS PROBABILITY PROPERTIES

The effect of some factors, such as bad weather conditions, heterogeneity of vehicles or unsatisfactory conditions of the road surface, make difficulties for the unimpeded motorway traffic. Under these conditions vehicles aren't able to overtake each other freely. So we can observe foundation of traffic groups (batches, motorcades). It doesn't allow us to consider traffic flow as the Poisson flow. Here the necessity of researching of the traffic flow spacial characteristic appears. First the spacial model of traffic flow of homogeneous vehicles was investigated with the usage of the Poisson process in [1]. Basing on the observation over the motorway traffic close to Gorky town [2] the following mechanism of traffic batch formation was proposed.

Every motorcade consists of one slow vehicle at the head of it and some fast vehicles that wait for the possibility of overtaking. That's why traffic batch was proposed to be considered as a service system with varying structure. So every slow vehicle can be interpreted as an servicing device for the fast vehicles. Here the service means overtaking slow vehicle by the fast one. We can take notice, that without slow vehicles fast vehicles move unimpeded enough to suppose their inflow in the motorcade having Poisson distribution. We will designate a random number of vehicles flowed into the the traffic batch for interval of time $[0,t)$ following the Poisson law with an parameter λ_0 as $\eta_0(\omega;t)$. We will consider basic probability space $(\Omega, \mathfrak{F}, \mathbf{P}(\cdot))$ below, where ω means elementary outcome or an element of the certain event Ω. Further we will designate as $\xi(\omega;t,\Delta t)$ a random number of fast vehicles overtook slow one for the interval of time $[t, t+\Delta t)$ and introduce a random variable $\varkappa(\omega;t)$ which counts a number of all vehicles in a traffic batch at the instant of time $t \geq 0$. Let $\varkappa(\omega;t)$ accept the values from the set $\{1, 2, \cdots, N\}$. It is possible if the intensity of overtaking slow vehicles by fast exceeds the intensity of the fast vehicles inflow into the motorcade considerably. In this case relatively small motorcade generates indeed. As far as the average overtaking time depends on a number of vehicles in the batch it is necessary to discriminate the following situations. Let μ_1^{-1} and μ_2^{-1} be an average overtaking time in case when the batch consists of two and three vehicles, respectively. Also let us assume that average time doesn't change if there more than three vehicles in the motorcade and we will designate it as μ_3^{-1}. Let the parameters μ_1, μ_2 and μ_3 term the overtaking intensities in cases mentioned above. We should impose constraint $\lambda_0 < \mu_3$ on the system parameters. This constraint means intensionally such condition when overtaking intensity exceeds the intensity of the vehicles inflow into the motorcade. Taking into account what the vehicles shouldn't be lost and what the batch can't consist of more than $N \geq 4$ vehicles we conjecture the fulfilment of the next constraint. If the fast vehicle catches up with the full batch of N vehicles, it will join the batch nevertheless. But at the same time the fast vehicle following the slow one will certainly overtake it. Now we can define in case of small $\Delta t > 0$ the conditional probability of events generated by the random variable $\xi(\omega; t, \Delta t)$ with the help of the following relations:

$$
\begin{gathered}
\mathbf{P}(\{\omega\colon \xi(\omega;t,\Delta t)=0\}|\{\omega\colon \varkappa(\omega;t)=1, \eta_0(\omega;t,\Delta t)=1\}) = 1 - O(\Delta t), \\
\mathbf{P}(\{\omega\colon \xi(\omega;t,\Delta t)=0\}|\{\omega\colon \varkappa(\omega;t)=2, \eta_0(\omega;t,\Delta t)=0\}) = 1 - \mu_1\Delta t + o(\Delta t), \\
\mathbf{P}(\{\omega\colon \xi(\omega;t,\Delta t)=1\}|\{\omega\colon \varkappa(\omega;t)=2, \eta_0(\omega;t,\Delta t)=0\}) = \mu_1\Delta t - o(\Delta t), \\
\mathbf{P}(\{\omega\colon \xi(\omega;t,\Delta t)=0\}|\{\omega\colon \varkappa(\omega;t)=3, \eta_0(\omega;t,\Delta t)=0\}) = 1 - \mu_2\Delta t + o(\Delta t), \\
\mathbf{P}(\{\omega\colon \xi(\omega;t,\Delta t)=1\}|\{\omega\colon \varkappa(\omega;t)=3, \eta_0(\omega;t,\Delta t)=0\}) = \mu_2\Delta t - o(\Delta t), \\
\mathbf{P}(\{\omega\colon \xi(\omega;t,\Delta t)=0\}|\{\omega\colon \varkappa(\omega;t)=k, \eta_0(\omega;t,\Delta t)=0\}) = 1 - \mu_3\Delta t + o(\Delta t), \\
\mathbf{P}(\{\omega\colon \xi(\omega;t,\Delta t)=1\}|\{\omega\colon \varkappa(\omega;t)=k, \eta_0(\omega;t,\Delta t)=0\}) = \mu_3\Delta t - o(\Delta t), \\
\mathbf{P}(\{\omega\colon \xi(\omega;t,\Delta t)=1\}|\{\omega\colon \varkappa(\omega;t)=N, \eta_0(\omega;t,\Delta t)=1\}) = 1,
\end{gathered} \tag{1}
$$

where $4 \le k \le N$. In the relation (1) symbols $O(\Delta t)$ and $o(\Delta t)$ are infinitesimal of the same order relative Δt and infinitesimal of higher order relative Δt, respectively.

We will designate as $Q(t,k)$ the probability $\mathbf{P}(\{\omega\colon \varkappa(\omega;t) = k\})$ that determines when $k = 1, 2, \ldots, N$ and $t \ge 0$. It should be noticed what the following equality takes place for every chance event of the form $\{\omega\colon \varkappa(\omega; t+\Delta t) = k\}$, $k = 1, 2, \ldots, N$:

$$
\{\omega\colon \varkappa(\omega;t+\Delta t)=k\} = \bigcup_{l=1}^{N}\bigcup_{m=0}^{\infty}\{\omega\colon \varkappa(\omega;t)=l, \eta_0(\omega;t,\Delta t)=m, \xi(\omega;t,\Delta t)=l+m-k\}.
$$

Basing on it and relations derived above we can write down the next equalities:

$$
\begin{gathered}
Q(t+\Delta t,1) = (1-\lambda_0\Delta t)Q(t,1) + \mu_1\Delta t Q(t,2) + o(\Delta t), \\
Q(t+\Delta t,2) = \lambda_0\Delta t Q(t,1) + (1-\lambda_0\Delta t - \mu_1\Delta t)Q(t,2) + \mu_2\Delta t Q(t,3) + o(\Delta t), \\
Q(t+\Delta t,3) = \lambda_0\Delta t Q(t,2) + (1-\lambda_0\Delta t - \mu_2\Delta t)Q(t,3) + \mu_3\Delta t Q(t,4) + o(\Delta t), \\
Q(t+\Delta t,k) = \lambda_0\Delta t Q(t,k-1) + (1-\lambda_0\Delta t - \mu_3\Delta t)Q(t,k) + \mu_3\Delta t Q(t,k+1) + o(\Delta t), \\
k = 1, 2, \ldots, N-1, \\
Q(t+\Delta t,N) = \lambda_0\Delta t Q(t,N-1) + (1-\lambda_0\Delta t - \mu_3\Delta t)Q(t,N) + \lambda_0\Delta t Q(t,N) + o(\Delta t).
\end{gathered}
$$

Proceeding to the limit $\Delta t \to 0$ we receive a system of N linear homogeneous Kolmogorov differential first-order equations with the constant coefficients and Jacobi matrix:

$$
\begin{gathered}
dQ(t,1)/dt = -\lambda_0 Q(t,1) + \mu_1 Q(t,2), \\
dQ(t,2)/dt = \lambda_0 Q(t,1) - (\lambda_0 + \mu_1)Q(t,2) + \mu_2 Q(t,3), \\
dQ(t,3)/dt = \lambda_0 Q(t,2) - (\lambda_0 + \mu_2)Q(t,3) + \mu_3 Q(t,4), \\
dQ(t,k)/dt = \lambda_0 Q(t,k-1) - (\lambda_0 + \mu_3)Q(t,k) + \mu_3 Q(t,k+1), 4 \le k < N, \\
dQ(t,N)/dt = \lambda_0 Q(t,N-1) - \mu_3 Q(t,N).
\end{gathered} \tag{2}
$$

This system describes probability distribution of the traffic batch length dynamics. We can consider the entry conditions of the form $Q(0,k) = \delta_{ik}$, i.e. $Q(0,i) = 1$ and $Q(0,k) = 0$ where $k \ne i$.

Solution of system (2) can be derived with the help of different methods of differential equations theory. However in general case it is too lengthy. Luckily in our case investigation of only some solution properties with tending $t \to \infty$ is really important. Let's designate $\lim_{t\to\infty} Q(t,k) = Q(k)$ for $k = 1, 2, \ldots, N$. According to Markov theorem these limits exist. Then the limits of functions $dQ(t,k)/dt$ exist too. Moreover all of them are equal to 0. Indeed, if such $j \in \{1, 2, \ldots, N\}$ that $\lim_{t\to\infty} \frac{dQ(t,j)}{dt} \neq 0$ had existed we would have gotten the infinite increase of absolute value of quantity $Q(t,j)$. Such increase is impossible because quantity $Q(t,j)$ defines the probability so it is limited. Relying on these reasonings we can pass from differential equations system (2) to system of linear equations (3):

$$\begin{gathered} 0 = -\lambda_0 Q(1) + \mu_1 Q(2), \quad 0 = \lambda_0 Q(1) - (\lambda_0 + \mu_1) Q(2) + \mu_2 Q(3), \\ 0 = \lambda_0 Q(2) - (\lambda_0 + \mu_2) Q(3) + \mu_3 Q(4), \\ 0 = \lambda_0 Q(k-1) - (\lambda_0 + \mu_3) Q(k) + \mu_3 Q(k+1), \quad k = 4,\ 5,\ \ldots,\ N-1, \\ 0 = \lambda_0 Q(N-1) - \mu_3 Q(N). \end{gathered} \tag{3}$$

This system determines so-called ergodic probability distribution $\{Q(k); 1 \le k \le N\}$. It characterizes stationary regime of batches motion. This regime describes intensionally such a situation when after the long-duration interval of time motorcades following one another move along the motorway and don't disturb their structure. Let's take notes that such distribution doesn't depend on the entry conditions and obeys the normalization condition: $\sum_{k=1}^{N} Q(k) = 1$. Expressing $Q(k)$ in terms of $Q(1)$ and substituting derived expressions in the normalization condition we can ascertain that ergodic probability distribution depends on only three essential parameters: $\nu_1 = \frac{\lambda_0}{\mu_1}, \nu_2 = \frac{\lambda_0}{\mu_2}, \nu_3 = \frac{\lambda_0}{\mu_3}$. Finally we derive the distribution of batches motion in the stationary regime:

$$\begin{gathered} Q(1) = (1 + \nu_1 + \nu_1 \nu_2 \frac{\nu_3^{N-2} - 1}{\nu_3 - 1})^{-1}, Q(2) = \nu_1 (1 + \nu_1 + \nu_1 \nu_2 \frac{\nu_3^{N-2} - 1}{\nu_3 - 1})^{-1}, \\ Q(3) = \nu_1 \nu_2 (1 + \nu_1 + \nu_1 \nu_2 \frac{\nu_3^{N-2} - 1}{\nu_3 - 1})^{-1}, \\ Q(k) = \nu_1 \nu_2 \nu_3^{k-3} (1 + \nu_1 + \nu_1 \nu_2 \frac{\nu_3^{N-2} - 1}{\nu_3 - 1})^{-1}, k = 4, 5, \ldots, N. \end{gathered}$$

In particular case when $N = 3$ and $\mu_2 = \mu_3$ system (2) takes on form:

$$\begin{gathered} dQ(t,1)/dt = -\lambda_0 Q(t,1) + \mu_1 Q(t,2), \\ dQ(t,2)/dt = \lambda_0 Q(t,1) - (\lambda_0 + \mu_1) Q(t,2) + \mu_2 Q(t,3), \\ dQ(t,3)/dt = \lambda_0 Q(t,2) - \mu_2 Q(t,3). \end{gathered}$$

Explicit solution of this system can be found in different ways. For example, one of them is adduced in monograph [3]. We should notice that explicit solution is a linear combination of three independent particular solutions of this system. The coefficients of the given combination should be derived from entry conditions. If we proceed to the limit $t \to \infty$ in the explicit expression of system (2) solution, the following equalities will obtain

$$Q(1) = (1 + \nu_1 + \nu_1\nu_2)^{-1}, Q(2) = \nu_1(1 + \nu_1 + \nu_1\nu_2)^{-1},$$
$$Q(3) = \nu_1\nu_2(1 + \nu_1 + \nu_1\nu_2)^{-1}.$$

We would have had the same results if we had proceeded to the limit in the very system as it has been shown above for the whatever N.

3. PROPERTIES OF THE FLOW

We have studied the single traffic batch so now let's return to the whole traffic flow. Watching the light traffic flow in practice we can observe that density of the slow vehicles exceeds density of the fast vehicles appreciably. At the same time the slow vehicles move independently in the stationary mode. Also we consider that on average distance between consecutive slow vehicles is lot more than maximal length of road section occupied by an average number of vehicles in non-homogeneous traffic batch. That's why it is possible to consider that flow of the slow vehicles is the Poisson flow with a parmeter λ and all the vehicles in the batch cross a transverse line in the stationary regime simultaneously. Such a flow is acceptedly called non-ordinary Poisson flow. Let's introduce a random variable $\eta(t) = \eta(\omega; t)$ which counts a number of all vehicles crossed a stop–line in the time interval $[0, t)$ and let's designate $P_k(t) = \mathbf{P}(\eta(t) = k)$. We will think traffic flow is non-ordinary Poisson flow in case when in every calling instant one arrival comes with a probability $p = (1 + \nu_1 + \nu_1\nu_2)^{-1}$, two arrivals – with a probability $q = \nu_1(1 + \nu_1 + \nu_1\nu_2)^{-1}$ and three – with a probability $s = \nu_1\nu_2(1 + \nu_1 + \nu_1\nu_2)^{-1}$.

Theorem 1. *For the probability generating function*

$$\Psi(t, z) = \sum_{k=0}^{\infty} P_k(t) z^k$$

of random variable $\eta(\omega; t)$ distribution the following equality takes place

$$\Psi(t, z) = e^{-\lambda t} \sum_{k=0}^{\infty} z^k \sum_{i=0}^{[\frac{k}{2}]} \sum_{j=0}^{[\frac{k-2i}{3}]} \binom{k-i-2j}{i, j, k-2i-3j} p^{k-2i-3j} q^i s^j \frac{(\lambda t)^{k-i-2j}}{(k-i-2j)!},$$

where $[x]$ means integer part of x.

Theorem 2. *For the one-dimensional distributions $P_k(t)$ the following equalities are true*

$$P_0(t) = e^{-\lambda t}, P_1(t) = \lambda t p e^{-\lambda t},$$

$$P_k(t) = e^{-\lambda t} \sum_{i=0}^{[\frac{k}{2}]} \sum_{j=0}^{[\frac{k-2i}{3}]} \binom{k-i-2j}{i, j, k-2i-3j} p^{k-2i-3j} q^i s^j \frac{(\lambda t)^{k-i-2j}}{(k-i-2j)!}, k = 2, 3, \ldots$$

Lemma 1. *The sum of m independent non-ordinary Poisson flows with parameters λ_j, p_j and q_j (where $j = 1, 2, \dots, m$) is non-ordinary Poisson flow with parameters*

$$\lambda = \sum_{j=0}^{m} \lambda_j, \; p = (\sum_{j=0}^{m} \lambda_j p_j)/(\sum_{j=0}^{m} \lambda_j), \; q = (\sum_{j=0}^{m} \lambda_j q_j)(\sum_{j=0}^{m} \lambda_j).$$

Lemma 2. *For the expectation and dispersion of random variable $\eta(\omega; t)$ the following qualities are true*

$$\mathbf{M}\eta(\omega; t) = \lambda t(1 + q + 2s), \; \mathbf{D}\eta(\omega; t) = \lambda t(1 + 3q + 8s).$$

4. CONCLUSION

Problem concerned construction and investigation of mathematical model of traffic flow spatial disposition on the crossroads-free motorway is solved. The nonlocal description method for the unconditioned flow is used. So not the probabilistic properties of every vehicle random disposition on the motorway but properties of some group of non-homogeneous vehicles random disposition are defined. For this purpose the easy mechanism of small traffic batches formation in case of bad weather and bad road conditions is proposed. It is proved that temporal characteristic of traffic flow stationary motion can be locally described as the non-ordinary Poisson flow with a limited number of the arrivals received in every calling instant. The superposition property of a finite number of such flows is proved. Easy formulas for basic numerical characteristics of such flows are found.

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