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Generalized Functions, Linear and Nonlinear Problems, I

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SCHRÖDINGER OPERATOR OF THE FORM $-\Delta u + a\delta u + b\frac{\partial\delta}{\partial x_1}u$

Abstract. The paper is devoted to the study of the formal differential expression of the form

$$Lu = -\Delta u + a\delta u + b\frac{\partial\delta}{\partial x_1}u$$

with generalized coefficients. Approximations of the singular part by means of a family of finite range operators are constructed and resolvent convergence of the approximations is investigated.

1. Introduction

The stationary Schrödinger operator with singular potential, symbolically written as

$$-\Delta u + a\,\delta u, \tag{1}$$

where δ is the Dirac δ-function, and a is the so-called coupling constant, models scattering on a particle located at the origin of coordinates.

The mathematical difficulties that appear during the investigation of expression (1) are related to the fact that the product $\delta \cdot u$ in (1) is not defined in the classical theory of distributions. Therefore, giving sense to the expression (1) as a self-adjoint operator in the space $L^2(\mathbb{R}^3)$ (which is usually necessary in quantum theory) requires overcoming some obstacles.

A mathematical interpretation of the expression (1) was given by F. Berezin and L. Faddeev in [4]. It looks as follows. Let $\mathring{L}$ be the restriction of the Laplace operator $-\Delta$ on the domain

$$D(\mathring{L}) = \{u \in H^2(\mathbb{R}^3),\ \ u(0) = 0\},$$

where $H^2(\mathbb{R}^3)$ is the Sobolev space. Then $\mathring{L}$ is a symmetric, but non-self-adjoint operator on $L^2(\mathbb{R}^3)$. All self-adjoint extensions $L^{(\alpha)}$ of the operator $\mathring{L}$ can be considered as possible perturbations of the Laplace operator by potentials, supported at zero. These self-adjoint extensions $L^{(\alpha)}$ are naturally parameterized by a single real parameter $\alpha \in (-\infty, +\infty]$, the value $\alpha = +\infty$ corresponds to the Laplace operator, i.e. $\alpha = +\infty$ if the perturbation does not influence the operator.

The expression (1) by itself does not contain the information as to what self-adjoint extension $L^{(\alpha)}$ corresponds to the concrete situation. In application, the expression (1) arises as a formal limit (as $\varepsilon \to 0$) of some family of operators L_ε. For example let

$$L_\varepsilon u = -\Delta u + q_\varepsilon(x)u, \tag{2}$$

where the potential $q_\varepsilon(x)$ is supported at ε-neighborhood of zero. Under the conditions

$$a(\varepsilon) = \int q_\varepsilon(x)dx \neq 0,$$

$$\int |q_\varepsilon(x)|dx \leq Ca(\varepsilon)$$

we have

$$\frac{1}{a(\varepsilon)} q_\varepsilon(x) \to \delta$$

and the family of potentials q_ε can be symbolically written as $a(\varepsilon)\delta$. Therefore the family (2) can be considered as an approximation of the formal expression (1).

The problem is to bring to light what self-adjoint extension corresponds to given approximation L_ε. As a rule, in usual sense the limit of L_ε does not exist and the resolvent convergence is considered here. Recall that one says that $L_\varepsilon \to L^{(\alpha)}$ in *resolvent sense*, if

$$\lim_{\varepsilon\to 0}(L_\varepsilon - \lambda I)^{-1} = (L^{(\alpha)} - \lambda I)^{-1}. \tag{3}$$

Different approximations of (1) were investigated in many papers (see [1, 2, 6] and references in [1]).

The main result looks as follows: if $a(\varepsilon) = a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots$, the limit (3) exists and defines an operator $L^{(\alpha)}$; this limit is a non-trivial extension ($\alpha \neq \infty$) only in the so-called *resonance cases*, when $a(\varepsilon) = a_1\varepsilon + a_2\varepsilon^2 + \cdots$ and the number a_1 belong to a discrete set Λ from $\mathbb{R}$, where Λ depends on the given approximation.

In more general cases the family of potentials q_ε can be symbolically written as

$$q_\varepsilon = a(\varepsilon)\delta + \sum_k b_k(\varepsilon)\frac{\partial\delta}{\partial x_k}, \tag{4}$$

and then the family L_ε is an approximation of the formal expression

$$Lu = -\Delta u + a\delta u + \sum_k b_k \frac{\partial\delta}{\partial x_k} u \tag{5}$$

Expressions of the form (5) were investigated early in the one-dimensional case [5, 8, 7].

In the present paper we consider some approximations of (5) in $L_2(\mathbb{R}^3)$ and calculate the limits (3). A new effect is discovered: *strong resonance cases* arise, when the limit (3) does not exist and the family L_ε cannot be interpreted as an operator in $L_2(\mathbb{R}^3)$.

2. Approximation using a family of finite rank operators

Let us consider the most simple approximation of the formal expression

$$Lu = -\Delta u + a\delta u + b\frac{\partial\delta}{\partial x_1}u. \tag{6}$$

Let $\varphi_1, \varphi_2 \in D(\mathbb{R}^3)$ such that $\varphi_1(x), \varphi_2(x) \in \mathbb{R}$ and $\int \varphi_i(x)\,dx = 1$, $i = 1,2$. The family of smooth functions

$$\varphi_{i,\varepsilon}(x) = \frac{1}{\varepsilon^3}\varphi_i\left(\frac{x}{\varepsilon}\right)$$

gives an approximation of δ as an element from the space of distributions $D'(\mathbb{R}^3)$. The family of linear functionals

$$\Phi_{k,\varepsilon}(u) = \int \varphi_{k,\varepsilon}(y)u(y)\,dy$$

gives an approximation of δ as a linear functional, since for smooth u

$$\Phi_{1,\varepsilon}(u)\varphi_{1,\varepsilon} \to u(0)\delta = \delta u,$$

the family of rank one operators

$$\Phi_{1,\varepsilon}(u)\varphi_{1,\varepsilon}$$

is an approximation of the operator of multiplication by δ. Let

$$\psi_\varepsilon(x) = \frac{\partial\varphi_{2,\varepsilon}(x)}{\partial x_1} = \frac{1}{\varepsilon^4}\frac{\partial\varphi_2}{\partial x_1}\left(\frac{x}{\varepsilon}\right).$$

In order to have below a uniform expression, we will use the notation

$$\varphi_3 = \frac{\partial\varphi_2(x)}{\partial x_1}, \quad \varphi_{3,\varepsilon} = \frac{1}{\varepsilon^3}\varphi_3(\frac{x}{\varepsilon}).$$

Then

$$\psi_\varepsilon = \frac{1}{\varepsilon}\varphi_{3,\varepsilon}.$$

The family of smooth functions ψ_ε gives an approximation of $\partial\delta/\partial x_1$ as an element from the space of distributions $D'(\mathbb{R}^3)$, the family of linear functionals

$$\Psi_\varepsilon(u) = \int \psi_\varepsilon(y)u(y)\,dy$$

gives an approximation of $\partial\delta/\partial x_1$ as a linear functional.

For a smooth function u, by definition

$$\frac{\partial\delta}{\partial x_1}u = -\frac{\partial u}{\partial x_1}(0)\delta + u(0)\frac{\partial\delta}{\partial x_1} = \langle\delta'; u\rangle\delta + \langle\delta; u\rangle\delta'$$

and the family of rank two operators $\Psi_\varepsilon(u)\varphi_{2,\varepsilon}(x)+\Phi_{2,\varepsilon}(u)\psi_\varepsilon(x)$ is an approximation of the operator of multiplication by $\partial\delta/\partial x_1$.

Therefore the family of operators

$$L_\varepsilon(u) = -\Delta u + T_\varepsilon u, \tag{7}$$

where

$$\begin{aligned} T_\varepsilon u &= a(\varepsilon)\varphi_{1,\varepsilon}(x)\int u(y)\varphi_{1,\varepsilon}(y)\,dy \\ &+ b(\varepsilon)\left[\varphi_{2,\varepsilon}(x)\int u(y)\psi_\varepsilon(y)\,dy + \psi_\varepsilon(x)\int u(y)\varphi_{2,\varepsilon}(y)\,dy\right], \end{aligned} \tag{8}$$

is an approximation of the formal expression (6).

The problem is to find the limit of these approximations in the sense of resolvent convergence.

For fixed $\varepsilon > 0$ the resolvent $R(\lambda,\varepsilon) = (L_\varepsilon - \lambda I)^{-1}$ can be constructed in explicit form by using results from [3].

Let

$$A(\varepsilon) = \begin{pmatrix} a(\varepsilon) & 0 & 0 \\ 0 & 0 & b(\varepsilon) \\ 0 & b(\varepsilon) & 0 \end{pmatrix}$$

be a matrix, generated by the coefficients $a(\varepsilon)$ and $b(\varepsilon)$. The inverse matrix is

$$A^{-1}(\varepsilon) = \begin{pmatrix} \frac{1}{a(\varepsilon)} & 0 & 0 \\ 0 & 0 & \frac{1}{b(\varepsilon)} \\ 0 & \frac{1}{b(\varepsilon)} & 0 \end{pmatrix}.$$

Let us introduce the fundamental solution

$$E_\lambda(x) = \frac{1}{4\pi\|x\|}e^{-\mu\|x\|},$$

where $\mu^2 = -\lambda$, $\operatorname{Re}\mu > 0$ and a vector function

$$\bar{E}(\varepsilon) = (E_1(\varepsilon); E_2(\varepsilon); E_3(\varepsilon)), \quad E_k(\varepsilon) \in L_2(\mathbb{R}^3),$$

where

$$E_1(\varepsilon) = E_\lambda * \varphi_{1,\varepsilon},\ \ E_2(\varepsilon) = E_\lambda * \varphi_{2,\varepsilon},\ \ E_3(\varepsilon) = E_\lambda * \psi_\varepsilon.$$

Denote

$$\langle u, v\rangle = \int u(x)v(x)dx,$$

$$\bar{F}(\varepsilon) = (f_1(\varepsilon); f_2(\varepsilon); f_3(\varepsilon)), \quad f_k(\varepsilon) \in \mathbb{C},$$

where

$$f_1(\varepsilon) = \langle \varphi_{1,\varepsilon}, E_\lambda * f\rangle, \quad f_2(\varepsilon) = \langle \varphi_{2,\varepsilon}, E_\lambda * f\rangle, \quad f_3(\varepsilon) = \langle \psi_\varepsilon, E_\lambda * f\rangle.$$

THEOREM 1. *Let* $\varepsilon > 0$ *and suppose that* $a(\varepsilon) \in \mathbb{R}$, $a(\varepsilon) \neq 0$, $b(\varepsilon) \in \mathbb{R}$, $b(\varepsilon) \neq 0$. *The resolvent* $R(\lambda,\varepsilon)$ *is determined for* $\operatorname{Re}\lambda \neq 0$ *and can be given by the expression*

$$R(\lambda,\varepsilon) f = f * E_\lambda - \left\langle \left[A^{-1}(\varepsilon) + B(\varepsilon,\lambda)\right]^{-1} \bar{F}(\varepsilon),\ \bar{E}(\varepsilon) \right\rangle, \tag{9}$$

where

$$B(\varepsilon,\lambda) = \begin{pmatrix} \langle \varphi_{1,\varepsilon}; E_1(\varepsilon)\rangle & \langle \varphi_{1,\varepsilon}; E_2(\varepsilon)\rangle & \langle \varphi_{1,\varepsilon}; E_3(\varepsilon)\rangle \\ \langle \varphi_{2,\varepsilon}; E_1(\varepsilon)\rangle & \langle \varphi_{2,\varepsilon}; E_2(\varepsilon)\rangle & \langle \varphi_{2,\varepsilon}; E_3(\varepsilon)\rangle \\ \langle \psi_{\varepsilon}; E_1(\varepsilon)\rangle & \langle \psi_{\varepsilon}; E_2(\varepsilon)\rangle & \langle \psi_{\varepsilon}; E_3(\varepsilon)\rangle \end{pmatrix}.$$

If $a(\varepsilon) \equiv 0$, $b(\varepsilon) \in \mathbb{R}$, $b(\varepsilon) \neq 0$, *then*

$$R(\lambda,\varepsilon) f = f * E_\lambda - \left\langle \left[A^{-1}(\varepsilon) + B(\varepsilon,\lambda)\right]^{-1} \bar{F}(\varepsilon), \bar{E}(\varepsilon) \right\rangle, \tag{10}$$

where

$$A(\varepsilon) = \begin{pmatrix} 0 & b(\varepsilon) \\ b(\varepsilon) & 0 \end{pmatrix}, \quad B(\varepsilon,\lambda) = \begin{pmatrix} \langle \varphi_{2,\varepsilon}; E_2(\varepsilon)\rangle & \langle \varphi_{2,\varepsilon}; E_3(\varepsilon)\rangle \\ \langle \psi_{\varepsilon}; E_2(\varepsilon)\rangle & \langle \psi_{\varepsilon}; E_3(\varepsilon)\rangle \end{pmatrix},$$

$$\bar{F}(\varepsilon) = (f_2(\varepsilon); f_3(\varepsilon)), \quad \bar{E}(\varepsilon) = (E_2(\varepsilon); E_3(\varepsilon)).$$

3. Resolvent convergence of approximations

According to (9), the behavior of the resolvent $R(\lambda,\varepsilon)$ depends on the behavior of the matrices A^{-1}, $B(\varepsilon,\lambda)$, and on the behavior of the vectors $\bar{E}(\varepsilon)$ and $\bar{F}(\varepsilon)$.

Let us denote

$$D(\varepsilon,\lambda) = A^{-1}(\varepsilon) + B(\varepsilon,\lambda)$$

and let

$$D^{-1}(\varepsilon,\lambda) = (d_{ij}).$$

Then

$$\begin{aligned} \left\langle \left[A^{-1}(\varepsilon) + B(\varepsilon,\lambda)\right]^{-1} \bar{F}(\varepsilon), \bar{E}(\varepsilon) \right\rangle &= \langle D^{-1}(\varepsilon,\lambda)\bar{F}(\varepsilon), \bar{E}(\varepsilon)\rangle \\ &= [d_{11}(\varepsilon) f_1(\varepsilon) + d_{12}(\varepsilon) f_2(\varepsilon) + d_{13}(\varepsilon) f_3(\varepsilon)]\, E_1(\varepsilon) \\ &\quad + [d_{21}(\varepsilon) f_1(\varepsilon) + d_{22}(\varepsilon) f_2(\varepsilon) + d_{23}(\varepsilon) f_3(\varepsilon)]\, E_2(\varepsilon) \\ &\quad + [d_{31}(\varepsilon) f_1(\varepsilon) + d_{32}(\varepsilon) f_2(\varepsilon) + d_{33}(\varepsilon) f_3(\varepsilon)]\, E_3(\varepsilon). \end{aligned}$$

Let us consider the behavior of the vectors $\bar{E}(\varepsilon)$ and $\bar{F}(\varepsilon)$ as $\varepsilon \to 0$.

It follows from the properties of functions $\varphi_{i,\varepsilon}$ that the limits

$$\lim_{\varepsilon\to 0} E_1(\varepsilon) = \lim_{\varepsilon\to 0} E_2(\varepsilon) = E_\lambda$$

exist in the space $L_2(\mathbb{R}^3)$, and for any $f \in L_2(\mathbb{R}^3)$ there exist the limits

$$\lim_{\varepsilon\to 0} f_1(\varepsilon) = \lim_{\varepsilon\to 0} f_2(\varepsilon) = (f * E_\lambda)(0).$$

In the distribution space $D'(\mathbb{R}^3)$ we have

$$\psi_\varepsilon \to \frac{\partial\delta}{\partial x_1}, \;\; E_3(\varepsilon) = E_\lambda * \psi_\varepsilon \to \frac{\partial E_\lambda}{\partial x_1}.$$

But in the space $L_2(\mathbb{R}^3)$

$$\|\psi_\varepsilon\| = \sqrt{\int \left(\frac{1}{\varepsilon^2}\psi\left(\frac{x}{\varepsilon}\right)\right)^2 dx} = \left(\int |\psi(t)|^2\,dt\right)^{\frac{1}{2}} \frac{1}{\varepsilon\sqrt{\varepsilon}}$$

the norm $\|E_3(\varepsilon)\|$ is increasing as $1/\varepsilon\sqrt{\varepsilon}$ and $E_3(\varepsilon)$ do not have a limit in the space $L_2(\mathbb{R}^3)$.

Similarly, it can be that for $f \in L_2(\mathbb{R}^3)$ the value $f_3(\varepsilon)$ increases and does not have a limit, but always $f_3(\varepsilon) = o(1/\varepsilon\sqrt{\varepsilon})$.

Therefore the finite limit of the resolvent (9) exists only if the elements $d_{13}(\varepsilon)$, $d_{23}(\varepsilon)$, $d_{33}(\varepsilon)$, as well as $(d_{31}(\varepsilon)f_1(\varepsilon)+d_{32}(\varepsilon)f_2(\varepsilon)+d_{33}(\varepsilon)f_3(\varepsilon))$ are small, namely

$$(11) \qquad d_{13}(\varepsilon) \sim o\big(\varepsilon^{\frac{3}{2}}\big); \quad d_{31}(\varepsilon)f_1(\varepsilon)+d_{32}(\varepsilon)f_2(\varepsilon)+d_{33}(\varepsilon)f_3(\varepsilon) \sim o\big(\varepsilon^{\frac{3}{2}}\big).$$

It follows that it is not enough to find the limit of the family of inverse matrices $D^{-1}(\varepsilon,\lambda)$, but it is also necessary to check subsequent terms (not only the main term) of the expansion of the matrix $D^{-1}(\varepsilon,\lambda)$, on which the behavior of expressions (11) depends.

Let us consider the behavior of d_{ij} as $\varepsilon \to 0$.

LEMMA 1. *The functions*

$$b_{lj}(\varepsilon,\lambda) = \langle \varphi_{l,\varepsilon}, E_\lambda * \varphi_{j,\varepsilon}\rangle$$

are analytic functions of two variables ε, μ ($\lambda \in \mathbb{C}\setminus\mathbb{R}_0^+$, $\lambda = -\mu^2$, *where* $\operatorname{Re}\mu > 0$, $\mu = (-\lambda)^{\frac{1}{2}}$, *a continuous branch of the function* $(-\lambda)^{\frac{1}{2}}$), *and admit an expansion*

$$b_{lj}(\varepsilon,\lambda) \equiv b_{lj}(\varepsilon,-\mu^2) = \frac{1}{\varepsilon}\sum_{k=0}^{\infty}(-1)^k(\varepsilon\mu)^k M_{lj}^{(k-1)} = \sum_{k=-1}^{\infty}\varepsilon^k(-\mu)^{k+1}M_{lj}^{(k)},$$

where

$$M_{lj}^{(k)} = \frac{1}{4\pi(k+1)!}\int(\varphi_l * \varphi_j)|x|^k\,dx.$$

In particular, according to the properties of functions $\varphi_i(x)$,

$$M_{11}^{(0)} = M_{12}^{(0)} = M_{21}^{(0)} = M_{22}^{(0)} = \frac{1}{4\pi},$$

$$M_{13}^{(0)} = M_{23}^{(0)} = M_{33}^{(0)} = M_{31}^{(0)} = M_{32}^{(0)} = 0.$$

THEOREM 2. *Let*

$$a(\varepsilon)=\varepsilon a_1+\varepsilon^2 a_2+\varepsilon^3 a_3+\dots,\quad b(\varepsilon)=\varepsilon^p b_p+\varepsilon^{p+1}b_{p+1}+\cdots,$$

where $a_1\neq 0,\ b_p\neq 0$.

I. *If* $a_1\neq -1/M_{11}^{(-1)}$, *then the resolvents (9) converge to the resolvent of the Laplace operator.*

II. *Suppose that the resonance condition* $a_1=-1/M_{11}^{(-1)}$ *holds.*

- *If* $p\geq 4$, *the resolvents (9) converge to the resolvent of the operator* A^α, *where* $\alpha=-a_2\big(M_{11}^{(-1)}\big)^2$.
- *If* $p=3$ *the resolvents (9) converge to the resolvent of the operator* A^α

$$R_\alpha(\mu)f=f*E_\lambda-\frac{4\pi}{4\pi\alpha-\mu}\left[(f*E_\lambda(0))\right]E_\lambda$$

where $\alpha=-a_2\big(M_{11}^{(-1)}\big)^2-b_3\big(M_{12}^{(-1)}M_{32}^{(-2)}+M_{21}^{(-1)}M_{13}^{(-2)}\big)$.

- *If* $p\leq 2$ *the limit of the family of resolvents (9) does not exist.*

Proof. The matrix $D(\varepsilon,\lambda)$ can be written in the form

$$\begin{pmatrix}\frac{1}{a(\varepsilon)}+\frac{1}{\varepsilon}M_{11}^{(-1)}-\frac{\mu}{4\pi}+\dots & \frac{1}{\varepsilon}M_{12}^{(-1)}-\frac{\mu}{4\pi}+\dots & \frac{1}{\varepsilon^2}M_{13}^{(-1)}+\dots\\ \frac{1}{\varepsilon}M_{21}^{(-1)}-\frac{\mu}{4\pi}+\dots & \frac{1}{\varepsilon}M_{22}^{(-1)}-\frac{\mu}{4\pi}+\dots & \frac{1}{b(\varepsilon)}+\frac{1}{\varepsilon^2}M_{23}^{(-1)}+\dots\\ \frac{1}{\varepsilon^2}M_{31}^{(-1)}+\dots & \frac{1}{b(\varepsilon)}+\frac{1}{\varepsilon^2}M_{32}^{(-1)}+\dots & \frac{1}{\varepsilon^3}M_{33}^{(-1)}+\dots\end{pmatrix}.$$

Let us consider the most interesting case $p=3$, when

$$\frac{1}{b(\varepsilon)}=\frac{1}{\varepsilon^3}b_{-3}+\frac{1}{\varepsilon^2}b_{-2}+\dots,$$

$$\frac{1}{a(\varepsilon)}=\frac{1}{\varepsilon}a_{-1}+\tilde{a}_0+\dots,$$

where $b_{-3}=\frac{1}{b_3},\ \ a_{-1}=\frac{1}{a_1}$. In this case the expansion of the matrix $D(\varepsilon,\lambda)$ is

$$\begin{pmatrix}\frac{1}{\varepsilon}(M_{11}^{(-1)}+a_{-1})-\frac{\mu}{4\pi}+\dots & \frac{1}{\varepsilon}M_{12}^{(-1)}-\frac{\mu}{4\pi}+\dots & \frac{1}{\varepsilon^2}M_{13}^{(-1)}+\dots\\ \frac{1}{\varepsilon}M_{21}^{(-1)}-\frac{\mu}{4\pi}+\dots & \frac{1}{\varepsilon}M_{22}^{(-1)}-\frac{\mu}{4\pi}+\dots & \frac{1}{\varepsilon^3}b_{-3}+\frac{1}{\varepsilon^2}(M_{23}^{(-1)}+b_{-2})+\dots\\ \frac{1}{\varepsilon^2}M_{31}^{(-1)}+\dots & \frac{1}{\varepsilon^3}b_{-3}+\frac{1}{\varepsilon^2}(M_{32}^{(-1)}+b_{-2})+\dots & \frac{1}{\varepsilon^3}M_{33}^{(-1)}+\dots\end{pmatrix}.$$

For the inverse matrix, we have expression

$$D^{-1}(\varepsilon,\lambda)=\frac{1}{\det D(\varepsilon,\lambda)}D^{\#}(\varepsilon,\lambda),$$

where

$$\det D(\varepsilon,\lambda) = -\frac{1}{\varepsilon^7}(b_{-3})^2(M_{11}^{(-1)}+a_{-1})$$
$$\frac{1}{\varepsilon^6}\left[\left(M_{12}^{(-1)}M_{31}^{(-1)}+M_{21}^{(-1)}M_{13}^{(-1)}-(M_{11}^{(-1)}+a_{-1})(M_{23}^{(-1)}+b_{-2})\right)b_{-3}\right.$$
$$\left.-\left(-\frac{\mu}{4\pi}+\tilde{a_0}\right)(b_{-3})^2-(M_{11}^{(-1)}+a_{-1})(M_{23}^{(-1)}+b_{-2})b_{-3}\right]+\ldots,$$

$$D^{\#}(\varepsilon,\lambda) = \begin{pmatrix} d_{11}^{\#} & d_{12}^{\#} & d_{13}^{\#} \\ d_{21}^{\#} & d_{22}^{\#} & d_{23}^{\#} \\ d_{31}^{\#} & d_{32}^{\#} & d_{33}^{\#} \end{pmatrix},$$

$$d_{11}^{\#} = -\frac{1}{\varepsilon^6}(b_{-3})^2+\ldots,$$

$$d_{12}^{\#} = -\frac{1}{\varepsilon^5}b_{-3}M_{13}^{(-1)}+\ldots,$$

$$d_{13}^{\#} = \frac{1}{\varepsilon^4}M_{12}^{(-1)}b_{-3}+\ldots,$$

$$d_{21}^{\#} = -\frac{1}{\varepsilon^5}b_{-3}M_{31}^{(-1)}+\ldots,$$

$$d_{22}^{\#} = \frac{1}{\varepsilon^4}\left(M_{33}^{(-1)}(M_{11}^{(-1)}+a_{-1})-M_{31}^{(-1)}M_{13}^{(-1)}\right)+\ldots$$

$$d_{23}^{\#} = -\frac{1}{\varepsilon^4}b_{-3}(M_{11}^{(-1)}+a_{-1})+\ldots$$

$$d_{31}^{\#} = \frac{1}{\varepsilon^4}M_{21}^{(-1)}b_{-3}+\ldots$$

$$d_{32}^{\#} = -\frac{1}{\varepsilon^4}b_{-3}(M_{11}^{(-1)}+a_{-1})+\ldots,$$

$$d_{33}^{\#} = \frac{1}{\varepsilon^2}\left(M_{22}^{(-1)}(M_{11}^{(-1)}+a_{-1})-M_{12}^{(-1)}M_{21}^{(-1)}\right)+\ldots.$$

If

$$(M_{11}^{(-1)}+a_{-1})(b_{-3})^2 \neq 0,$$

then

$$D^{-1}(\varepsilon,\lambda)\to 0,\ \ d_{13}=O(\varepsilon^3),\ \ d_{33}=O(\varepsilon^5),$$

and the condition (11) fulfilled. It follows that the resolvents (9) converge to the resolvent of the Laplace operator.

The limit of the matrix $D^{-1}(\varepsilon,\lambda)$ can be non-zero, if *the resonance condition*

$$(M_{11}^{(-1)} + a_{-1})(b_{-3})^2 = 0, \tag{12}$$

is fulfilled. This condition is equivalent to

$$a_1 = -1/M_{11}^{(-1)}$$

and the resonance is possible only if the coefficient $a(\varepsilon)$ admits an expansion

$$a(\varepsilon) = \varepsilon a_1 + \varepsilon^2 a_2 + o(\varepsilon^2),$$

where $a_1 = -1/M_{11}^{(-1)}$.

Under this condition

$$\det D(\varepsilon,\lambda) = \frac{1}{\varepsilon^6}\left[\left(M_{12}^{(-1)}M_{31}^{(-1)} + M_{21}^{(-1)}M_{13}^{(-1)}\right)b_{-3} - \left(-\frac{\mu}{4\pi} + \tilde{a_0}\right)(b_{-3})^2\right] + \dots$$

and $D^{-1}(\varepsilon,\lambda)$ is a matrix of the form

$$\begin{pmatrix} \frac{-(b_{-3})^2}{\left(M_{12}^{(-1)}M_{31}^{(-1)}+M_{21}^{(-1)}M_{13}^{(-1)}\right)b_{-3}-\left(-\frac{\mu}{4\pi}+\tilde{a_0}\right)(b_{-3})^2} + \varepsilon(\cdots) & \varepsilon(\cdots) & \varepsilon^2(\cdots) \\ \varepsilon(\cdots) & \varepsilon^2(\cdots) & \varepsilon^2(\cdots) \\ \varepsilon^2(\cdots) & \varepsilon^2(\cdots) & \varepsilon^4(\cdots) \end{pmatrix}. \tag{13}$$

So

$$\lim_{\varepsilon\to 0} D^{-1}(\varepsilon,\lambda) = \begin{pmatrix} \frac{4\pi}{4\pi\alpha-\mu} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$\alpha = -a_2\left(M_{11}^{(-1)}\right)^2 - b_3\left(M_{12}^{(-1)}M_{32}^{(-1)} + M_{21}^{(-1)}M_{13}^{(-1)}\right).$$

It follows from (13) that the condition (11) are fulfilled and the limit of the family of resolvents (9) is the resolvent of the operator A^α:

$$R_\alpha(\mu)f = f * E_\lambda - \frac{4\pi}{4\pi\alpha - \mu}\left[(f * E_\lambda(0))\right]E_\lambda.$$

If $p \neq 3$ the calculations are similar. □

We emphasize that a new effect arises here: it can be that the finite limit of the resolvents (9) does not exist. Let us demonstrate this effect in detail for the case $a(\varepsilon) = 0$.

THEOREM 3. *Let* $a(\varepsilon)=0$ *and* $b(\varepsilon)=\varepsilon^p b_p+\varepsilon^{p+1}b_{p+1}+\cdots$, *where* $b_p\neq 0$.

- *If* $p\geq 3$, *then the limit of the family (4) in resolvent sense is the Laplace operator.*
- *If* $p=2$ *and*

$$M_{33}^{(-3)}M_{22}^{(-1)}-\left(M_{23}^{(-2)}+R_2\right)\left(M_{32}^{(-2)}+R_2\right)\neq 0,$$

then the limit of the family (4) in resolvent sense is the Laplace operator.

- *If* $p=2$ *and resonance condition*

$$M_{33}^{(-3)}M_{22}^{(-1)}-\left(M_{23}^{(-2)}+R_2\right)\left(M_{32}^{(-2)}+R_2\right)=0$$

is fulfilled, then limit of the family of resolvents (6) does not exist.

Proof. If $a(\varepsilon)=0$, then the matrix $D(\varepsilon,\lambda)$ is

$$D(\varepsilon,\lambda)=\begin{pmatrix} \frac{1}{\varepsilon}M_{22}^{(-1)}-\frac{\mu}{4\pi}+\dots & \frac{1}{b(\varepsilon)}+\frac{1}{\varepsilon^2}M_{23}^{(-1)}+\dots \\ \frac{1}{b(\varepsilon)}+\frac{1}{\varepsilon^2}M_{32}^{(-1)}+\dots & \frac{1}{\varepsilon^3}M_{33}^{(-1)}+\dots \end{pmatrix}.$$

Remark that

$$\frac{1}{b(\varepsilon)}=\frac{1}{\varepsilon^p}R_p+\frac{1}{\varepsilon^{p-1}}R_{p-1}+\dots,$$

where $R_p=1/b_p$.

If $p>3$, then the main term in the expansion of the matrix $D(\varepsilon,\lambda)$ is the invertible matrix

$$\frac{1}{\varepsilon^p}\begin{pmatrix} 0 & R_p \\ R_p & 0 \end{pmatrix}$$

and $D^{-1}(\varepsilon,\lambda)\to 0$ as ε^p.

If $p=3$, then the expansion of the matrix $D(\varepsilon,\lambda)$ begins from the invertible matrix

$$\frac{1}{\varepsilon^3}\begin{pmatrix} 0 & R_p \\ R_p & M_{33}^{(-1)} \end{pmatrix}$$

and thus $D^{-1}(\varepsilon,\lambda)\to 0$ as ε^3 when $\varepsilon\to 0$.

This means that if $p\geq 3$, then the conditions (11) are fulfilled and the limit of the family (7) in the resolvent sense is the Laplace operator.

If $p=2$, then the matrix $D(\varepsilon,\lambda)$ can be written in the form

$$\begin{pmatrix} \frac{1}{\varepsilon}M_{22}^{(-1)}-\frac{\mu}{4\pi}+\dots & \frac{1}{\varepsilon^2}\left(M_{23}^{(-1)}+R_2\right)+\frac{1}{\varepsilon}\left(M_{23}^{(-1)}+R_1\right)+\dots \\ \frac{1}{\varepsilon^2}\left(M_{32}^{(-1)}+R_2\right)+\frac{1}{\varepsilon}R_1+\dots & \frac{1}{\varepsilon^3}M_{33}^{(-1)}+\dots \end{pmatrix}.$$

If
$$M_{33}^{(-1)}M_{22}^{(-1)}-\left(M_{23}^{(-1)}+R_2\right)\left(M_{32}^{(-1)}+R_2\right)\neq 0,$$
then $D^{-1}(\varepsilon,\lambda)\to 0$ when $\varepsilon\to 0$ as ε^3 and the limit of the family (4) in resolvent sense is the Laplace operator.

Set
$$d_3 = M_{22}^{(-1)}M_{33}^{(-1)}-\frac{\mu}{4\pi}M_{33}^{(-1)}-\left(M_{23}^{(-1)}+R_2\right)\left(M_{32}^{(-1)}+R_1\right) - \left(M_{23}^{(-1)}+R_1\right)\left(M_{32}^{(-1)}+R_2\right).$$

If the *resonance condition*
$$M_{33}^{(-3)}M_{22}^{(-1)}-\left(M_{23}^{(-2)}+R_2\right)\left(M_{32}^{(-2)}+R_2\right)=0$$
is fulfilled and $d_3\neq 0$, the matrix $D^{-1}(\varepsilon,\lambda)$ can be written in the form
$$\frac{1}{d_3}\begin{pmatrix} M_{33}^{(-1)}+\dots & -\varepsilon\left(M_{32}^{(-1)}+R_2\right)-\varepsilon^2R_1+\dots \\ -\varepsilon\left(M_{23}^{(-1)}+R_2\right)-\varepsilon^2R_1+\dots & \varepsilon^2M_{22}^{(-1)}-\varepsilon^3\frac{\mu}{4\pi}+\dots \end{pmatrix}$$
and
$$D^{-1}(\varepsilon,\lambda)\to\begin{pmatrix} \frac{M_{33}^{(-1)}}{d_3} & 0 \\ 0 & 0\end{pmatrix}.$$
But in this case conditions, similar to the conditions (11), are not fulfilled. In particular the expression from (11) includes the term of the form
$$C\varepsilon^2 f_3(\varepsilon)E_3(\varepsilon),$$
which for some $f\in L_2(\mathbb{R}^3)$ can satisfy
$$\left\|\varepsilon^2 f_3(\varepsilon)E_3(\varepsilon)\right\|_{L_2}\to+\infty.$$
So the finite limit of the family of resolvents (6) does not exist. □

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