LIMIT THEOREMS FOR CLOSED QUEUING NETWORKS WITH EXCESS OF SERVERS

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In this paper limit theorems for closed queuing networks with excess of servers are formulated and proved. First theorem is a variant of the central limit theorem and is proved using classical results of V.I. Romanovskiy for discrete Markov chains. Second theorem considers a convergence to chi square distribution. These theorems are mainly based on an assumption of servers excess in queuing nodes.

Keywords: Closed queuing network, central limit theorem, chi square distribution

1. INTRODUCTION

In $[2, 3]$ problems of a formulation and a proof of central limit theorem for queuing systems and networks are considered. At the international conference "Probability theory and its applications" the author of this manuscript had useful and productive discussion with A.A. Nazarov which stimulated an interest to this problem.

In this paper a model of closed queuing network with an excess of servers in its nodes and with singular service time or geometrically distributed service time is considered. For loads of this model nodes central limit theorem is formulated and is proved using classical results of V.I. Romanovskiy [4] in discrete Markov chains. It is suggested to define parameters of limit normal distribution using Monte-Carlo simulations of single customer motion along nodes of the network. This approach allows to decrease calculation time significantly because it is not necessary to simulate process of loads in nodes of this network.

2. FORMULATION AND PROOF OF MAIN RESULTS

2.1. Central Limit Theorem. In this paper we consider closed queuing network S with the set of nodes $N = \{1, \dots, n\}$ and with m customers. Assume that in each node there are m servers. The network works in discrete time $0, 1, \dots,$ and service time of each customer equals unit. Suppose that $\Theta = ||\theta_{i,j}||_{i,j=1}^{n}$ is route matrix of the network S which satisfies the condition

(A) for any $i, j \in N$ there are $i_1, \dots, i_k \in N$ so that the product $\theta_{i,i_1} \cdot \theta_{i_1,i_2} \cdot \ldots \cdot \theta_{i_{k-1},i_k} \cdot \theta_{i_k,j} > 0$.

Consider discrete Markov chain $x_t,\ t = 0, 1, \dots$ with the set of states N and with the matrix of transition probabilities Θ. From Condition $(\mathbf{A})$ we have that this chain is ergodic [1, chapter XV], denote its limit distribution by $\pi_i,\ i \in N$. This distribution does not depend on initial state x_0 of the chain $x_t,\ t = 0, 1, \dots$ Designate $\tau_i(T) = \sharp\{t:\ x_t = i,\ t = 0, \dots, T\}$ sojourn time of the chain x_t in the state i on time interval $0, \dots, T$. Introduce n - dimensional random vector $\left(\frac{\tau_i(T) - \pi_i T}{\sqrt{T}},\ i \in N\right)$. In [4, chapters IV, V] it is proved that there is n - dimensional and normally distributed random vector $R = (r_i,\ i \in N)$ with zero mean and with covariance matrix $\mathcal{B}$ so that for any real numbers $\tau_1, \dots, \tau_n$ independently on initial state x_0 for $T \to \infty$ we have the tendence

$$P\left(\frac{\tau_i(T) - \pi_i T}{\sqrt{T}} < \tau_i,\ i \in N\right) \to P\left(r_i < \tau_i,\ i \in N\right). \tag{1}$$

Return now to closed queuing network S and enumerate customers of the network by $1, \dots, m$. Denote $x_t^j,\ t \geq 0,\ j = 1, \dots, m$ trajectories of the network S customers along its. These trajectories are independent Markov chains with the set of states N and with matrix of transition probabilities Θ. Assume that $\tau_i^j(T) = \sharp\{t:\ x_t^j = i,\ t = 0, \dots, T\}$ is sojourn time of the customer j in the node i on time interval $0, \dots, T$. Introduce random vectors $\left(\frac{\tau_i^j(T) - \pi_i T}{\sqrt{T}},\ i \in N\right),\ j = 1, \dots, m$, which are independent because trajectories of different customers are independent also. Analogously with Formula (1) obtain for arbitrary real $\tau_1, \dots, \tau_m$ for $T \to \infty$

$$P\left(\frac{\tau_i^j(T) - \pi_i T}{\sqrt{T}} < \tau_i,\ i \in N\right) \to P\left(r_i^j < \tau_i,\ i \in N\right), \tag{2}$$

where n - dimensional random vectors $(r_i^j,\ i \in N),\ j = 1, \dots, m$ are independent and have normal distribution with zero mean and covariance matrix $\mathcal{B}$. Consequently for $T \to \infty$ we have

$$P\left(\sum_{j=1}^{m} \frac{\tau_i^j(T) - \pi_i T}{\sqrt{T}} < \tau_i,\ i \in N\right) \to P\left(R_i < \tau_i,\ i \in N\right), \tag{3}$$

where $(R_i,\ i \in N)$ is n - dimensional and normally distributed randomvector with zero mean and with covariance matrix $m\mathcal{B}$. Formula (3) may be rewritten for $T \to \infty$ as follows

$$P\left(\frac{T_i(T) - m\pi_i T}{\sqrt{T}} < \tau_i,\ i \in N\right) \to P\left(r_i < \sqrt{m}\tau_i,\ i \in N\right). \tag{4}$$

Here random variables $T_i(T) = \sum_{j=1}^{m} \tau_i^j(T),\ i \in N$, designate total loads on nodes $i \in N$ of the network S on time interval $t = 0, \dots, T$. It is clear that $T_i(T) = \sum_{t=0}^{T} m_i(t)$, where $m_i(t)$ is a number of servers of the node i, busy by customers at moment t.

So central limit theorem for discrete Markov chain with finite set of states may be transferred onto random vector $(T_i(T),\ i \in N)$, consisted of loads of the network S nodes. This result may be generalized in different directions. Assume that random service time η_i

in the node i has geometrical distribution $P(\eta_i = k) = (1 - a_i)a_i^{k-1},\ k = 1, 2, \ldots$ Then redefining route matrix Θ by the formulas:

$$\theta_{i,i} := \frac{\theta_{i,i} + a_i}{1 + a_i},\ \theta_{i,j} := \frac{\theta_{i,j}}{1 + a_i},\ j \neq i,\ j \in N, \tag{5}$$

it is possible to obtain results represented by Formula (4). Moreover Formula (4) may be transformed into formula which characterizes loads in some not all nodes $1, \ldots, n_1 < n$ of the network S for $T \to \infty$:

$$P\left(\frac{T_i(T) - m\pi_i T}{\sqrt{T}} < \tau_i,\ 1 \leq i \leq n_1\right) \to P\left(r_i < \sqrt{m}\tau_i,\ 1 \leq i \leq n_1\right). \tag{6}$$

To estimate covariance matrix $\mathcal{B}$ using in Formulas (1) - (4), (6) it is possible to estimate the matrix $||cov(\tau_i(T), \tau_l(T))||_{i,l=1}^{n}$ using Monte-Carlo simulations with sufficiently large T. This estimate is based on independent realizations of Markov chain $x_t,\ t = 0, 1, \ldots, T$. Formulas (4), (6) are constructed for total loads of the network S nodes and are not connected with a motion of single customer.

Consider now an aggregation of nodes in this model. To aggregate nodes of closed queuing network is vary complicated because of difficult symbolic calculations. So it is interesting to consider this problem from a view of the central limit theorem. For a simplicity of a consideration divide the set of nodes $N = \{1, \ldots, n\}$ into two subsets $N_1 = \{1, \ldots, n_1\},\ N_2 = \{n_1 + 1, \ldots, n\},\ 1 \leq n_1 < n$. Then total loads $T^1(T),\ T^2(T)$ on the sets $N_1,\ N_2$ of nodes are defined by the equalities

$$T^1(T) = \sum_{i \in N_1} T_i(T),\ T^2(T) = \sum_{i \in N_2} T_i(T).$$

Consequently covariance matrix $||cov(T^k, (T), T^r(T))||_{k,r=1}^{2}$ may be calculated by covariance matrix $||cov(T_i(T), T_l(T))||_{i,l \in N}$ using simple equalities

$$cov(T^k(T), T^r(T)) = \sum_{i \in N_k,\ k \in N_r} cov(T_i(T), T_k(T)),\ k, r = 1, 2.$$

And in a case of two nodes we have

$$cov(T^1(T), T^1(T)) = cov(T^2(T), T^2(T)) = -cov(T^1(T), T^2(T)) = DT^1(T).$$

Consequently an aggregation of nodes in closed network leads to simple and clear formulas for covariances of loads in aggregated nodes. And central limit theorem for nodes of the network S are transformed into central limit theorem for aggregated nodes of this network: for any real $\tau^1,\ \tau^2$ and $T \to \infty$

$$P\left(\frac{T^k(T) - m\sum_{i \in N_k} \pi_i}{\sqrt{T}} < \tau^k,\ k = 1, 2\right) \to P\left(\sum_{i \in N_k} r_i < \sqrt{m}\tau^k,\ k = 1, 2\right).$$

2.2. Chi Square Limit Distribution. Assume that service times at all nodes are unit and $\pi_i,\ i \in N$, is distribution of ergodic and stationary Markov chain $x_t,\ t \geq 0$. Introduce random variables

$$b_m(o) = \sum_{i=1}^{n} \frac{(\frac{m_i(0)}{m} - \pi_i)^2}{m\pi_i}.$$

Then from [5, chapter III, pp. 169-172] we obtain the following statement. For any positive τ and for $m \to \infty$

$$P(b_m(0) < \tau) \to P(\xi^2_{n-1} < \tau).$$

Here $m_i(0)$ is the number of customers in the node i at the moment 0 and ξ^2_{n-1} is random variable with chi square distribution and with $n-1$ degree of freedom. This statement may be generalized onto arbitrary stationary stochastic sequence $x_t,\ t = 0, 1, \ldots$, with stationary distribution $\pi_i,\ i \in N$.

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