

Crossed product of a C^* -algebra by an endomorphism, coefficient algebras and transfer operators

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Abstract. The paper presents a construction of the crossed product of a C^* -algebra by an endomorphism generated by partial isometry.

Bibliography: 26 titles.

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§ 1. Introduction

Given a C^* -algebra A and an endomorphism α there are a number of ways to construct a new C^* -algebra (an extension of A) called the crossed product. Among the successful constructions of this sort one should mention, for example, the constructions developed by Cuntz and Krieger [1], [2], Paschke [3], Stacey [4], Murphy [5], Exel [6] and Kwasniewski [7]. Here Exel's crossed product [6] is the most general one since all the others can be reduced to it by means of some or other procedure. At the same time the foregoing statement is 'not completely true'. By saying this we mean the following. Exel's construction (see Definition 4.10) starts with two objects—an endomorphism α of a unital C^* -algebra A and a transfer operator \mathcal{L} satisfying the prescribed relations. The crossed product is then defined as the universal enveloping C^* -algebra generated by A and the operator S that generates the transfer operator and satisfies the given relations. By contrast, if one considers Kwasniewski's crossed product (see § 4.6) then there is no transfer operator among the starting objects. The same is true for the Cuntz-Krieger algebra (see § 4.3). The recent paper by Exel [8] makes the situation even more intriguing (in this paper endomorphism disappears). Exel's crossed product also possesses a certain 'drawback': as is shown by Brownlowe and Raeburn [9] it does not always contain the initial algebra A and therefore it is not always an extension of A . All this means that the natural question: "What is the 'construction' of the crossed product in general?" is still waiting for an answer.

In this article we investigate this question from one more side.

In the paper [10] by Lebedev and Odziejewicz the notion of the so-called coefficient algebra was introduced, and it was shown that this object plays a principal role in the extensions of C^* -algebras by partial isometries. In the paper [11] by Bakhtin and Lebedev the criterion for a C^* -algebra to be a coefficient algebra associated with a given endomorphism was obtained. On the basis of the results of these papers one naturally arrives at the construction of a certain crossed product. This

construction is simpler and ‘more natural’ than Exel’s. Presentation and discussion of this crossed product is the main theme of the article.

The paper is organized as follows.

In § 2 we recall the necessary results on coefficient algebras and transfer operators borrowed from [6], [10] and [11] and introduce the notion of the crossed product. § 3 is devoted to the description of the internal structure of the crossed product introduced. Here, in particular, we give a criterion for a representation to be a faithful representation of the crossed product and present the regular representation. In § 4 we compare the crossed product introduced in this article with the already existing successful constructions. It appears that it covers ‘almost all’ of them, in particular, in the ‘most popular situation’ when all the powers of the transfer operator are generated by partial isometries it coincides with Exel’s crossed product but with different algebra, different endomorphism and different transfer operator. The final § 5 is devoted to the discussion of the ‘general crossed product construction’. Though this section does not contain ‘exact’ results we consider it as being among the important parts of this article since it presents the ‘general’ crossed product ‘philosophy’ (from our point of view).

The paper is a journal version of the e-print [12].

§ 2. Coefficient algebras, transfer operators, crossed product

For the sake of completeness of the presentation we start by recalling some definitions and facts concerning transfer operators and coefficient algebras. The corresponding material is borrowed from [6], [10] and [11].

Let \mathcal{A} be a C^* -algebra with an identity 1 and let $\delta: \mathcal{A} \rightarrow \mathcal{A}$ be an endomorphism of this algebra. A linear map $\delta_*: \mathcal{A} \rightarrow \mathcal{A}$ is called a *transfer operator* for the pair (\mathcal{A}, δ) if it is continuous and positive and such that

$$\delta_*(\delta(a)b) = a\delta_*(b), \quad a, b \in \mathcal{A}. \quad (2.1)$$

Proposition 2.1 ([6], Proposition 2.3). *Let δ_* be a transfer operator for the pair (\mathcal{A}, δ) . Then the following are equivalent:*

- (i) *the composition $E = \delta \circ \delta_*$ is a conditional expectation onto $\delta(\mathcal{A})$;*
- (ii) $\delta \circ \delta_* \circ \delta = \delta$;
- (iii) $\delta(\delta_*(1)) = \delta(1)$.

If the equivalent conditions in Proposition 2.1 hold, then Exel calls δ_* a *non-degenerate* transfer operator.

The transfer operator δ_* is called *complete* if

$$\delta\delta_*(a) = \delta(1)a\delta(1), \quad a \in \mathcal{A}. \quad (2.2)$$

Observe that a complete transfer operator is nondegenerate. Indeed, (2.2) implies

$$\delta\delta_*\delta(a) = \delta(1)\delta(a)\delta(1) = \delta(a)$$

and so condition (ii) in Proposition 2.1 is satisfied.

The next result presents criteria for the existence of a complete transfer operator.

Theorem 2.2 ([11], Theorem 2.8). *Let \mathcal{A} be a C^* -algebra with an identity 1 and let $\delta: \mathcal{A} \rightarrow \mathcal{A}$ be an endomorphism of \mathcal{A} . The following are equivalent:*

- 1) *there exists a complete transfer operator δ_* (for (\mathcal{A}, δ));*
- 2) *there exists a nondegenerate transfer operator δ_* and*
 - (ii) *$\delta(\mathcal{A})$ is a hereditary subalgebra of \mathcal{A} ;*
- 3) *(i) there exists a central orthogonal projection $P \in \mathcal{A}$ such that*
 - a) $\delta(P) = \delta(1)$,
 - b) *the mapping $\delta: P\mathcal{A} \rightarrow \delta(\mathcal{A})$ is a $*$ -isomorphism,*
and
 - (ii) $\delta(\mathcal{A}) = \delta(1)\mathcal{A}\delta(1)$.

Moreover the transfer operator δ_ in 1) and 2), as well as the central projection P in 3) are defined in a unique way and*

$$P = \delta_*(1), \tag{2.3}$$

$$\delta_*(a) = \delta^{-1}(\delta(1)a\delta(1)), \quad a \in \mathcal{A}, \tag{2.4}$$

where $\delta^{-1}: \delta(\mathcal{A}) \rightarrow P\mathcal{A}$ is the inverse mapping to $\delta: P\mathcal{A} \rightarrow \delta(\mathcal{A})$.

Definition 2.3. Let $A \subset L(H)$ be a $*$ -subalgebra containing the identity 1 of $L(H)$ and let $V \in L(H)$. We call A the *coefficient algebra* of the C^* -algebra $C^*(A, V)$ generated by A and V if A and V satisfy the following three conditions:

$$Va = VAV^*V, \quad a \in A; \tag{2.5}$$

$$VaV^* \in A, \quad a \in A; \tag{2.6}$$

and

$$V^*aV \in A, \quad a \in A. \tag{2.7}$$

It was shown in [10] and [11] that instead of (2.5) one can use equivalently the condition

$$V^*V \in Z(A), \tag{2.8}$$

where $Z(A)$ is the centre of A , or the condition

$$V^*VaV^*bV = aV^*bV, \quad a, b \in A. \tag{2.9}$$

It is worth mentioning that conditions (2.5)–(2.7) imply that V is a partial isometry and the mapping $A \ni a \mapsto VaV^*$ is an endomorphism ([10], Proposition 2.2). Thus (recalling (2.1)) we see that a C^* -algebra $A \subset L(H)$ containing the identity of $L(H)$ is the coefficient algebra for $C^*(A, V)$ if and only if the mapping $V \cdot V^*: A \rightarrow A$ is an endomorphism and the mapping $V^* \cdot V: A \rightarrow A$ is a transfer operator for $V \cdot V^*$.

Definition 2.4. Let δ be an endomorphism of an (abstract) unital C^* -algebra \mathcal{A} . We say that the pair (\mathcal{A}, δ) is *finely representable* if there exists a triple (H, π, U) consisting of a Hilbert space H , a faithful nondegenerate representation

$\pi: \mathcal{A} \rightarrow L(H)$ and a linear continuous operator $U: H \rightarrow H$ such that for every $a \in \mathcal{A}$ the following conditions are satisfied:

$$\pi(\delta(a)) = U\pi(a)U^*, \quad U^*\pi(a)U \in \pi(\mathcal{A}), \tag{2.10}$$

$$U\pi(a) = \pi(\delta(a))U, \quad a \in \mathcal{A}. \tag{2.11}$$

That is, $\pi(\mathcal{A})$ is the coefficient algebra for $C^*(\pi(\mathcal{A}), U)$ under the fixed endomorphism $U \cdot U^*$. In this case we also say that \mathcal{A} is a *coefficient algebra associated with δ* .

By the discussion following Definition 2.3, instead of condition (2.11) one can use equivalently the condition

$$U^*U \in Z(\pi(\mathcal{A})) \tag{2.12}$$

or the condition

$$U^*U\pi(a)U^*\pi(b)U = \pi(a)U^*\pi(b)U, \quad a, b \in \mathcal{A}. \tag{2.13}$$

In particular it is clear that the finely representable pair (\mathcal{A}, δ) can also be defined as a pair such that there exists a triple (H, π, U) where $\pi: \mathcal{A} \rightarrow L(H)$ is a faithful nondegenerate representation, $U \in L(H)$ and the mapping $U \cdot U^*$ coincides with the endomorphism δ on $\pi(\mathcal{A})$ while the mapping $U^* \cdot U$ is a transfer operator for δ .

Since δ is an endomorphism it follows that $\delta(1)$ is a projection, and since (2.10) implies that UU^* is a projection, so U is a partial isometry.

Theorem 2.5 ([11], Theorem 3.1). *A pair (\mathcal{A}, δ) is finely representable if and only if there exists a complete transfer operator δ_* .*

Definition 2.6. Let (\mathcal{A}, δ) be a finely representable pair. The *crossed product* of A and δ (which we denote by $\mathcal{A} \times_\delta \mathbb{Z}$ or simply $\mathcal{A} \times \mathbb{Z}$ when δ is clear) is the universal unital C^* -algebra generated by a copy of \mathcal{A} and a partial isometry U subject to the relations

$$\delta(a) = UaU^*, \quad \delta_*(a) = U^*aU, \quad a \in \mathcal{A}, \tag{2.14}$$

where δ_* is the complete transfer operator for (\mathcal{A}, δ) (this δ_* does exist by Theorem 2.5 and is unique by Theorem 2.2). The algebra \mathcal{A} will be called the *coefficient algebra* for $\mathcal{A} \times_\delta \mathbb{Z}$.

Remark 2.7. The reason why we use \mathbb{Z} in the notation for the crossed product but not \mathbb{N} (as in a number of other sources) will be uncovered in the next section.

Theorems 2.2 and 2.5 imply the nondegeneracy of Definition 2.6 (there exists a nonzero representation for $\mathcal{A} \times_\delta \mathbb{Z}$ and \mathcal{A} is a C^* -subalgebra of $\mathcal{A} \times_\delta \mathbb{Z}$). The further investigation of the structure of $\mathcal{A} \times_\delta \mathbb{Z}$ is presented in the next section.

§ 3. Faithful and regular representations of the crossed product

This section is devoted to the description of the internal structure of $\mathcal{A} \times_\delta \mathbb{Z}$. Among the main technical results here are the results of [10] and [11].

3.1. Property (*). Let \widehat{a} and \widehat{U} be the canonical images of $a \in \mathcal{A}$ and U in $\mathcal{A} \times \mathbb{Z}$, respectively. Note that we can identify \widehat{a} with a which justifies the use of the notations $\delta(\widehat{a})$ and $\delta_*(\widehat{a})$.

By Definition 2.4 and the definition of the crossed product we have that $\widehat{\mathcal{A}}$ is a coefficient algebra of $\mathcal{A} \times_\delta \mathbb{Z} = C^*(\widehat{\mathcal{A}}, \widehat{U})$ (see Definition 2.3).

Proposition 3.1 ([10], Proposition 2.3). *Let A be the coefficient algebra of $C^*(A, V)$. Then the vector space B_0 consisting of finite sums*

$$x = V^{*N}a_{-N} + \dots + V^*a_{-1} + a_0 + a_1V + \dots + a_NV^N, \tag{3.1}$$

where $a_k \in A$ and $N \in \mathbb{N} \cup \{0\}$, is a dense $*$ -subalgebra of $C^*(A, V)$.

Since any C^* -algebra can faithfully be represented as a C^* -subalgebra of operators acting in some Hilbert space Proposition 3.1 implies the next

Proposition 3.2. *The vector space C_0 consisting of finite sums*

$$x = \widehat{U}^{*N}\widehat{a}_{-N} + \dots + \widehat{U}^*\widehat{a}_{-1} + \widehat{a}_0 + \widehat{a}_1\widehat{U} + \dots + \widehat{a}_N\widehat{U}^N, \tag{3.2}$$

where $a_k \in \mathcal{A}$ and $N \in \mathbb{N} \cup \{0\}$, is a dense $*$ -subalgebra of $\mathcal{A} \times_\delta \mathbb{Z}$.

Definition 3.3. We say that the algebra $C^*(A, V)$ mentioned in Definition 2.3 possesses property (*) if for any $x \in B_0$ (given by (3.1)) the inequality

$$\|a_0\| \leq \|x\| \tag{3.3}$$

holds.

Remark 3.4. Observe that $\mathcal{A} \times_\delta \mathbb{Z} = C^*(\widehat{\mathcal{A}}, \widehat{U})$ possesses property (*). Indeed, take any faithful nondegenerate representation $\pi: \mathcal{A} \rightarrow L(H)$ and a partial isometry $U: H \rightarrow H$ mentioned in Definition 2.4. Consider the space $\mathcal{H} = l^2(\mathbb{Z}, H)$ and the representation $\nu: C^*(\widehat{\mathcal{A}}, \widehat{U}) \rightarrow L(\mathcal{H})$ given by the formulae

$$\begin{aligned} (\nu(\widehat{a})\xi)_n &= \pi(a)(\xi_n), \quad \text{where } a \in \mathcal{A}, \quad l^2(\mathbb{Z}, H) \ni \xi = \{\xi_n\}_{n \in \mathbb{Z}}; \\ (\nu(\widehat{U})\xi)_n &= U(\xi_{n-1}), \quad (\nu(\widehat{U}^*)\xi)_n = U^*(\xi_{n+1}). \end{aligned}$$

Routine verification shows that $\nu(\widehat{\mathcal{A}})$ and $\nu(\widehat{U})$ satisfy all the conditions mentioned in Definition 2.4 (for $\pi(\mathcal{A})$ and U).

Now take any $x \in C^*(\widehat{\mathcal{A}}, \widehat{U})$ given by (3.2) and for a given $\varepsilon > 0$ choose a vector $\eta \in H$ such that

$$\|\eta\| = 1, \quad \|\pi(a_0)\eta\| > \|\pi(a_0)\| - \varepsilon. \tag{3.4}$$

Set $\xi \in l^2(\mathbb{Z}, H)$ to be defined by $\xi_n = \delta_{0n}\eta$, where δ_{ij} is the Kronecker symbol. We have $\|\xi\| = 1$ and the explicit form of $\nu(x)\xi$ and (3.4) imply

$$\|\nu(x)\xi\|^2 \geq \|\pi(a_0)\eta\|^2 > (\|\pi(a_0)\| - \varepsilon)^2,$$

which by the arbitrariness of ε proves the desired inequality

$$\|x\| \geq \|\pi(a_0)\| = \|\widehat{a}_0\|.$$

Remark 3.4 provides us with the possibility to exploit all the main results of [10] to uncover the structure of the crossed product and we start to do this.

The next theorem shows the crucial value of property (*) in the crossed product: this property is a criterion for a representation of the crossed product to be faithful.

Theorem 3.5. *Let \mathcal{A} be a C^* -algebra with an identity 1 and let $\delta: \mathcal{A} \rightarrow \mathcal{A}$ be an endomorphism such that there exists a complete transfer operator δ_* for (\mathcal{A}, δ) and let (H, π, U) be a triple mentioned in Definition 2.4. Then the map*

$$\Phi(\widehat{U}) = U, \quad \Phi(\widehat{a}) = \pi(a), \quad a \in \mathcal{A}, \tag{3.5}$$

gives rise to the isomorphism between the algebras $\mathcal{A} \times_\delta \mathbb{Z} = C^(\widehat{\mathcal{A}}, \widehat{U})$ and $C^*(\pi(\mathcal{A}), U)$ if and only if the algebra $C^*(\pi(\mathcal{A}), U)$ possesses property (*).*

This follows from Remark 3.4 and [10], Theorem 2.13.

Remark 3.6. The representation $\nu: C^*(\widehat{\mathcal{A}}, \widehat{U}) \rightarrow L(\mathcal{H})$ described in Remark 3.4 possesses property (*) so it is a faithful representation of $\mathcal{A} \times_\delta \mathbb{Z}$.

Remark 3.7. It is worth mentioning that the value of property (*) in the theory of crossed products of C^* -algebras by discrete groups (semigroups) of automorphisms (endomorphisms) has been observed by a number of authors along with the proofs of the results of the type of Theorem 3.5. (Probably) for the first time the importance of property (*) was clarified by O’Donovan [13] in connection with the description of C^* -algebras generated by weighted shifts. The most general result establishing the crucial role of this property in the theory of crossed products of C^* -algebras by discrete groups of automorphisms was obtained in [14] (see also [15], Chs. 2, 3 for complete proofs and various applications) for an arbitrary C^* -algebra and amenable discrete group. The relation of the corresponding property to faithful representations of crossed products by endomorphisms generated by isometries was investigated in [16] and [17]. The role of properties of (*) type in the theory of algebras of Fell bundles was studied in [18]. Properties of this sort proved to be of great value not only in pure C^* -theory but also in various applications such as, for example, the construction of symbolic calculus and developing the solvability theory of functional differential equations (see [19] and [20]). We shall also exploit this property heavily in §3.2.

3.2. Regular representation of the crossed product. The representation mentioned in Remark 3.6 being faithful is not defined explicitly (in terms of $\mathcal{A}, \delta, \delta_*$), in fact we have only established its existence. Now we shall present a faithful representation of $\mathcal{A} \times_\delta \mathbb{Z}$ that will be written explicitly in terms of $\mathcal{A}, \delta, \delta_*$. Keeping in mind the standard regular representations for the various known versions of crossed products it is reasonable to call it the *regular representation of $\mathcal{A} \times_\delta \mathbb{Z}$* . In fact the construction of this representation has been obtained in the proof of Theorem 3.1 in [11].

First we construct the desired Hilbert space H by means of the elements of the initial algebra \mathcal{A} in the following way. Let $\langle \cdot, \cdot \rangle$ be a certain nonnegative inner product on \mathcal{A} (differing from the ordinary inner product only in that for certain nonzero elements $v \in \mathcal{A}$ the expression $\langle v, v \rangle$ may be equal to zero). For example, this inner product may have the form $\langle v, u \rangle = f(u^*v)$ where f is some positive linear

Equalities (3.9) and (3.10) guarantee that the operators U and U^* are well defined (that is, they preserve factorization and completion by means of which the spaces H_n^f were built from the algebra \mathcal{A}) and are mutually adjoint.

Now let us define the representation $\pi: \mathcal{A} \rightarrow L(H)$. For any $a \in \mathcal{A}$ the operator $\pi(a): H \rightarrow H$ will leave invariant all the subspaces $H^f \subset H$ and also all the subspaces $H_n^f \subset H^f$. If $h_n \in H_n^f$ then we set

$$\pi(a)h_n = \begin{cases} ah_n, & n \geq 0, \\ \delta^{|n|}(a)h_n, & n \leq 0. \end{cases} \tag{3.13}$$

The scheme of the action of the operator $\pi(a)$ is presented in the second line of the diagram given above.

In the proof of Theorem 3.1 in [11] it was verified that the triple (H, π, U) described above satisfies all the conditions of Definition 2.4 and property (*) here can be proved in the same way as in Remark 3.4.

3.3. Coefficients of elements of the crossed product. The results of [10] give us an opportunity to say more on the structure and properties of elements in $\mathcal{A} \times_\delta \mathbb{Z}$ and henceforth we present these properties.

Since $\widehat{U} \cdot \widehat{U}^*: \widehat{\mathcal{A}} \rightarrow \widehat{\mathcal{A}}$ is an endomorphism it follows that for any $k \in \mathbb{N}$ the map $\widehat{U}^k \cdot \widehat{U}^{*k}: \widehat{\mathcal{A}} \rightarrow \widehat{\mathcal{A}}$ is an endomorphism as well and therefore $\widehat{U}^k \widehat{U}^{*k}$ is a projection and thus $\widehat{U}^k, k \in \mathbb{N}$, is a partial isometry. So $\widehat{U}^k = \widehat{U}^k \widehat{U}^{*k} \widehat{U}^k$ and $\widehat{U}^{*k} = \widehat{U}^{*k} \widehat{U}^k \widehat{U}^{*k}$. Therefore one can always choose coefficients \widehat{a}_k and \widehat{a}_{-k} of (3.2) in such a way that

$$\widehat{a}_k \widehat{U}^k \widehat{U}^{*k} = \widehat{a}_k \quad \text{and} \quad \widehat{U}^k \widehat{U}^{*k} \widehat{a}_{-k} = \widehat{a}_{-k}. \tag{3.14}$$

The forthcoming proposition shows that the assumption that $\widehat{a}_k, \widehat{a}_{-k} \in \widehat{\mathcal{A}}, k = 1, \dots, N$, satisfy (3.14) guarantees their uniqueness in the expansion in (3.2), and moreover, any element of $\mathcal{A} \times_\delta \mathbb{Z}$ can be determined in a unique way by means of these coefficients (this is the subject of Theorem 3.10).

Proposition 3.8. *If the coefficients of x in (3.2) satisfy (3.14) then*

$$\|\widehat{a}_k\| = \|a_k\| \leq \|x\|, \quad \|\widehat{a}_{-k}\| = \|a_{-k}\| \leq \|x\| \tag{3.15}$$

for $k \in \{0, 1, \dots, N\}$. In particular, these coefficients are uniquely defined.

This follows from [10], Proposition 2.6 since $\mathcal{A} \times_\delta \mathbb{Z}$ possesses property (*).

Proposition 3.8 means that one can define the continuous linear maps $\mathcal{N}_k: C_0 \rightarrow \mathcal{A}$ and $\mathcal{N}_{-k}: C_0 \rightarrow \mathcal{A}, k \in \mathbb{N}$:

$$\mathcal{N}_k(x) = a_k \in \mathcal{A}U^kU^{*k} \subset \mathcal{A}, \quad \mathcal{N}_{-k}(x) = a_{-k} \in U^kU^{*k}\mathcal{A} \subset \mathcal{A}. \tag{3.16}$$

By continuity these maps can be extended to the whole of $\mathcal{A} \times_\delta \mathbb{Z}$ thus defining the ‘coefficients’ of an arbitrary element $x \in \mathcal{A} \times_\delta \mathbb{Z}$.

Theorem 3.9 presented below shows that the norm of an element $x \in C_0$ can be calculated only in terms of elements of \mathcal{A} (the 0-degree coefficients of various powers of xx^*).

Theorem 3.9. *For any element $x \in C_0$ of the form (3.2) we have*

$$\|x\| = \lim_{k \rightarrow \infty} \sqrt[4k]{\|\mathcal{N}_0[(xx^*)^{2k}]\|}, \tag{3.17}$$

where \mathcal{N}_0 is the mapping defined by (3.16).

This follows from [10], Theorem 2.11.

We finish this section with the statement showing that the mapping \mathcal{N}_0 is a conditional expectation from $\mathcal{A} \times_\delta \mathbb{Z}$ onto \mathcal{A} and any element $x \in \mathcal{A} \times_\delta \mathbb{Z}$ can be ‘recovered’ from its coefficients $\mathcal{N}_k(x)$ and $\mathcal{N}_{-k}(x)$.

Theorem 3.10. *Let $x \in \mathcal{A} \times_\delta \mathbb{Z}$. Then the following conditions are equivalent:*

- (i) $x = 0$;
- (ii) $\mathcal{N}_k(x) = 0, k \in \mathbb{Z}$;
- (iii) $\mathcal{N}_0(x^*x) = 0$.

This follows from [10], Theorem 2.15.

§ 4. Various crossed products

In this section we list a number of successful constructions of crossed products associated with some or other sort of C^* -algebras and endomorphism and discuss their interrelations with the crossed product introduced.

4.1. Monomorphisms with hereditary range. We start with the crossed product under special hypotheses which have been considered in [5].

Let \mathcal{A} be a C^* -algebra with an identity 1 and let $\delta: \mathcal{A} \rightarrow \mathcal{A}$ be a monomorphism with hereditary range. We shall denote by $\mathcal{U}(\mathcal{A}, \delta)$ the universal unital C^* -algebra generated by \mathcal{A} and an isometry T subject to the relation

$$\delta(a) = TaT^*, \quad a \in \mathcal{A}. \tag{4.1}$$

This $\mathcal{U}(\mathcal{A}, \delta)$ was proposed in [5] as the definition for the crossed product of \mathcal{A} by δ .

Since $\delta(\mathcal{A})$ is a hereditary subalgebra of \mathcal{A} it follows ([6], Proposition 4.1) that $\delta(\mathcal{A}) = \delta(1)\mathcal{A}\delta(1)$. So we find ourselves under the conditions of 3) of Theorem 2.2 with $P = 1$. Therefore for the endomorphism δ under consideration there is a unique complete transfer operator δ_* given by (2.4). Thus we can take the crossed product $\mathcal{A} \times_\delta \mathbb{Z}$. Definition 2.6 of the crossed product $\mathcal{A} \times_\delta \mathbb{Z}$, the universal property of $\mathcal{U}(\mathcal{A}, \delta)$ and (4.1) imply that $\mathcal{A} \times_\delta \mathbb{Z}$ is a representation of $\mathcal{U}(\mathcal{A}, \delta)$ and since \mathcal{A} is embedded in $\mathcal{A} \times_\delta \mathbb{Z}$ it is embedded in $\mathcal{U}(\mathcal{A}, \delta)$ as well. We shall therefore view \mathcal{A} as a subalgebra of $\mathcal{U}(\mathcal{A}, \delta)$.

Proposition 4.1. *Let \mathcal{A} be a C^* -algebra with an identity 1 and let $\delta: \mathcal{A} \rightarrow \mathcal{A}$ be a monomorphism with hereditary range. Then the mapping*

$$\varphi: \mathcal{U}(\mathcal{A}, \delta) \rightarrow \mathcal{A} \times_\delta \mathbb{Z}$$

such that $\varphi(T) = \widehat{U}$ and $\varphi(a) = \widehat{a}$ for all $a \in \mathcal{A}$, where \widehat{a} and \widehat{U} are the canonical images of $a \in \mathcal{A}$ and U in $\mathcal{A} \times_\delta \mathbb{Z}$, respectively, is a $*$ -isomorphism.

Proof. The argument preceding the proposition implies that φ is a $*$ -epimorphism and to finish the proof it is enough to observe that

$$T^*aT = \delta_*(a), \quad a \in \mathcal{A}.$$

But this follows from (4.1), (2.4), equality $T^*T = 1$ and the relations

$$T^*aT = T^*TT^*aTT^*T = T^*\delta(1)a\delta(1)T = T^*\delta\delta_*(a)T = T^*T\delta_*(a)T^*T = \delta_*(a).$$

4.2. Partial crossed product. The notion of the partial crossed product of a C^* -algebra by a group \mathbb{Z} of partial automorphisms was introduced by Exel [21]; it was generalized by McClanahan [22] to the crossed product of a C^* -algebra by partial actions of discrete groups. Faithful representations of these crossed products were discussed in [23].

Within the framework of the present article it is natural to confine ourselves to the group \mathbb{Z} . Let us recall the notion of the partial crossed product [21].

Definition 4.2. A *partial automorphism* of a C^* -algebra A is a triple $\Theta = (\theta, I, J)$ where I and J are closed two sided ideals in A and $\theta: I \rightarrow J$ is a $*$ -isomorphism.

Given a partial isomorphism one can consider, for each integer n , the set D_n , the domain of θ^{-n} (which is equal to the image of θ^n). By convention we put $D_0 = A$ and θ^0 is the identity automorphism of A .

For each integer n , D_n is an ideal in A ([21], Proposition 3.2).

For the sake of convenience of the presentation we give hereafter the definition of the partial crossed product which differs from Exel’s original definition but in fact is equivalent to it (cf. [21], §§ 3, 5).

The *partial crossed product* for $\Theta = (\theta, I, J)$ is the universal enveloping C^* -algebra $C^*(A, \Theta)$ generated by the finite sums

$$a_{-n}V^{*n} + \dots + a_{-1}V^* + a_0 + a_1V + \dots + a_nV^n, \quad n \in \mathbb{N}, \quad (4.2)$$

where $a_i \in D_i$, $i \in \mathbb{Z}$ and V is a partial isometry such that

- (i) the initial space for V is the closure of $I\mathcal{H}$ and the final space for V is the closure of $J\mathcal{H}$ (here \mathcal{H} is the space where A acts);
- (ii) $VaV^* = \theta(a)$, $a \in I$.

Remark 4.3. Condition (i) implies in particular that both the projections V^*V and VV^* belong to the commutant of A . In addition conditions (i) and (ii) imply that $V^*aV = \theta^{-1}(a)$, $a \in J$.

The next proposition describes the conditions on a coefficient algebra under which the crossed product introduced in this article is a partial crossed product.

Proposition 4.4. *Let δ_* be a complete transfer operator for (\mathcal{A}, δ) and $\delta(1) \in Z(\mathcal{A})$; then*

- 1) *the triple $\Theta = (\delta, \delta_*(1)\mathcal{A}, \delta(1)\mathcal{A})$ is a partial automorphism;*
- 2) *$\delta_*: \mathcal{A} \rightarrow \mathcal{A}$ is an endomorphism, and*
- 3) *the mapping*

$$\phi: C^*(\mathcal{A}, \Theta) \rightarrow \mathcal{A} \times_{\delta} \mathbb{Z}$$

such that $\varphi(V) = \widehat{U}$ and $\varphi(a) = \widehat{a}$ for all $a \in \mathcal{A}$, where \widehat{a} and \widehat{U} are the canonical images of $a \in \mathcal{A}$ and U in $\mathcal{A} \times_{\delta} \mathbb{Z}$, respectively, is a $$ -isomorphism.*

Proof. 1) and 2) are proved in [11], Proposition 2.9.

By applying equality (2.2) and the condition $\delta(1) \in Z(\mathcal{A})$, for any $a, b \in \mathcal{A}$ we obtain

$$\delta(\delta_*(a)b) = \delta\delta_*(a)\delta(b) = \delta(1)a\delta(1)\delta(b) = a\delta(b).$$

This means that δ is a transfer operator for δ_* . Therefore for every $n \in \mathbb{N}$ we have $\delta^n(1) \in Z(\mathcal{A})$ and $\delta_*^n(1) \in Z(\mathcal{A})$, and it is easy to see that $D_n = \mathcal{A}\delta^n(1)$ and $D_{-n} = \mathcal{A}\delta_*^n(1)$ (here the D_n are the ideals mentioned in Definition 4.2).

Now 3) follows from the definition of $C^*(\mathcal{A}, \Theta)$ along with the definition of $\mathcal{A} \times_{\delta} \mathbb{Z}$ and Proposition 3.2 (here one should note that since $\delta^n(1) = \widehat{U}^n \widehat{U}^{*n} \in Z(\widehat{\mathcal{A}})$ we have

$$\begin{aligned} \widehat{U}^{*n} \widehat{a} &= \widehat{U}^{*n} \widehat{U}^n \widehat{U}^{*n} \widehat{a} = \widehat{U}^{*n} \widehat{a} \widehat{U}^n \widehat{U}^{*n} = \delta_*^n(\widehat{a}) \widehat{U}^{*n} \widehat{U}^n \widehat{U}^{*n} = \delta_*^n(\widehat{a}) \delta_*^n(1) \widehat{U}^{*n}, \\ \widehat{a} \widehat{U}^n &= \widehat{a} \widehat{U}^n \widehat{U}^{*n} \widehat{U}^n = \widehat{a} \delta^n(1) \widehat{U}^n, \end{aligned}$$

and therefore the form of (3.2) coincides with the form of (4.2)).

Proposition 4.4 is ‘almost invertible’. That is, by a ‘slight’ (natural) extension of a partial automorphism one can always obtain a coefficient algebra satisfying the conditions of Proposition 4.4.

Indeed, let A be a unital algebra and let $\Theta = (\theta, I, J)$ be the triple mentioned in Definition 4.2. Set $A_1 = \{A, VV^*, V^*V\}$ to be the enveloping C^* -algebra generated by A, VV^* and V^*V (V is the partial isometry mentioned in the definition of the partial crossed product). The sets VV^*A and V^*VA are ideals in A_1 and the isomorphism $\theta: I \rightarrow J$ extends to the isomorphism

$$\tilde{\theta}: V^*VA \rightarrow VV^*A.$$

Evidently the mapping

$$\delta: A_1 \rightarrow A_1, \quad \delta(\cdot) = \tilde{\theta}(V^*V\cdot),$$

is an endomorphism of A_1 and the mapping

$$\delta_*: A_1 \rightarrow A_1, \quad \delta_*(\cdot) = \tilde{\theta}^{-1}(VV^*\cdot),$$

is a complete transfer operator for (A_1, δ) .

Since $\delta(1) = VV^* \in Z(A_1)$ it follows that (A_1, δ, δ_*) satisfy all the conditions of Proposition 4.4.

Thus by a slight abuse of language one can say that the partial crossed product is the crossed product introduced in this article under the additional condition $\delta(1) \in Z(\mathcal{A})$.

4.3. Cuntz-Krieger algebras. Throughout this subsection we shall let A be an $n \times n$ matrix with $A(i, j) \in \{0, 1\}$ for all i and j , and such that no row and no column of A is identically zero. The Cuntz-Krieger algebra \mathcal{O}_A (see [2]) is a C^* -algebra generated by partial isometries $S_i, i \in 1, \dots, n$, that act on a Hilbert space in such a way that their support projections $Q_i = S_i^* S_i$ and their range projections $P_i = S_i S_i^*$ satisfy the relations

$$P_i P_j = 0, \quad i \neq j; \quad Q_i = \sum_{r=1}^n A(i, r) P_r, \quad i, j \in 1, \dots, n. \tag{4.3}$$

Algebras of this sort arise naturally as the objects associated with topological Markov chains and serve as a source of inspiration for numerous investigations.

We shall show that \mathcal{O}_A can naturally be considered as a certain crossed product of the type introduced.

The sum of the range projections P_i is a unit in \mathcal{O}_A . If $\mu = (i_1, \dots, i_k)$ is a multi-index with $i_j \in 1, \dots, n$, we denote by $|\mu|$ the length k of μ and write $S_\emptyset = 1$, $S_\mu = S_{i_1} S_{i_2} \cdots S_{i_k}$ (\emptyset is also considered as a multi-index). It is shown in [2] that all S_μ are partial isometries.

We recall in this connection that the product of two partial isometries is not necessarily a partial isometry. The criterion for the product of partial isometries to be a partial isometry is given in the next proposition.

Proposition 4.5 ([24], Lemma 2). *Let S and T be partial isometries. Then ST is a partial isometry if and only if S^*S commutes with TT^* .*

The symbols P_μ and Q_μ will stand for the range and support projections of S_μ , respectively. The foregoing observation means that for any two given multi-indices μ and ν the projections P_μ and Q_ν commute.

In addition by Lemma 2.1 in [2], for $|\mu| = |\nu|$ we have

$$S_\mu^* S_\nu \neq 0 \implies \mu = \nu, \quad S_\mu^* S_\nu = Q_\mu = Q_{i_k}, \quad \mu = (i_1, \dots, i_k), \tag{4.4}$$

which implies

$$P_\mu P_\nu = \delta_{\mu, \nu} P_\mu, \tag{4.5}$$

and since $\sum_i P_i = 1$, it also follows that

$$\sum_{|\mu|=k} P_\mu = 1. \tag{4.6}$$

Let \mathcal{F}_A be the C^* -algebra generated by all the elements of the form $S_\mu P_i S_\nu^*$ where $|\mu| = |\nu| = k$, $k = 0, 1, \dots$; $i \in 1, \dots, n$. Clearly,

$$S_i \mathcal{F}_A S_i^* \subset \mathcal{F}_A, \quad i \in 1, \dots, n, \tag{4.7}$$

and it is also shown in [2], Lemma 2.2 that

$$S_i^* \mathcal{F}_A S_i \subset \mathcal{F}_A, \quad i \in 1, \dots, n. \tag{4.8}$$

Moreover, [2], Proposition 2.8 asserts that any element X of the $*$ -algebra \mathcal{O}_A generated by the S_i , $i \in 1, \dots, n$, can be written as a finite sum

$$X = \sum_{|\nu| \geq 1} X_\nu S_\nu^* + X_0 + \sum_{|\mu| \geq 1} S_\mu X_\mu, \tag{4.9}$$

where $X_\nu, X_0, X_\mu \in \mathcal{F}_A$ and for X in (4.9) the next inequality is true:

$$\|X\| \geq \|X_0\|. \tag{4.10}$$

For every projection P_i , $i \in 1, \dots, n$, we set

$$\alpha(P_i) = \sum_{j=1}^n S_j P_i S_j^* = \sum_{j=1}^n P_{j,i}. \tag{4.11}$$

Clearly $\alpha(P_i) \in \mathcal{F}_A$. Moreover (4.11), (4.5) and (4.6) imply

$$\alpha(P_i)\alpha(P_j) = 0, \quad i \neq j, \tag{4.12}$$

and

$$\sum_i \alpha(P_i) = 1. \tag{4.13}$$

We extend formula (4.11) by linearity to

$$\alpha(Q_i) = \sum_{r=1}^n A(i, r)\alpha(P_r). \tag{4.14}$$

Set

$$Q = \sum_{i=1}^n Q_i, \tag{4.15}$$

and

$$\alpha(Q) = \sum_{i=1}^n \alpha(Q_i). \tag{4.16}$$

Obviously $\alpha(Q_i), Q, \alpha(Q) \in \mathcal{F}_A$ and since no column of the matrix A is zero it follows from (4.3) along with (4.15) and from (4.13), (4.14) and (4.16) that

$$Q \geq 1, \quad \alpha(Q) \geq 1. \tag{4.17}$$

Set

$$S = \alpha(Q)^{-1/2} \sum_{i=1}^n S_i. \tag{4.18}$$

From this and (4.4) we get

$$S_i = P_i \alpha(Q)^{1/2} S. \tag{4.19}$$

Observe now that S is an isometry. Indeed, by (4.15), (4.16) and (4.14) we have

$$\alpha(Q) = \sum_{r=1}^n \gamma_r \alpha(P_r), \tag{4.20}$$

where $\gamma_r \neq 0, r = 1, \dots, n$, and

$$Q = \sum_{r=1}^n \gamma_r P_r, \tag{4.21}$$

and therefore

$$\alpha(Q)^{-1} = \sum_{r=1}^n \gamma_r^{-1} \alpha(P_r), \tag{4.22}$$

and

$$Q^{-1} = \sum_{r=1}^n \gamma_r^{-1} P_r. \tag{4.23}$$

Now we have by (4.18) and (4.4):

$$S^* S = \sum_{i,j} S_i^* \alpha(Q)^{-1} S_j = \sum_j S_j^* \alpha(Q)^{-1} S_j$$

and substituting (4.22) into the last formula and recalling (4.11), (4.23) and (4.4) we obtain

$$\begin{aligned} S^* S &= \sum_j S_j^* \sum_r \gamma_r^{-1} \alpha(P_r) S_j = \sum_j \sum_r \gamma_r^{-1} \sum_i S_j^* S_i P_r S_i^* S_j \\ &= \sum_j \sum_r \gamma_r^{-1} Q_j P_r = \sum_j Q_j \sum_r \gamma_r^{-1} P_r = Q Q^{-1} = 1. \end{aligned}$$

Thus S is an isometry.

In view of (4.18), (4.7) and (4.8) we conclude that

$$S \mathcal{F}_A S^* \subset \mathcal{F}_A \quad \text{and} \quad S^* \mathcal{F}_A S \subset \mathcal{F}_A. \tag{4.24}$$

Since S is an isometry it follows that the mapping $\delta: \mathcal{F}_A \rightarrow \mathcal{F}_A$ defined by

$$\delta(\cdot) = S(\cdot) S^*, \tag{4.25}$$

is an endomorphism of \mathcal{F}_A .

Observe that (4.24) and (4.25) mean that \mathcal{F}_A is a coefficient algebra for the C^* -algebra $C^*(\mathcal{F}_A, S)$ and

$$\delta_*(\cdot) = S^*(\cdot) S. \tag{4.26}$$

Now we are ready to establish the desired crossed product structure of the Cuntz-Krieger C^* -algebra \mathcal{O}_A .

Proposition 4.6. *For every $n \times n$ matrix A with no zero rows or columns we have*

$$\mathcal{O}_A = C^*(\mathcal{F}_A, S) \cong \mathcal{F}_A \times_{\delta} \mathbb{Z},$$

where the isomorphism

$$\varphi: C^*(\mathcal{F}_A, S) \rightarrow \mathcal{F}_A \times_{\delta} \mathbb{Z}$$

is such that $\varphi(S) = \widehat{U}$ and $\varphi(a) = \widehat{a}$ for all $a \in \mathcal{F}_A$, where \widehat{a} and \widehat{U} are the canonical images of $a \in \mathcal{F}_A$ and U in $\mathcal{F}_A \times_{\delta} \mathbb{Z}$, respectively, and δ and δ_* are defined by (4.25) and (4.26).

Proof. The equality $\mathcal{O}_A = C^*(\mathcal{F}_A, S)$ follows from (4.18), (4.19), (4.9) along with the observation that the $P_i, i \in 1, \dots, n$, and $\alpha(Q)$ belong to \mathcal{F}_A .

Note now that (4.10) is nothing else than property (*) for $C^*(\mathcal{F}_A, S)$. Therefore the desired result follows from Theorem 3.5.

4.4. Paschke-type crossed product. While analysing the simplicity of the Cuntz algebra (considered in [1]) Paschke has found a certain condition on the action of an endomorphism generated by an isometry S under which the C^* -algebra $C^*(A, S)$ generated by an initial C^* -algebra A and S is on the one hand isomorphic to a certain crossed product and on the other hand is simple. His result is stated as follows.

Theorem 4.7 ([3], Theorem 1). *Let A be a strongly amenable unital C^* -algebra acting on a Hilbert space H . Suppose that S is a nonunitary isometry (that is, $S^*S = 1 \neq SS^*$) in $L(H)$ such that*

- (i) $SAS^* \subset A, S^*AS \subset A$; and
 - (ii) *if a proper (two-sided) ideal J satisfies the condition $SJS^* \subseteq J$ then $J = \{0\}$.*
- Then $C^*(A, S)$ is simple.*

We shall now see that Paschke’s result can easily be generalized to the situation considered in this article. First we note the next lemma.

Lemma 4.8. *Let A be the coefficient algebra for $C^*(A, V)$ (see Definition 2.3). Suppose that A is strongly amenable, $VV^* \neq 1$ and the only proper (two-sided) ideal J for which $VJV^* \subseteq J$ is the zero ideal. Then $C^*(A, V)$ possesses property (*).*

Proof. The proof can be obtained by a word-for-word repetition of the corresponding parts of the proofs of [3], Lemmas 2 and 3 using when necessary instead of the Paschke’s condition $S^*S = 1$ the coefficient-algebra condition $V^*V \in Z(A)$. So we omit it.

As a corollary of this lemma we obtain the following generalization of Theorem 4.7.

Theorem 4.9. *Let A be the coefficient algebra of $C^*(A, V)$ (see Definition 2.3). Suppose that A is strongly amenable, $VV^* \neq 1$ and if a proper (two-sided) ideal J satisfies the condition $VJV^* \subseteq J$ then it is zero. Then*

- 1) $C^*(A, V) \cong A \times_{\delta} \mathbb{Z}$ where $\delta(\cdot) = V(\cdot)V^*$ and $\delta_*(\cdot) = V^*(\cdot)V$, and the isomorphism

$$\varphi: C^*(A, V) \rightarrow A \times_{\delta} \mathbb{Z}$$

is such that $\varphi(V) = \widehat{U}$ and $\varphi(a) = \widehat{a}$ for all $a \in A$, where \widehat{a} and \widehat{U} are the canonical images of $a \in A$ and U in $A \times_{\delta} \mathbb{Z}$, respectively;

- 2) $C^*(A, V)$ is simple.

Proof. 1) By Lemma 4.8 $C^*(A, V)$ possesses property (*). Thus the result follows from Theorem 3.5.

2) Let I be any proper ideal in $C^*(A, V)$. Then $J = I \cap A$ is a proper ideal in A (since if $J = A$ then $I = C^*(A, V)$). Clearly $VJV^* \subseteq J$ and therefore

$$I \cap A = J = \{0\}. \tag{4.27}$$

Consider the canonical mapping $\pi: C^*(A, V) \rightarrow C^*(A, V)/I$. It is easy to see that $\pi(C^*(A, V)) = C^*(\pi(A), \pi(V))$, and for every $a \in A$ we have

$$\pi(V^*aV) = \pi(V)^*\pi(a)\pi(V), \quad \pi(VaV^*) = \pi(V)\pi(a)\pi(V)^*$$

and

$$\pi(V^*V) \in Z(\pi(A)).$$

Equality (4.27) implies

$$\pi(A) \cong A/(A \cap I) \cong A$$

and therefore $\pi(C^*(A, V)) = C^*(\pi(A), \pi(V))$ satisfies all the assumptions of Lemma 4.8 with $\pi(A)$ and $\pi(V)$ substituted for A and V . Thus by the already proved first part of the theorem

$$\pi(C^*(A, V)) \cong A \times_{\delta} \mathbb{Z} \cong C^*(A, V).$$

It follows that $I = \{0\}$.

4.5. Exel’s crossed product. Recently Exel proposed in [6] a new definition for the crossed product of a unital C^* -algebra by an endomorphism α and a certain transfer operator \mathcal{L} . He also proved in [6] that this new construction generalizes many of the previously known constructions among which are the above mentioned monomorphisms with hereditary range, Paschke’s crossed product and Cuntz-Krieger algebras (in fact we believe that the main inspiring motivation for Exel was to generalize the Cuntz-Krieger construction, and he perfectly succeeded).

In this subsection we analyse the interrelations between Exel’s crossed product and ours. We show that on the one hand the crossed product introduced in this article is a special case of Exel’s and on the other hand in the most natural situations (when all the powers of \mathcal{L} are generated by partial isometries) Exel’s crossed product is of the type introduced here but with different algebra, different endomorphism and different transfer operator. In fact in the general situation the crossed product ‘philosophy’ is even more peculiar and subtle and this will be the theme of the subsequent sections.

Throughout this subsection A will be a unital C^* -algebra, $\alpha: A \rightarrow A$ will be a $*$ -endomorphism and \mathcal{L} will be a certain transfer operator for the pair (A, α) , that is, $\mathcal{L}: A \rightarrow A$ is a continuous positive operator satisfying condition (2.1) with δ substituted for α and δ_* for \mathcal{L} .

Definition 4.10. The (Exel’s) *crossed product* $A \times_{\alpha, \mathcal{L}} \mathbb{N}$ is the universal C^* -algebra generated by a copy of A and element S subject to the relations

- (i) $Sa = \alpha(a)S, a \in A;$
- (ii) $S^*aS = \mathcal{L}(a), a \in A;$
- (iii) if $(a, k) \in A\alpha(A)A \times \overline{ASS^*A}$ is such that

$$abS = kbS \tag{4.28}$$

for all $b \in A$ then $a = k$; here $\overline{A\alpha(A)A}$ is the closed linear space generated by $A\alpha(A)A$ (the two-sided ideal in A generated by $\alpha(A)$) and $\overline{ASS^*A}$ is the closed linear space generated by ASS^*A .

Any pair (a, k) mentioned in (iii) is called a *redundancy*.

Notation 4.11. We will denote by $\mathcal{T}(A, \alpha, \mathcal{L})$ the universal algebra satisfying (i) and (ii) in Definition 4.10.

Remark 4.12. Exel showed [6], 3.5 that for any endomorphism α and a transfer operator \mathcal{L} the algebra $\mathcal{T}(A, \alpha, \mathcal{L})$ is nondegenerate (that is, the canonical map $A \rightarrow \mathcal{T}(A, \alpha, \mathcal{L})$ is injective).

Therefore one can think of the crossed product $A \times_{\alpha, \mathcal{L}} \mathbb{N}$ as the quotient of $\mathcal{T}(A, \alpha, \mathcal{L})$ by the closed two-sided ideal \mathcal{I} generated by the set of differences $a - k$ for all redundancies (a, k) .

By contrast with the situation with $\mathcal{T}(A, \alpha, \mathcal{L})$, not for all α and \mathcal{L} is there a natural inclusion of A in $A \times_{\alpha, \mathcal{L}} \mathbb{N}$. The conditions that ensure the canonical map $A \rightarrow A \times_{\alpha, \mathcal{L}} \mathbb{N}$ to be injective were found by Brownlowe and Raeburn [9]. They are formulated in terms of Cuntz-Pimzner algebras.

We start with the observation that the crossed product introduced in this paper is a special case of Exel’s crossed product. (This is stated in Theorem 4.16.)

The argument here goes absolutely in the same way as the corresponding argument in Section 4 of [6] and we give the necessary proofs simply for the sake of completeness of presentation.

The product $\mathcal{A} \times_{\delta} \mathbb{Z}$ is ‘not bigger’ than $\mathcal{T}(\mathcal{A}, \delta, \delta_*)$, as is stated in the next proposition.

Proposition 4.13. *Let the pair (\mathcal{A}, δ) be finely representable and let δ_* be the corresponding transfer operator. Then there exists a unique $*$ -epimorphism*

$$\psi: \mathcal{T}(\mathcal{A}, \delta, \delta_*) \rightarrow \mathcal{A} \times_{\delta} \mathbb{Z}$$

such that $\psi(S) = \widehat{U}$ and $\psi(a) = \widehat{a}$ for all $a \in \mathcal{A}$.

Proof. Since $\delta(\cdot) = U(\cdot)U^*$ and $U^*U \in Z(\mathcal{A})$ it follows that, for all $a \in \mathcal{A}$,

$$Ua = UU^*Ua = UaU^*U = \delta(a)U.$$

In other words, relations (i), (ii) in Definition 4.10 hold for $\widehat{\mathcal{A}}$ and \widehat{U} within $\mathcal{A} \times_{\delta} \mathbb{Z}$ and hence the conclusion follows from the universal property of $\mathcal{T}(\mathcal{A}, \delta, \delta_*)$.

To establish the coincidence between $\mathcal{A} \times_{\delta} \mathbb{Z}$ (the crossed product introduced in this paper) and $\mathcal{A} \times_{\delta, \delta_*} \mathbb{N}$ (Exel’s crossed product) we observe first the following fact.

Proposition 4.14. *Let $(\mathcal{A}, \delta, \delta_*)$ satisfy the hypothesis of Proposition 4.13. Then*

- (i) *the canonical element $S \in \mathcal{T}(\mathcal{A}, \delta, \delta_*)$ is a partial isometry and hence also its image $\widehat{S} \in \mathcal{A} \times_{\delta, \delta_*} \mathbb{N}$;*
- (ii) *for every $a \in \mathcal{A}$ one has that $(\delta(a), SaS^*)$ is a redundancy.*

Proof. (i) By (2.3) we have that $\delta_*(1)$ is a projection; thus S is a partial isometry.

(ii) For any $b \in \mathcal{A}$ we have

$$SaS^*bS = Sa\delta_*(b) = \delta(a)\delta\delta_*(b)S = \delta(a)\delta(1)b\delta(1)S = \delta(a)bS,$$

where we have used the completeness of δ_* and the fact that $\delta(1)S = S1 = S$. It remains to notice that $SaS^* = \delta(a)SS^* \in \overline{\mathcal{A}SS^*\mathcal{A}}$.

Corollary 4.15. *Under the hypothesis of Proposition 4.14 there exists a unique $*$ -epimorphism*

$$\phi: \mathcal{A} \times_{\delta} \mathbb{Z} \rightarrow \mathcal{A} \times_{\delta, \delta_*} \mathbb{N},$$

such that $\phi(\widehat{U}) = \dot{S}$ and $\phi(\widehat{a}) = \dot{a}$ for all $a \in \mathcal{A}$, where \dot{S} and \dot{a} are the canonical images of S and a in $\mathcal{A} \times_{\delta, \delta_*} \mathbb{N}$, respectively, and \widehat{a} and \widehat{U} are the canonical images of $a \in \mathcal{A}$ and U in $\mathcal{A} \times_{\delta} \mathbb{Z}$ (here $\mathcal{A} \times_{\delta, \delta_*} \mathbb{N}$ is Exel’s crossed product and $\mathcal{A} \times_{\delta} \mathbb{Z}$ is the crossed product introduced in this paper).

Proof. By Proposition 4.14, (ii) we have $\delta(\dot{a}) = \dot{S}\dot{a}\dot{S}^*$. Hence the conclusion follows from the universal property of $\mathcal{A} \times_{\delta} \mathbb{Z}$.

Thus $\mathcal{A} \times_{\delta, \delta_*} \mathbb{N}$ is not ‘bigger’ than $\mathcal{A} \times_{\delta} \mathbb{Z}$. The next result shows the desired coincidence between $\mathcal{A} \times_{\delta} \mathbb{Z}$ and $\mathcal{A} \times_{\delta, \delta_*} \mathbb{N}$.

Theorem 4.16. *Let (\mathcal{A}, δ) be a finely representable pair with the complete transfer operator δ_* . Then the map ϕ in Corollary 4.15 is a $*$ -isomorphism between $\mathcal{A} \times_{\delta} \mathbb{Z}$ and $\mathcal{A} \times_{\delta, \delta_*} \mathbb{N}$.*

Proof. Observe first that the map ψ of Proposition 4.13 vanishes on the ideal \mathcal{I} mentioned in Remark 4.12. Indeed, let $(a, k) \in \overline{\mathcal{A}\delta(\mathcal{A})\mathcal{A}} \times \overline{\mathcal{A}SS^*\mathcal{A}}$ be a redundancy. Then for all $b \in \mathcal{A}$ one has $abS = kbS$. By applying ψ to both sides one obtains

$$\widehat{ab}\widehat{U} = \psi(k)\widehat{b}\widehat{U}.$$

Since $\delta(1) = UU^*$, for all $b, c \in \mathcal{A}$ we have

$$\widehat{ab}\widehat{\delta(1)}\widehat{c} = \widehat{ab}\widehat{U}\widehat{U}^*\widehat{c} = \psi(k)\widehat{b}\widehat{U}\widehat{U}^*\widehat{c} = \psi(k)\widehat{b}\widehat{\delta(1)}\widehat{c}.$$

It follows that $\widehat{ax} = \psi(k)\widehat{x}$ for all $x \in \overline{\mathcal{A}\delta(1)\mathcal{A}}$. Since $k \in \overline{\mathcal{A}SS^*\mathcal{A}}$, we have $\psi(k) \in \overline{\mathcal{A}\widehat{U}\widehat{U}^*\mathcal{A}} = \overline{\mathcal{A}\widehat{\delta(1)}\mathcal{A}}$. Finally, by the completeness of δ_* we have

$$a \in \overline{\mathcal{A}\delta(\mathcal{A})\mathcal{A}} = \overline{\mathcal{A}\delta(1)\mathcal{A}\delta(1)\mathcal{A}} = \overline{\mathcal{A}\delta(1)\mathcal{A}}.$$

Thus $\widehat{a} = \psi(k)$ and it follows that $\psi(a - k) = 0$ and hence ψ vanishes on \mathcal{I} as claimed. By passage to the quotient we get a map

$$\widetilde{\psi}: \mathcal{A} \times_{\delta, \delta_*} \mathbb{N} \rightarrow \mathcal{A} \times_{\delta} \mathbb{Z},$$

which is the inverse of the map ϕ in Corollary 4.15.

Thus we have established that the crossed product introduced in the present article is a special case of Exel’s. Now we shall move in the opposite direction and show that in ‘the most popular’ situation when all the operators S^n , $n = 1, 2, \dots$, are partial isometries (which is equivalent to the condition that all the elements $\mathcal{L}^n(1)$, $n = 1, 2, \dots$, are projections) Exel’s crossed product is of type introduced here but with different algebra, different endomorphism and different transfer operator.

We start with the result showing that under the mentioned hypothesis even $\mathcal{T}(A, \alpha, \mathcal{L})$ possesses the structure of a certain crossed product.

Theorem 4.17. *Let A be a unital C^* -algebra, $\alpha: A \rightarrow A$ an endomorphism, and \mathcal{L} a transfer operator such that all the elements $\mathcal{L}^n(1)$, $n = 1, 2, \dots$, are projections. Let \mathcal{A} be the C^* -algebra generated by A and $S^k S^{*k}$, $k = 1, 2, \dots$, where S is the universal operator satisfying relations (i) and (ii) in Definition 4.10. Then the map*

$$\nu: \mathcal{A} \times_{\delta} \mathbb{Z} \rightarrow \mathcal{T}(A, \alpha, \mathcal{L}),$$

such that $\nu(\widehat{U}) = (S)$ and $\nu(\widehat{a}) = a$ for all $a \in A$, $\delta: \mathcal{A} \rightarrow \mathcal{A}$ is given by $\delta(\cdot) = S(\cdot)S^*$ and $\delta_*: \mathcal{A} \rightarrow \mathcal{A}$ is given by $\delta_*(\cdot) = S^*(\cdot)S$, establishes a $*$ -isomorphism.

Proof. As has already been mentioned the condition that all the elements $\mathcal{L}^n(1)$, $n = 1, 2, \dots$, are projections is equivalent to the condition that all the operators S^k , $k = 1, 2, \dots$, are partial isometries. In view of Proposition 4.5 this condition implies that all the operators $S^{*k}S^k$, $S^j S^{*j}$, $k, j = 1, 2, \dots$, commute with one another and therefore S^*S belongs to the commutant of \mathcal{A} . Observe now that for any $a \in A$ and any $k = 1, 2, \dots$ we have

$$SaS^* = \alpha(a)SS^* \in \mathcal{A}, \quad S(S^k S^{*k})S^* = S^{k+1}S^{*k+1} \in \mathcal{A}$$

and for any $c, d \in \mathcal{A}$ we have

$$ScdS^* = SS^*ScdS^* = ScS^*SdS^*, \tag{4.29}$$

where we have used the mentioned fact that S^*S belongs to the commutant of \mathcal{A} .

The foregoing observations imply

$$S\mathcal{A}S^* \subset \mathcal{A} \quad \text{and} \quad S^*S \in Z(\mathcal{A}). \tag{4.30}$$

Note also that (4.29) implies that $S(\cdot)S^*$ is an endomorphism of \mathcal{A} .

A routine computation shows also that

$$S^*\mathcal{A}S \subset \mathcal{A}. \tag{4.31}$$

In view of the definition of $\mathcal{T}(A, \alpha, \mathcal{L})$ and \mathcal{A} we have

$$\mathcal{T}(A, \alpha, \mathcal{L}) = C^*(\mathcal{A}, S). \tag{4.32}$$

And (4.30) and (4.31) mean that \mathcal{A} is the coefficient algebra for $C^*(\mathcal{A}, S)$ with $\delta(\cdot) = S(\cdot)S^*$ and $\delta_*(\cdot) = S^*(\cdot)S$. Now the universal property of $\mathcal{A} \times_{\delta} \mathbb{Z}$ implies that the mapping

$$\nu: \mathcal{A} \times_{\delta} \mathbb{Z} \rightarrow \mathcal{T}(A, \alpha, \mathcal{L}),$$

where $\nu(\widehat{a}) = a$, $a \in A$ and $\nu(\widehat{U}) = S$, is a $*$ -epimorphism.

To prove that this mapping is in fact a $*$ -isomorphism it is enough to show that the C^* -algebra $C^*(\mathcal{A}, S)$ in (4.32) possesses property $(*)$ (in this case one can apply Theorem 3.5). So let us verify the latter property. Let A and S be the universal algebra and element that generate $\mathcal{T}(A, \alpha, \mathcal{L}) = C^*(\mathcal{A}, S)$. They satisfy relations (4.30) and (4.31), and without loss of generality we can assume that $C^*(\mathcal{A}, S)$ is a C^* -subalgebra of $L(H)$ for some Hilbert space H and that the identity of A

is the identity of $L(H)$. Consider the space $\mathcal{H} = l^2(\mathbb{Z}, H)$ and the representation $\mu: \mathcal{T}(A, \alpha, \mathcal{L}) \rightarrow L(\mathcal{H})$ given by the formulae

$$\begin{aligned} (\mu(a)\xi)_n &= a(\xi_n), & a \in A, & \quad l^2(\mathbb{Z}, H) \ni \xi = \{\xi_n\}_{n \in \mathbb{Z}}; \\ (\mu(S)\xi)_n &= S(\xi_{n-1}), & (\mu S^*)\xi_n &= S^*(\xi_{n+1}). \end{aligned}$$

It is easy to see that $\mu(\mathcal{A}) \cong \mathcal{A}$ and that $\mu(\mathcal{A})$ and $\mu(S)$ satisfy the same relations (4.30) and (4.31), and $\mu(S)$ generates the same mappings δ and δ_* on $\mu(\mathcal{A})$. Note now that the algebra $C^*(\mu(\mathcal{A}), \mu(S))$ possesses property (*) (recall the argument of Remark 3.4). But this means that the algebra $C^*(\mathcal{A}, S) = \mathcal{T}(A, \alpha, \mathcal{L})$ possesses this property as well. The proof is finished.

Now we establish the desired isomorphism between Exel’s crossed product and the crossed product introduced in this article.

Theorem 4.18. *Let the hypothesis of Theorem 4.17 be satisfied. Let $\dot{\mathcal{A}}$ be the C^* -algebra generated by \dot{A} and the $\dot{S}^k \dot{S}^{*k}$, $k = 1, 2, \dots$, where \dot{S} and \dot{A} are the canonical images of S and A in $A \times_{\alpha, \mathcal{L}} \mathbb{N}$, respectively. Then the map*

$$\gamma: \dot{\mathcal{A}} \times_{\delta} \mathbb{Z} \rightarrow A \times_{\alpha, \mathcal{L}} \mathbb{N},$$

such that $\gamma(\widehat{U}) = \dot{S}$ and $\gamma(\widehat{a}) = \dot{a}$ for all $a \in A$, and $\delta: \dot{\mathcal{A}} \rightarrow \dot{\mathcal{A}}$ is given by $\delta(\cdot) = \dot{S}(\cdot)\dot{S}^*$ and $\delta_*: \dot{\mathcal{A}} \rightarrow \dot{\mathcal{A}}$ is given by $\delta_*(\cdot) = \dot{S}^*(\cdot)\dot{S}$, establishes a *-isomorphism.

Proof. In view of (4.30)–(4.32) we have

$$A \times_{\alpha, \mathcal{L}} \mathbb{N} = C^*(\dot{\mathcal{A}}, \dot{S}) \tag{4.33}$$

and $\dot{\mathcal{A}}$ is the coefficient algebra for $C^*(\dot{\mathcal{A}}, \dot{S})$ with $\delta(\cdot) = \dot{S}(\cdot)\dot{S}^*$ and $\delta_*(\cdot) = \dot{S}^*(\cdot)\dot{S}$. Now the universal property of $\dot{\mathcal{A}} \times_{\delta} \mathbb{Z}$ implies that the mapping

$$\gamma: \dot{\mathcal{A}} \times_{\delta} \mathbb{Z} \rightarrow A \times_{\alpha, \mathcal{L}} \mathbb{N},$$

where $\gamma(\widehat{a}) = \dot{a}$, $a \in A$ and $\gamma(\widehat{U}) = \dot{S}$, is a *-epimorphism.

To prove that this mapping is in fact a *-isomorphism it is enough to show that the C^* -algebra $C^*(\dot{\mathcal{A}}, \dot{S})$ in (4.33) possesses property (*).

Let \dot{A} and \dot{S} be the universal algebra and element that generate $A \times_{\alpha, \mathcal{L}} \mathbb{N} = C^*(\dot{\mathcal{A}}, \dot{S})$. They satisfy relations (4.30) and (4.31) (for $\dot{\mathcal{A}}$ and \dot{S}), and without loss of generality we can assume that $C^*(\dot{\mathcal{A}}, \dot{S})$ is a C^* -subalgebra of $L(H)$ for some Hilbert space H and that the identity of \dot{A} is the identity of $L(H)$. Consider the space $\mathcal{H} = l^2(\mathbb{Z}, H)$ and the representation $\nu: \mathcal{T}(A, \alpha, \mathcal{L}) \rightarrow L(\mathcal{H})$ given by the formulae

$$\begin{aligned} (\nu(a)\xi)_n &= \dot{a}(\xi_n), & a \in A, & \quad l^2(\mathbb{Z}, H) \ni \xi = \{\xi_n\}_{n \in \mathbb{Z}}; \\ (\nu(S)\xi)_n &= \dot{S}(\xi_{n-1}), & (\nu(S^*)\xi)_n &= \dot{S}^*(\xi_{n+1}). \end{aligned}$$

It is easy to see that $\nu(\mathcal{A}) \cong \dot{\mathcal{A}}$ (here \mathcal{A} is the C^* -algebra mentioned in Theorem 4.17) and that $\nu(\mathcal{A})$ and $\nu(S)$ satisfy the same relations (4.30) and (4.31) (for $\dot{\mathcal{A}}$ and \dot{S}) and $\nu(S)$ generates the same mappings δ and δ_* on $\nu(\mathcal{A})$. Moreover

since \dot{A} and \dot{S} satisfy relations (i), (ii) and (iii) in Definition 4.10, by the construction of ν we have that $\nu(A)$ and $\nu(S)$ satisfy these relations as well. Therefore we can consider ν as a representation of $A \times_{\alpha, \mathcal{L}} \mathbb{N} = C^*(\dot{\mathcal{A}}, \dot{S})$. But $C^*(\nu(\dot{\mathcal{A}}), \nu(\dot{S}))$ possesses property (*) (by the argument of Remark 3.4). Thus $C^*(\dot{\mathcal{A}}, \dot{S})$ possesses this property as well.

4.6. Kwasniewski’s crossed product. This subsection is devoted to the description of the crossed product structure developed recently by Kwasniewski [7] and the discussion of the interrelation between this structure and the crossed product introduced in the present paper. To give the motivation of Kwasniewski’s crossed product we start with two simple examples.

Example 4.19. Consider the Hilbert space $H = L^2(\mathbb{R})$. Let $A \subset L(H)$ be the C^* -algebra of operators of multiplication by continuous bounded functions on \mathbb{R} that are constant on $\mathbb{R}_- = \{x : x \leq 0\}$. Set the unitary operator $U \in L(H)$ by the formula

$$(Uf)(x) = f(x - 1), \quad f(\cdot) \in H. \tag{4.34}$$

Routine verification shows that the mapping

$$A \ni a \mapsto UaU^* \tag{4.35}$$

is an endomorphism of A of the form

$$UaU^*(x) = a(x - 1), \quad a(\cdot) \in A, \tag{4.36}$$

and

$$U^*aU(x) = a(x + 1), \quad a(\cdot) \in A. \tag{4.37}$$

Clearly the mapping $A \ni a \mapsto U^*aU$ does not preserve A .

Let $C^*(A, U)$ be the C^* -algebra generated by A and U . It is easy to show that

$$C^*(A, U) = C^*(\mathcal{A}, U), \tag{4.38}$$

where $\mathcal{A} \subset L(H)$ is the C^* -algebra of operators of multiplication by continuous bounded functions on \mathbb{R} that have limits as $x \rightarrow -\infty$.

In addition we have

$$U\mathcal{A}U^* \subset \mathcal{A} \quad \text{and} \quad U^*\mathcal{A}U \subset \mathcal{A}, \tag{4.39}$$

and the corresponding actions $\delta(\cdot) = U(\cdot)U^*$ and $\delta_*(\cdot) = U^*(\cdot)U$ on \mathcal{A} are given by formulae (4.36) and (4.37).

Thus \mathcal{A} is a coefficient algebra for $C^*(\mathcal{A}, U)$.

Moreover one can easily check that $C^*(\mathcal{A}, U)$ possesses property (*). Therefore by Theorem 3.5 we conclude that

$$C^*(A, U) = C^*(\mathcal{A}, U) \cong \mathcal{A} \times_{\delta} \mathbb{Z}. \tag{4.40}$$

We would like to emphasize that the situation considered does not satisfy the conditions under which Exel’s crossed product was defined since we started with the

algebra A and the operator U such that the mapping $A \ni a \mapsto U^*aU$ (4.37) is not a transfer operator (it does not preserve A). But after extending A up to \mathcal{A} we have obtained the coefficient algebra and thus found ourselves under the main assumptions of the crossed product construction of the present article.

In fact by a slight modification of the example considered it is easy to ‘worsen’ the situation even further.

Example 4.20. Let H and U be the same as in the previous example and let $A \subset L(H)$ be the C^* -algebra of operators of multiplication by continuous bounded functions on \mathbb{R} that are constant on \mathbb{R}_- and are constant for $x \geq \pi$. Then we have

$$C^*(A, U) = C^*(\mathcal{A}, U), \tag{4.41}$$

where \mathcal{A} is the C^* -algebra of operators of multiplication by continuous bounded functions on \mathbb{R} that have limits at $\pm\infty$.

As in the previous example, here \mathcal{A} is the coefficient algebra for $C^*(\mathcal{A}, U)$ and we have

$$C^*(A, U) = C^*(\mathcal{A}, U) \cong \mathcal{A} \times_{\delta} \mathbb{Z},$$

where δ and δ_* are given by the same formulae. But by contrast with the previous example, here even the mapping $A \ni a \mapsto UaU^*$ does not preserve A (thus this mapping is not an endomorphism).

These examples show that neither a transfer operator nor even a certain endomorphism are among the starting objects that lead to coefficient algebras and crossed products. In fact in both these examples the principal moment was a certain procedure of extension of the initial algebra A up to a coefficient algebra \mathcal{A} (this procedure is generated by the mappings $A \ni a \mapsto UaU^*$ and $A \ni a \mapsto U^*aU$). After the implementation of this procedure and obtaining the coefficient algebra \mathcal{A} the final step, the construction of the crossed product, goes smoothly (in accordance with the scheme introduced in the present article).

The general procedures of extension of initial C^* -algebras up to coefficient algebras were given in [10]. The maximal ideal spaces of the commutative coefficient algebras arising in this way were described in [25]. By developing the technique of [10] and [25] along with a number of new ideas Kwasniewski has described the extension procedure as for the initial commutative C^* -algebra A so also for the action ($A \ni a \mapsto UaU^*$ and $A \ni a \mapsto U^*aU$) up to obtaining a commutative coefficient algebra \mathcal{A} and the corresponding action (in fact the partial action) which leads to the corresponding crossed product (in fact the partial crossed product). Kwasniewski’s crossed product should naturally be considered as the most general crossed product of the type presented in Example 4.19.

We emphasize that there is no transfer operator among the starting objects of this construction and therefore one should consider Kwasniewski’s construction to be qualitatively different from Exel’s crossed product.

Hereafter we describe in brief Kwasniewski’s construction and its interrelation with the crossed product introduced in the present article.

Let A be a commutative unital C^* -algebra and let δ be an endomorphism of A . As A is commutative we can use the Gelfand transform in order to identify A with the algebra $C(X)$ of continuous functions on the maximal ideal space X of A .

Within this identification the endomorphism δ generates (see, for example, [25], Proposition 2.1) a continuous mapping $\gamma: \Delta \rightarrow X$, where $\Delta \subset X$ is clopen and the following formula holds:

$$\delta(a)(x) = \begin{cases} a(\gamma(x)), & x \in \Delta, \\ 0, & x \notin \Delta, \end{cases} \quad a \in C(X). \tag{4.42}$$

Mappings γ of this sort are called *partial mappings* (of X). Thus we have a one-to-one correspondence between the pairs (A, δ) and the pairs (X, γ) , where X is compact and γ is a partial continuous mapping with a clopen domain. In [7] (X, γ) is called a *partial dynamical system*.

In [7] the author developed the crossed product construction for an ‘almost arbitrary’ endomorphism δ , namely the only presumed constraint was that the image $\gamma(\Delta)$ of the partial mapping γ is open. As Proposition 4.24 and Remark 4.23 assert, this constraint is completely insignificant.

If A is a unital commutative C^* -subalgebra of $L(H)$ for some Hilbert space H and U is a partial isometry such that $U^*U \in A'$ and $UAU^* \subset A$, then clearly $\delta(\cdot) = U(\cdot)U^*$ is an endomorphism of A . Although in this situation A is not necessarily a coefficient algebra of $C^*(A, U)$, there is a natural way to construct one by passing to a bigger C^* -algebra \mathcal{A} generated by $\{A, U^*AU, U^{2*}AU^2, \dots\}$. This is stated in the next proposition.

Proposition 4.21 ([10], Proposition 4.1). *If $\delta(\cdot) = U(\cdot)U^*$ is an endomorphism of a C^* -algebra A , U^*U belongs to its commutant, and \mathcal{A} is the C^* -algebra generated by $\bigcup_{n=0}^\infty U^{*n}AU^n$, then \mathcal{A} is commutative and both the mappings $\delta: \mathcal{A} \rightarrow \mathcal{A}$ and $\delta_*: \mathcal{A} \rightarrow \mathcal{A}$ (where $\delta_*(\cdot) = U^*(\cdot)U$) are endomorphisms.*

Thus in the situation described \mathcal{A} is the coefficient algebra for $C^*(\mathcal{A}, U) = C^*(A, U)$.

It is of primary importance here that \mathcal{A} is commutative and that due to [25] we can describe its maximal ideal space, denoted further by \mathcal{X} , in terms of the maximal ideals in A . Let us recall this description.

To start with we have to introduce some notation.

Hereafter in this subsection A denotes a commutative unital C^* -algebra, X denotes its maximal ideal space (that is, a compact Hausdorff topological space), δ is an endomorphism of A , while γ stands for the continuous partial mapping $\gamma: \Delta \rightarrow X$, where $\Delta \subset X$ is clopen and formula (4.42) holds. When dealing with partial mappings γ^n , $n = 1, 2, \dots$, for $n > 0$ we denote the domain of γ^n by $\Delta_n = \gamma^{-n}(X)$ and its image by $\Delta_{-n} = \gamma^n(\Delta_n)$; for $n = 0$ we set $\gamma^0 = \text{Id}$, $\Delta_0 = X$, and thus, for $n, m \in \mathbb{N}$ we have

$$\gamma^n: \Delta_n \rightarrow \Delta_{-n}, \tag{4.43}$$

$$\gamma^n(\gamma^m(x)) = \gamma^{n+m}(x), \quad x \in \Delta_{n+m}. \tag{4.44}$$

Note that in terms of multiplicative functionals on A , γ is given by

$$x \in \Delta_1 \iff x(\delta(1)) = 1, \tag{4.45}$$

$$\gamma(x) = x \circ \delta, \quad x \in \Delta_1. \tag{4.46}$$

In [25] the authors calculated the maximal ideal space \mathcal{X} of \mathcal{A} in terms of (A, δ) , or, more precisely, in terms of the generated partial dynamical system (X, γ) . This description is presented in Theorem 4.22.

With every $\tilde{x} \in \mathcal{X}$ we associate a sequence of functionals $\xi_{\tilde{x}}^n : A \rightarrow \mathbb{C}$, $n \in \mathbb{N}$, defined by the condition

$$\xi_{\tilde{x}}^n(a) = \delta_*^n(a)(\tilde{x}), \quad a \in A. \tag{4.47}$$

The sequence $\xi_{\tilde{x}}^n$ determines \tilde{x} uniquely because $\mathcal{A} = C^*(\bigcup_{n=0}^\infty \delta_*^n(A))$. Since δ_* is an endomorphism of \mathcal{A} the functionals $\xi_{\tilde{x}}^n$ are linear and multiplicative on A . So either $\xi_{\tilde{x}}^n \in X$ (X is the spectrum of A) or $\xi_{\tilde{x}}^n \equiv 0$. It follows then that the mapping

$$\mathcal{X} \ni \tilde{x} \rightarrow (\xi_{\tilde{x}}^0, \xi_{\tilde{x}}^1, \dots) \in \prod_{n=0}^\infty (X \cup \{0\}) \tag{4.48}$$

is an injection and the following statement is true.

Theorem 4.22 ([25], Theorems 3.1 and 3.3). *Let $\delta(\cdot) = U(\cdot)U^*$ be an endomorphism of A , $U^*U \in A$, and let $\gamma : \Delta_1 \rightarrow X$ be the partial mapping generated by δ . Then the maximal ideal space \mathcal{X} of the algebra $\mathcal{A} = C^*(\bigcup_{n=0}^\infty \delta_*^n(A))$ is given by*

$$\mathcal{X} = \bigcup_{N=0}^\infty \mathcal{X}_N \cup \mathcal{X}_\infty,$$

$$\mathcal{X}_N = \{ \tilde{x} = (x_0, \dots, x_N, 0, \dots) : x_n \in \Delta_n, \gamma(x_n) = x_{n-1}, 1 \leq n \leq N, x_N \notin \Delta_{-1} \},$$

$$\mathcal{X}_\infty = \{ \tilde{x} = (x_0, x_1, \dots) : x_n \in \Delta_n, \gamma(x_n) = x_{n-1}, n \geq 1 \}.$$

The topology on \mathcal{X} is defined by a fundamental system of neighbourhoods of points $\tilde{x} \in \mathcal{X}_N$ of the form

$$O(a_1, \dots, a_k, \varepsilon) = \{ \tilde{y} \in \mathcal{X}_N : |a_i(x_N) - a_i(y_N)| < \varepsilon, i = 1, \dots, k \}$$

and of points $\tilde{x} \in \mathcal{X}_\infty$ of the form

$$O(a_1, \dots, a_k, n, \varepsilon) = \left\{ \tilde{y} \in \bigcup_{N=n}^\infty \mathcal{X}_N \cup \mathcal{X}_\infty : |a_i(x_n) - a_i(y_n)| < \varepsilon, i = 1, \dots, k \right\},$$

where $\varepsilon > 0$, $a_i \in A$ and $k, n \in \mathbb{N}$.

Remark 4.23. The topology on X is $*$ -weak. One immediately sees then (see (4.48)) that the topology on $\bigcup_{N \in \mathbb{N}} \mathcal{X}_N \cup \mathcal{X}_\infty$ is in fact the product topology inherited from $\prod_{n=0}^\infty (X \cup \{0\})$, where $\{0\}$ is clopen.

Theorem 4.22 motivates us to take a closer look at the condition $U^*U \in A$.

Observe ([25], Proposition 3.5) that if U^*U belongs to the commutant of A then δ is an endomorphism of the C^* -algebra $A_1 = C^*(A, U^*U)$ and we also have

$$\mathcal{A} = C^* \left(\bigcup_{n=0}^\infty \delta_*^n(A) \right) = C^* \left(\bigcup_{n=0}^\infty \delta_*^n(A_1) \right).$$

Thus the mentioned condition simply means that when calculating the maximal ideal space \mathcal{X} of \mathcal{A} one should start with the C^* -algebra A_1 rather than from A .

Furthermore, the condition $U^*U \in A$ is closely related to the openness of Δ_{-1} (as Δ_1 is compact and γ is continuous, Δ_{-1} is always closed).

Proposition 4.24 ([7], Proposition 2.4). *Let $P_{\Delta_{-1}} \in A$ be the projection corresponding to a characteristic function $\chi_{\Delta_{-1}} \in C(X)$. If $U^*U \in A$, then Δ_{-1} is open and $U^*U = P_{\Delta_{-1}}$. If U^*U belongs to the commutant of A , Δ_{-1} is open and A acts nondegenerately on H , then $U^*U \leq P_{\Delta_{-1}}$.*

Remark 4.25. As is shown in [7], the inequality in the second part of Proposition 4.24 cannot be replaced by an equality.

By virtue of Proposition 4.21 the mappings δ and δ_* are endomorphisms of the C^* -algebra \mathcal{A} . Making a start from Theorem 4.22 we can now find the form of the partial mappings they generate. The initial hint here is the following

Proposition 4.26 ([25], Proposition 2.5). *Let $\delta(\cdot) = U(\cdot)U^*$ and $\delta_*(\cdot) = U^*(\cdot)U$ be endomorphisms of A and let γ be the partial mapping of X generated by δ . Then Δ_1 and Δ_{-1} are clopen and $\gamma: \Delta_1 \rightarrow \Delta_{-1}$ is a homeomorphism. Moreover, the endomorphism δ_* is given on $C(X)$ by the formula*

$$(\delta_*f)(x) = \begin{cases} f(\gamma^{-1}(x)), & x \in \Delta_{-1}, \\ 0, & x \notin \Delta_{-1}. \end{cases} \tag{4.49}$$

This proposition along with Theorem 4.22 leads to

Theorem 4.27 ([7], Theorem 2.8). *Let the hypotheses of Theorem 4.22 hold. Then*

(i) *the sets*

$$\begin{aligned} \tilde{\Delta}_1 &= \{\tilde{x} = (x_0, x_1, \dots) \in \mathcal{X} : x_0 \in \Delta_1\}, \\ \tilde{\Delta}_{-1} &= \{\tilde{x} = (x_0, x_1, \dots) \in \mathcal{X} : x_1 \neq 0\} \end{aligned}$$

are clopen subsets of \mathcal{X} ;

(ii) *the endomorphism δ generates on \mathcal{X} the partial homeomorphism*

$$\tilde{\gamma}: \tilde{\Delta}_1 \rightarrow \tilde{\Delta}_{-1}$$

given by the formula

$$\tilde{\gamma}(\tilde{x}) = \tilde{\gamma}(x_0, x_1, \dots) = (\gamma(x_0), x_0, x_1, \dots), \quad \tilde{x} \in \tilde{\Delta}_1; \tag{4.50}$$

(iii) *the partial mapping generated by δ_* is the inverse of $\tilde{\gamma}$, that is,*

$$\tilde{\gamma}^{-1}: \tilde{\Delta}_{-1} \rightarrow \tilde{\Delta}_1,$$

where

$$\tilde{\gamma}^{-1}(\tilde{x}) = \tilde{\gamma}^{-1}(x_0, x_1, \dots) = (x_1, x_2, \dots), \quad \tilde{x} \in \tilde{\Delta}_{-1}. \tag{4.51}$$

The pair $(\mathcal{X}, \tilde{\gamma})$ is called in [7] the *reversible extension* of the partial dynamical system (X, γ) .

Along with the description of the maximal ideal space of $\mathcal{A} = C(\mathcal{X})$ and the action of $\tilde{\gamma}$ on it the author presented in [7] an explicit algebraic construction of \mathcal{A} in terms of (A, δ) . Here is this construction.

Observe first that the family $\{A_n\}_{n \in \mathbb{N}}$, where $A_n = \delta^n(1)A$, is a decreasing family of closed two-sided ideals. Since the operator $\delta^n(1)$ corresponds to $\chi_{\Delta_n} \in C(X)$, one can consider A_n as $C_{\Delta_n}(X)$ (we denote by $C_K(X)$ the algebra of continuous functions on X vanishing outside the set $K \subset X$). Let $\mathcal{E}_*(A)$ be the set consisting of the sequences $a = \{a_n\}_{n \geq 0}$ where $a_n \in A_n$ and only a finite number of the functions a_n are nonzero. Namely,

$$\mathcal{E}_*(A) = \left\{ a \in \prod_{n=0}^{\infty} A_n : \exists N > 0 \forall n > N \ a_n \equiv 0 \right\}.$$

Let $a = \{a_n\}_{n \geq 0}, b = \{b_n\}_{n \geq 0} \in \mathcal{E}_*(A)$ and $\lambda \in \mathbb{C}$. We define addition, multiplication by scalar, convolution multiplication and involution on $\mathcal{E}_*(A)$ as follows:

$$(a + b)_n = a_n + b_n, \tag{4.52}$$

$$(\lambda a)_n = \lambda a_n, \tag{4.53}$$

$$(a \cdot b)_n = a_n \sum_{j=0}^n \delta^j(b_{n-j}) + b_n \sum_{j=1}^n \delta^j(a_{n-j}), \tag{4.54}$$

$$(a^*)_n = \bar{a}_n. \tag{4.55}$$

These operations are well defined and very natural, except maybe the multiplication of two elements of $\mathcal{E}_*(A)$. We point out here that the index in one of the sums in (4.54) starts running from 0.

Proposition 4.28 ([7], Proposition 4.5). *The set $\mathcal{E}_*(A)$ with operations (4.52)–(4.55) becomes a commutative algebra with involution.*

Now let us define a morphism

$$\varphi: \mathcal{E}_*(A) \rightarrow C(\mathcal{X}).$$

To this end, let $a = \{a_n\}_{n \geq 0} \in \mathcal{E}_*(A)$ and $\tilde{x} = (x_0, x_1, \dots) \in \mathcal{X}$. We set

$$\varphi(a)(\tilde{x}) = \sum_{n=0}^{\infty} a_n(x_n), \tag{4.56}$$

where $a_n(x_n) = 0$ whenever $x_n = 0$. The mapping φ is well defined as only a finite number of functions a_n are nonzero.

Theorem 4.29 ([7], Theorem 4.6). *The mapping $\varphi: \mathcal{E}_*(A) \rightarrow C(\mathcal{X})$ given by (4.56) is a morphism of algebras with involution. The image of φ is dense in $C(\mathcal{X})$, that is,*

$$\overline{\varphi(\mathcal{E}_*(A))} = C(\mathcal{X}).$$

Let us consider the quotient space $\mathcal{E}_*(A)/\ker \varphi$ and the corresponding quotient mapping $\phi: \mathcal{E}_*(A)/\ker \varphi \rightarrow C(\mathcal{X})$, that is, $\phi(a + \ker \varphi) = \varphi(a)$. Clearly ϕ is an injective mapping onto a dense $*$ -subalgebra of $C(\mathcal{X})$. Let us set

$$E_*(A) := \phi(\mathcal{E}_*(A)/\ker \varphi), \quad (4.57)$$

$$[a] := \phi(a + \ker \varphi), \quad a \in \mathcal{E}_*(A). \quad (4.58)$$

Definition 4.30 ([7], Definition 4.7). $E_*(A)$ is called a *coefficient $*$ -algebra* of the pair (A, δ) . We shall write

$$[a] = [a_0, a_1, \dots] \in C(\mathcal{X})$$

for $a = (a_0, a_1, a_2, \dots) \in \mathcal{E}_*(A)$.

The natural injection $A \ni a_0 \rightarrow [a_0, 0, 0, \dots] \in E_*(A)$ enables us to treat A as a C^* -subalgebra of $E_*(A)$ and hence also of $\mathcal{A} = C(\mathcal{X})$:

$$A \subset E_*(A) \subset \mathcal{A}, \quad \overline{E_*(A)} = \mathcal{A}.$$

Once the extension of A to the coefficient algebra \mathcal{A} is implemented the further construction of the crossed product in [7], §5 goes smoothly. Since here δ and δ_* are endomorphisms of \mathcal{A} they are partial automorphisms (Proposition 4.26) and their actions are described in Theorem 4.27. Therefore the crossed product in [7] is naturally defined as the partial crossed product developed in [21]. Clearly in the situation considered it coincides with $\mathcal{A} \times_\delta \mathbb{Z}$.

To finish this subsection we would like to mention a number of interesting observations made in [7] about the dependence of \mathcal{X} on γ .

If γ is surjective then $\mathcal{X}_N = \emptyset$, $n \in \mathbb{N}$, and $\mathcal{X} = \mathcal{X}_\infty$; in this case \mathcal{X} can be defined as a projective limit ([7], Proposition 3.10).

If γ is injective then a natural continuous projection $\Phi: \mathcal{X} \rightarrow X$ given by the formula

$$\Phi(x_0, x_1, \dots) = x_0, \quad (4.59)$$

is a homeomorphism ([7], Proposition 2.3).

If (X, γ) is the one-sided topological Markov chain then its reversible extension $(\mathcal{X}, \tilde{\gamma})$ is the two-sided topological Markov chain ([7], Example 2.8).

The latter example shows in particular that Kwasniewski's crossed product is qualitatively different from the Cuntz-Krieger algebra. Though both these crossed products start from the one-sided topological Markov chain, in the Cuntz-Krieger construction we arrive at the coefficient algebra \mathcal{F}_A (see §4.3) while Kwasniewski's construction leads to the commutative coefficient algebra $C(\mathcal{X})$ where $(\mathcal{X}, \tilde{\gamma})$ is the two-sided topological Markov chain.

§ 5. Crossed product, coefficient algebras, transfer operators etc. (interrelations)

The constructions presented in the previous section being different (and sometimes even 'qualitatively' different, here we would like to 'counterpose' Exel's and Kwasniewski's crossed products) all turned out to be of the type of the crossed

product introduced in this article: in all of them there arise a finely representable pair (\mathcal{A}, δ) and the corresponding complete transfer operator δ_* .

But since nevertheless these constructions are different one naturally arrives at the question: what object should be named the ‘crossed product’? Or, perhaps, the ‘more exact’ question: ‘what’ are we crossing with ‘what’? The objects of the previous section along with the results of §§ 2 and 3 lead to the ‘natural answer’ which we are giving below.

5.1. Crossed product construction. The crossed product construction consists of two steps.

Step 1 (initial object and extension procedure). There should be given a certain C^* -algebra A (or even only a certain set of elements of A) and an extension procedure by means of which one can extend the algebra A to a coefficient C^* -algebra \mathcal{A} and define an endomorphism $\delta: \mathcal{A} \rightarrow \mathcal{A}$ in such a way that the pair (\mathcal{A}, δ) is finely representable (this is equivalent to the existence of a complete transfer operator δ_* for (\mathcal{A}, δ) which is unique by Theorem 2.2).

Step 2 (crossing the coefficient algebra with the endomorphism δ). Once \mathcal{A}, δ and δ_* are given the crossed product (of \mathcal{A} and δ) is defined according to Definition 2.6.

So ‘in essence’ we are crossing the coefficient algebra \mathcal{A} with the endomorphism δ , while the initial algebra A and the extension procedure serve as an instrument to construct \mathcal{A} and δ .

5.2. Discussion. I. *Extension.* In Step 1 of the crossed product construction the extension procedure was mentioned. What is it? Is there any possibility to describe it explicitly? In the general setting (unfortunately) we do not see a way to give the complete description. Looking through the crossed product structures considered in § 4 one can notice that neither a transfer operator (see, for example, Kwasniewski’s crossed product) nor even an endomorphism of A (see Example 4.20) is necessary for this procedure. Moreover if we look at the Cuntz-Krieger algebra, then among the starting objects we find the projections $Q_i = S_i^* S_i$ and $P_i = S_i S_i^*$ the set of which does not even form an algebra and there is no transfer operator among the starting objects either (the necessary algebra and transfer operator arise here as the outcome of the extension procedure). We would like also to mention in this connection the paper [26] by Lindiarni and Raeburn where the starting objects are a C^* -algebra A and the positive cone Γ^+ of a totally ordered Abelian group acting on A by endomorphisms (that are extendible onto the multiplier algebra of A); no transfer operator(s) is(are) presumed. Assuming that these endomorphisms are generated by partial isometries $V_s, s \in \Gamma^+$, given by $A \ni a \mapsto V_s a V_s^*$, in such a way that the $V_s^* V_s$ belong to the commutant of A the authors define the crossed product as the universal enveloping algebra C^* -algebra generated by A and V_s . Clearly the corresponding coefficient algebra \mathcal{A} (the extension of A) is the universal C^* -algebra generated by $A, V_s^* A V_s, s \in \Gamma^+$, and also $\delta_s(\cdot) = V_s(\cdot) V_s^*$ and $\delta_{*s}(\cdot) = V_s^*(\cdot) V_s$.

In fact in the general situation the extension procedure is given by the mappings $A \ni a \mapsto U a U^*$ and $A \ni a \mapsto U^* a U$ that depend on the origin of the algebra A and the origin of the operator U (in general neither the mapping $U(\cdot) U^*$ nor the mapping $U^*(\cdot) U$ preserves A so they are neither endomorphisms nor transfer operators). In principle the extension procedure reduces to a certain axiomatic

description of these mappings. The main structures and objects that will appear in this way are described in [10], §3.

Here we would like also to remind the already mentioned ‘contrast’ between the Cuntz-Krieger algebra and Kwasniewski’s crossed product for the one-sided topological Markov chain. Both these crossed products start from the one-sided topological Markov chain. Then the operators S_i in the Cuntz-Krieger algebra generate the isometry S (4.18), which leads to the extension procedure given by $S(\cdot)S^*$ and $S^*(\cdot)S$ and ending with the coefficient algebra \mathcal{F}_A (while the extension procedure for Kwasniewski’s crossed product leads to the commutative coefficient algebra $C(\mathcal{X})$, where $(\mathcal{X}, \tilde{\gamma})$ is the two-sided topological Markov chain).

In the recent paper [8] by Exel certain extension procedures are discussed that have the form

$$U^*aU = \mathcal{H}(a)U^*U \quad \text{and} \quad UaU^* = \mathcal{V}(a)UU^*, \quad a \in A,$$

where A is a C^* -algebra, U is a partial isometry and $\mathcal{V}, \mathcal{H}: A \rightarrow A$ are some positive linear maps. After finishing a series of fascinating calculations, at the end of [8], §7 Exel exclaims: “The reader may be struck with the impression that the wild juggling of covariant representations... is a bit exaggerated and that something must be done to stop it. I agree.” We agree as well. But all the foregoing reasoning convinces us that there is no universal way to describe an arbitrary extension procedure. We repeat once more that there are as many extension procedures as types of C^* -algebras A and mappings $A \ni a \mapsto UaU^*$ and $A \ni a \mapsto U^*aU$.

II. *Coefficient algebras and transfer operators.* In this article we have started with the coefficient algebras (§2). But in fact as is clear from the above discussion the coefficient algebra is not a starting object but rather the ending (intermediate) one—it is the target of the extension procedure.

Concerning the interrelations between the coefficient algebras and transfer operators we have to emphasize that at the beginning (when we are starting to construct a coefficient algebra) a transfer operator may or may not arise (it depends on the extension procedure we are confronted with) while at the end (when we have already constructed the coefficient algebra) this operator arises necessarily and is unique. It is worth mentioning that in Exel’s construction of the crossed product (see Definition 4.10) the transfer operator \mathcal{L} is not defined in a unique way (it is not totally defined by α ; see, for example [11], Remark 2.4). On the other hand, when by means of Exel’s construction one obtains a coefficient algebra \mathcal{A} (like in Theorems 4.17 and 4.18) then the arising transfer operator δ_* is unique (it is totally defined by δ ; see Theorem 2.2). Therefore the difference between various Exel’s transfer operators \mathcal{L} indicates the difference between the starting objects for the various corresponding extension procedures (we repeat that not all of the extension procedures can be defined by means of a certain transfer operator).

III. *Crossed product.* The construction given in §2 does not cover all the already existing crossed-product type constructions (in particular Exel’s crossed product associated with the transfer operator \mathcal{L} that is not related to a partial isometry is not embedded into it). On the other hand, we believe (and the material of the article convinces us in this) that once one is confronted with the crossed product

structure related to partial isometries one will necessarily come to the crossed product construction described in §5.1 and therefore at the end to the crossed product given in Definition 2.6.

This crossed product is quite satisfactory in a number of ways: it covers all the most successful crossed product structures developed earlier (see §4), and it possesses good internal structural properties (described in §3). Therefore recalling Exel's exclamation: "... something must be done to stop it ..." we believe that this crossed product may be considered as a reasonable (at least intermediate) stop.

Bibliography

- [1] J. Cuntz, "Simple C^* -algebra generated by isometries", *Comm. Math. Phys.* **57**:2 (1977), 173–185.
- [2] J. Cuntz and W. Krieger, "A class of C^* -algebras and topological Markov chains", *Invent. Math.* **56**:3 (1980), 251–268.
- [3] W. L. Paschke, "The crossed product of a C^* -algebra by an endomorphism", *Proc. Amer. Math. Soc.* **80**:1 (1980), 113–118.
- [4] P. J. Stacey, "Crossed products of C^* -algebras by $*$ -endomorphisms", *J. Aust. Math. Soc. Ser. A* **54**:2 (1993), 204–212.
- [5] G. J. Murphy, "Crossed products of C^* -algebras by endomorphisms", *Integral Equations Operator Theory* **24**:3 (1996), 298–319.
- [6] R. Exel, "A new look at the crossed-product of a C^* -algebra by an endomorphism", *Ergodic Theory Dynam. Systems* **23**:6 (2003), 1733–1750.
- [7] B. K. Kwaśniewski, "Covariance algebra of a partial dynamical system", *Cent. Eur. J. Math.* **3**:4 (2005), 718–765.
- [8] R. Exel, "Interactions", *J. Funct. Anal.* **244**:1 (2007), 26–62.
- [9] N. Brownlowe and I. Raeburn, "Exel's crossed product and relative Cuntz–Pimsner algebras", *Math. Proc. Cambridge Philos. Soc.* **141**:3 (2006), 497–508.
- [10] A. V. Lebedev and A. Odziejewicz, "Extensions of C^* -algebras by partial isometries", *Mat. Sb.* **195**:7 (2004), 37–70; English transl. in *Sb. Math.* **195**:7 (2004), 951–982.
- [11] V. I. Bakhtin and A. V. Lebedev, "When a C^* -algebra is a coefficient algebra for a given endomorphism", arXiv: [math.OA/0502414](https://arxiv.org/abs/math.OA/0502414).
- [12] A. B. Antonevich, V. I. Bakhtin and A. V. Lebedev, "Crossed product of a C^* -algebra by an endomorphism, coefficient algebras and transfer operators", arXiv: [math.OA/0502415](https://arxiv.org/abs/math.OA/0502415).
- [13] D. P. O'Donovan, "Weighted shifts and covariance algebras", *Trans. Amer. Math. Soc.* **208** (1975), 1–25.
- [14] A. V. Lebedev, "On certain C^* -methods that are used while investigating algebras associated with automorphisms and endomorphisms", VINITI, no. 5351-B87 (Russian), 1987.
- [15] A. Antonevich and A. Lebedev, *Functional differential equations: I. C^* -theory*, Pitman Monographs Surveys Pure Appl. Math., vol. 70, Longman, Harlow 1994.
- [16] S. Boyd, N. Keswani and I. Raeburn, "Faithful representations of crossed products by endomorphisms", *Proc. Amer. Math. Soc.* **118**:2 (1993), 427–436.
- [17] S. Adji, M. Laca, M. Nilsen and I. Raeburn, "Crossed products by semigroups of endomorphisms and the Toeplitz algebras of ordered groups", *Proc. Amer. Math. Soc.* **122**:4 (1994), 1133–1141.
- [18] R. Exel, "Amenability for Fell bundles", *J. Reine Angew. Math.* **492** (1997), 41–73.

- [19] A. Antonevich, M. Belousov and A. Lebedev, *Functional differential equations. II. C^* -applications: Part 1. Equations with continuous coefficients*, Pitman Monographs Surveys Pure Appl. Math., vol. 94, Longman, Harlow 1998.
- [20] A. Antonevich, M. Belousov and A. Lebedev, *Functional differential equations. II. C^* -applications: Part 2. Equations with discontinuous coefficients and boundary value problems*, Pitman Monographs Surveys Pure Appl. Math., vol. 95, Longman, Harlow 1998.
- [21] R. Exel, “Circle actions on C^* -algebras, partial automorphisms, and a generalized Pimsner–Voiculescu exact sequence”, *J. Funct. Anal.* **122**:2 (1994), 361–401.
- [22] K. McClanahan, “ K -theory for partial crossed products by discrete groups”, *J. Funct. Anal.* **130**:1 (1995), 77–117.
- [23] A. V. Lebedev, “Topologically free partial actions and faithful representations of crossed products”, *Funktsional. Anal. i Prilozhen.* **39**:3 (2005), 54–63; English transl. in *Funct. Anal. Appl.* **39**:3 (2005), 207–214.
- [24] P. R. Halmos and L. J. Wallen, “Powers of partial isometries”, *J. Math. Mech.* **19**:8 (1970), 657–663.
- [25] B. K. Kwaśniewski and A. V. Lebedev, “Reversible extensions of irreversible dynamical systems: C^* -method”, *Mat. Sb.* **199**:11 (2008), 45–74; English transl. in *Sb. Math.* **199**:11 (2008), 1621–1648.
- [26] J. Lindiarni and I. Raeburn, “Partial-isometric crossed products by semigroups of endomorphisms”, *J. Operator Theory* **52**:1 (2004), 61–87.

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