SYSTEM STATE DISTRIBUTIONS IN ONE FINITE SOURCE UNRELIABLE RETRIAL QUEUE

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The object of this paper is to study joint and marginal distributions of the system states at any arbitrary time moment for a single server, finite source retrial queue, in which the server can sustain breakdowns during service times. The server life times as well as the intervals between repetitions are exponentially distributed, while the repair and the service times are generally distributed. Unlike the unreliable model studied by J. Wang et al. [23], in which the interrupted customer waits for the server back from repair, to accomplish the remaining service, in our model this customer goes to the orbit, losing the service time, elapsed before the breakdown occurs.

Keywords: closed queueing system, retrial queue, unreliable server.

1. INTRODUCTION

Retrial queues are widely used to model many real situations in telephone, computer, communication and other systems. The most obvious example of a retrial queue appears in a telephone call, when a person phones and finds the line engaged. Usually in such a situation the subscriber repeats his attempts until he is satisfied. A description of other situations in which retrials arise can be found in [11], and more recent applications of retrial models in [6].

There is a large literature devoted to retrial queues in the past twenty years, but most papers assume that the population of potential customers is very large so that input stream is Poisson. However, in many practical situations, the number of users who access the system is finite. Thus, it is important to take into account the fact that the rate of generation of new calls changes, depending of the number of customers in the system. This can be done with the help of finite - source or quasi - random input models. Finite source retrial models arise in various practical areas as magnetic disk memory systems, [18], a star-like local area networks, [12] and [17], a local area networks with nonpersistent CSMA/CD protocol, [16], telephone networks, [14], etc. A systematic account of the fundamental methods and results on this topic can be found in [4–6, 11] and the literature therein.

Another very important component of queueing models is the reliability of the server/servers. More and more applications of these models in telephone, banking and airline companies depend on the correct and timely operation of the system and require a particular quality of service. Although the reliability study is of great importance, there are only few works that take into consideration retrial phenomenon involving the reliability of the server/servers. Most of them deal with unreliable retrial queues with an infinite customers source [1, 3, 7, 15, 20–22, 24]. Finite source retrial systems with unreliable servers are still an interesting topic. In fact, in the retrial queueing literature we found only a few papers by Sztrik et al. [2, 19] where various types of unreliable systems are studied by using the software MOSEL (Modeling, Specification and Evaluation Language) and a paper of Garbi [13], that studies unreliable retrial systems with several homogeneous servers by means of GSPNs (Generalized Stochastic Petri Nets). To the best of our knowledge the only paper, providing an exhaustive theoretical analysis of a single server finite source retrial queue with unreliable server is the paper [23]. With the help of supplementary variables method combined with discrete transformations approach they investigate the stationary characteristics of the model, including the queue length process, busy period and waiting time process.

In the model, considered by Wang et al., [23] it is assumed that the server sustains breakdowns only when it is busy and immediately after the breakdown is sent for repair. The customer, being served just before the breakdown occurs waits for the server back to accomplish its remained service. In practice there are many other opportunities regarding the breakdown occurrences, the behavior of the server and of the customer, in case of server breakdown.

In present paper we consider a single server unreliable finite source retrial model in which, like in the model studied in [23] breakdowns occur only when the server is busy and after the breakdown the server is immediately sent for repair. But in our model, unlike the model in [23] this customer is sent back to the orbit and repeats his attempts for a new service. The service time, spent before the breakdown is just "lost". For convenience further in the paper we shall call the model studied in [23] "type 1 unreliable model/system" and the model, considered in present paper - "type 2 unreliable model/system".

The main objective of the paper is to establish close/recurrent formulas for computing stationary distributions of the orbit size and the server state and their main characteristics. It is done using the supplementary variables method as well as the discrete transformation method.

2. MODEL DESCRIPTION AND NOTATION

We consider a retrial queueing system of type $M/G/1//N$ in Kendall's notation with server breakdowns. This means that the system has one server, which serves N customers (subscribers), $2 \leq N < \infty$ and each one of them, in his free state produces a Poisson process of demands (calls) with rate λ. These customers are identified as primary customers or sources of primary calls. Thus, when a source is free at time moment t (i.e. is not being served and is not waiting for service) it may generate a primary call during interval $(t,\ t+dt)$ with probability λdt. If the server is free at the instant of a primary call arrival, then the call starts to be served. During the service time the source cannot generate a new primary call. After service the source moves into the free state and can generate a new primary call.

When an arriving primary customer finds the server busy, he leaves the service area and repeats the request after some random time. Between trials a customer is said to be in orbit (queue) or to be a source of secondary calls, a secondary subscriber. Each secondary subscriber generates a Poisson flow of repeated calls with intensity μ until he finds the server free. As in the case of a primary call, after service the source becomes free and can generate a new primary call.

The server has an exponentially distributed lifetime with failure rate α during service processes. When the server breaks down it is sent immediately for repair and the customer being served just before server breakdown goes to the orbit, "losing" the service time, spent before the breakdown. The behavior of the interrupted customer differs our model (as it is called in the introduction "type 2 unreliable model") from the unreliable queueing model studied in [23], "type 1 unreliable model", where the interrupted customer waits for the server back to accomplish his remaining service.

We assume that primary calls, repeated attempts, service times, breakdowns and repairs are mutually independent. The service times are identically distributed both for the primary and secondary subscribers with probability distribution function $G(x)$, $G(0) = 0$, hazard rate function $\gamma(x)$,

$$\gamma(x) = \frac{G'(x)}{1 - G(x)},$$

Laplace - Stieltjes transform $g(s)$ and expected value ν^{-1}. The server repair times follow an arbitrary distribution with distribution function, hazard rate function, Laplace - Stieltjes transform and expected value $B(x)$, $\beta(x)$, $b(s)$ and κ^{-1}, respectively.

The state of the system at time t can be described by the process $\{X(t),\ t \geq 0\} = \{C(t),\ R(t),\ z(t),\ w(t),\ t \geq 0\}$, where $C(t)$ describes the server state,

$$C(t) = \begin{cases} 0, & \text{if the server is free,} \\ 1, & \text{if the server is busy,} \\ 2, & \text{if the server is down (under repair),} \end{cases}$$

$R(t)$ is the number of sources of repeated calls at time t (secondary subscribers, orbit size, queue length), $z(t)$ and $w(t)$ are supplementary variables, equal to the elapsed service time (when $C(t) = 1$) and elapsed repair time (when $C(t) = 2$), respectively. It is obvious that $\{X(t),\ t \geq 0\}$ is a Markov process and since the situation $C(t) = 0,\ R(t) = N$ is impossible, its state space is the set $\{0, 1, 2\} \times \{0, 1, \ldots, N-1\}$.

We define the following probabilities (densities):

$$p_{1n}(x)dx = \lim_{t \to \infty} P\left\{C(t) = 1,\ R(t) = n,\ x \leq z(t) < x + dx\right\}, \tag{1}$$

$$p_{2n}(x)dx = \lim_{t \to \infty} P\left\{C(t) = 2,\ R(t) = n, x \leq w(t) < x + dx\right\}, \tag{2}$$

$$p_{in} = \lim_{t \to \infty} P\left\{C(t) = i, R(t) = n\right\}, \tag{3}$$

$i = 0, 1, 2,\ \ n = 0, 1, \ldots, N-1$ and obviously

$$p_{1n} = \int_0^\infty p_{1n}(x)dx,\ p_{2n} = \int_0^\infty p_{2n}(x)dx.$$

3. DISTRIBUTIONS OF THE ORBIT SIZE AND THE SERVER STATE AT ANY ARBITRARY TIME MOMENT

In this section we derive formulas for computing joint distributions $p_{in}(x), i = 1, 2,\ p_{0n}$ of the orbit size and the server state at any arbitrary time moment, so called an outside observer's distribution. To this end we obtain in a general way the equations of statistical equilibrium,

$$[(N-n)\lambda + n\mu]p_{0n} = \int_0^\infty p_{1n}(x)\gamma(x)dx + \int_0^\infty p_{2n}(x)\beta(x)dx, \tag{4}$$

$$\frac{dp_{1n}(x)}{dx} = -[(N-n-1)\lambda + \alpha + \gamma(x)]p_{1n}(x) + (N-n)\lambda p_{1,n-1}(x), \tag{5}$$

$$\frac{dp_{2n}(x)}{dx} = -[(N-n)\lambda + \beta(x)]p_{2n}(x) + (N-n+1)\lambda p_{2,n-1}(x), \tag{6}$$

with boundary conditions

$$p_{1n}(0) = (N-n)\lambda p_{0n} + (n+1)\mu p_{0,n+1}, \tag{7}$$

$$p_{2n}(0) = \alpha \int_0^\infty p_{1,n-1}(x)dx = \alpha p_{1,n-1}, \tag{8}$$

$$n = 0, 1, \ldots, N-1,\ p_{1,-1}(x) = p_{2,-1}(y) = p_{0,N} = 0.$$

As in most cases of finite queues investigation (see [9,10,23]) we will solve these equations with the help of so-called "discrete transformations" approach. Namely, to simplify the differential equations (5) we introduce variables $q_{1m}(x)$ and q_{im}, "discrete transformations" of $p_{1m}(x)$ and p_{im}, $i = 0, 1$, related with them by the equations

$$p_{1n}(x) = \sum_{m=0}^{n} (-1)^m \binom{N-n-1+m}{m} q_{1,N-n-1+m}(x), \tag{9}$$

$$p_{in} = \sum_{m=0}^{n} (-1)^m \binom{N-n-1+m}{m} q_{i,N-n-1+m}, i = 0, 1. \tag{10}$$

Unfortunately, unlike the reliable model, studied in [10] and the type 1 unreliable model, studied in [23], these transformations cannot simplify to the desired form all equations of statistical equilibrium, especially the differential equations (6). To simplify them we introduce variables $q_{2m}(x)$ and q_{2m}, "discrete transformations" of $p_{2m}(x)$ and p_{2m}, defined by the formulas

$$p_{2n}(x) = \sum_{m=0}^{n} (-1)^m \binom{N-n+m}{m} q_{2,N-n-1+m}(x), \tag{11}$$

$$p_{2n} = \sum_{m=0}^{n} (-1)^m \binom{N-n+m}{m} q_{2,N-n-1+m}. \tag{12}$$

Proposition. *The quantities $q_{1m}(x)$ and $q_{2m}(x)$ are solutions of the following systems differential equations:*

$$\frac{dq_{1m}(x)}{dx} = -[m\lambda + \alpha + \gamma(x)]q_{1m}(x), \tag{13}$$

$$\frac{dq_{2m}(x)}{dx} = -[(m+1)\,\lambda + \beta(x)]q_{2m}(x), \tag{14}$$

with boundary conditions

$$q_{1m}(0) = [(m+1)\lambda + (N-2m-1)\mu]\, q_{0m}$$

$$+ (N-m)\mu q_{0,m-1} + (m+1)(\lambda-\mu)q_{0,m+1}, \tag{15}$$

$$q_{2m}(0) = \alpha \int_0^\infty q_{1,m-1}(x)dx = \alpha q_{1,m-1}, \tag{16}$$

$$m = 0, 1, \ldots, N-1.$$

Equations, relating quantities q_{0m} with $q_{1m}(x)$ and $q_{2m}(x)$ as well as the normalizing condition are respectively:

$$[(m+1)\lambda + (N-m-1)\,\mu]q_{0m} + (m+1)(\lambda-\mu)q_{0,m+1}$$

$$= \int_0^\infty q_{1m}(x)\gamma(x)dx + \sum_{k=m}^{N-1}(-1)^{k-m}\int_0^\infty q_{2k}(x)\beta(x)dx, \tag{17}$$

$$q_{00} + q_{10} + \sum_{m=0}^{N-1}(-1)^m q_{2m} = 1. \tag{18}$$

Here

$$q_{0,-1} = q_{0N} = q_{1,-1} = q_{1,-1}(x) = 0.$$

Solving the system (13) - (18) we obtain formulas for computing quantities $q_{im}(x)$, $m = 1, 2$ and $q_{im}, i = 0, 1, 2$.

Corollary 1. *Quantities $q_{im}(x), i = 1, 2$ and $q_{im}, i = 0, 1, 2$ can be calculated by the formulas*

$$q_{1m}(x) = q_{1m}(0)e^{-(m\lambda+\alpha)x}(1 - G(x)), \tag{19}$$

$$q_{2m}(x) = q_{2m}(0)e^{-(m+1)\lambda x}(1 - B(x)), \tag{20}$$

for $q_{2m}(0)$ it holds

$$q_{2m}(0) = \alpha q_{1,m-1}(0)\frac{1 - g((m-1)\,\lambda + \alpha)}{(m-1)\,\lambda + \alpha}, \tag{21}$$

while $q_{1m}(0)$ and q_{0m} are related by the linear equations

$$[(m+1)\lambda + (N-m-1)\,\mu]q_{0m} + (m+1)(\lambda-\mu)q_{0,m+1} = q_{1m}(0)g(m\lambda+\alpha)$$

$$+\alpha\sum_{k=m}^{N-1}(-1)^{k-m}q_{1,k-1}(0)\frac{1-g((k-1)\lambda+\alpha)}{(k-1)\lambda+\alpha}b((k+1)\lambda) \tag{22}$$

and equations (15)*,* $0\leq m\leq N-1$*. In addition,* $q_{1m}(0)$ *and* q_{0m}*,* $0\leq m\leq N-1$ *have to satisfy the following normalizing condition:*

$$q_{00}+q_{10}(0)(1-g(\alpha))\left(\frac{1}{\alpha}-\frac{b(2\lambda)}{2\lambda}\right)$$

$$-\alpha\sum_{k=1}^{N-2}(-1)^{k}q_{1k}(0)\frac{1-g(k\lambda+\alpha)}{k\lambda+\alpha}\frac{1-b((k+2)\lambda)}{(k+2)\lambda}=1. \tag{23}$$

All of the quantities $q_{im}(x), i=1,2$ *and* $q_{im}, i=0,1,2$ *for which* $m<0$ *or* $m>N-1$ *are equal to 0.*

Thus, the computational difficulty for calculating each of the quantities $q_{im}(x), m=1,2$ and $q_{im}, i=0,1,2$ is to solve the linear equations (15), (22), (23). To solve them we can, for example, eliminate from (15), (22) $q_{1m}(0)$ and get equations of the type:

$$\sum_{k=m-2}^{N-1}Q_{km}q_{0k}=0,\ 1\leq m\leq N-1, Q_{1,-1}=0. \tag{24}$$

From here we express all $q_{0k}, k=0,...,N-3$ in terms of $q_{0,N-1}$ and $q_{0,N-2}$ and then find $q_{0,N-1}$ and $q_{0,N-2}$ from the normalizing condition (23)) and equation (24) for $m=1$.

Corollary 2. *With the help of quantities* q_{0m} *and/or* $q_{1m}(0)$ *we can calculate many of the basic system characteristics, without calculating the probabilities* $p_{im}, i=0,1,2$:

- *The stationary probability that the server is idle,* $P_{0,}$

$$P_0=\sum_{n=0}^{N-1}p_{0n}=q_{00};$$

- *The stationary probability that the server is working,* $P_{1,}$

$$P_1=\sum_{n=0}^{N-1}p_{1n}=q_{10}=q_{10}(0)\frac{1-g(\alpha)}{\alpha};$$

- *The stationary probability that the server is under repair,* $P_{2,}$

$$P_2=\sum_{n=0}^{N-1}p_{2n}=\sum_{m=0}^{N-1}(-1)^m q_{2m}$$

$$=\sum_{l=0}^{N-2}(-1)^{l+1}\alpha q_{1l}(0)\frac{1-g(l\lambda+\alpha)}{l\lambda+\alpha}\frac{1-b((l+2)\lambda)}{(l+2)\lambda};$$

- *The stationary mean orbit size,* $E[R]$*,*

$$E[R]=\sum_{n=0}^{N-1}n\left(p_{0n}+p_{1n}+p_{2n}\right)$$

$$=N-\left(q_{00}+q_{10}+q_{20}+q_{01}+q_{11}\right). \tag{25}$$

4. CONCLUSION

In present paper we establish formulas for computing the stationary distributions of the orbit size and server state at any arbitrary time moment and some of their main characteristics in one finite source, unreliable retrial queue. These formulas are the basis for any further study of the model, like system state distributions at the moments of a primary call arrivals, waiting time process and other. They also allow numerical studies of the system characteristics dependence on the system parameters to be carried out.

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