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# WEIGHTED TRANSLATION OPERATORS GENERATED BY MAPPINGS WITH SADDLE POINTS: A MODEL CLASS

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Abstract. A particular class of weighted translation operators $B$ generated by mappings with saddle points are considered. For $\lambda$ belonging to the spectrum of the operator $B$, a description of properties of the operator $B - \lambda I$ is found. In particular, necessary and sufficient conditions of one-side invertibility are found. It follows from the obtained results that weighted translation operators generated by mappings with saddle points have principally different spectral properties compared to weighted translation operators generated by mappings without saddle points (investigated earlier).

It is proved that the operator $B - I$ is one-side invertible if and only if a certain property of a linear extension associated with the operator $B$ holds.

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## 1. Introduction

A linear bounded operator $B$ acting in a Banach space $F(X)$ of functions (or vector-functions) defined in an arbitrary set $X$ is called a *weighted translation operator* (WTO) if it can be represented as

$$Bu(x) = a_0(x)u(\alpha(x)), \quad x \in X, \tag{1.1}$$

where $\alpha : X \to X$ is a mapping, while $a_0(x)$ is a scalar or matrix-valued function defined in $X$.

Such operators and operator algebras generated by them as well as related functional equations were studied by many authors (in various function spaces) as an independent object and in various applications such as dynamical system theory, integrodifferential, functional-differential, functional, and differential-difference equations, automorphisms and endomorphisms of Banach algebras, nonlocal problems, nonclassical boundary-value problems for the string equation, general theory of operator algebras, etc.

First of all, properties of WTO depend on the dynamics of the mapping $\alpha$, i.e., on the behavior of point orbits under the mapping iterations action. The *orbit* of a point $x_0$ is the point sequence $x_k = \alpha_k(x_0),\ k \in \mathbb{Z}, \alpha_k(x) = \alpha(\alpha_{k-1}(x))$. In this area, the main research direction is links between the

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spectral properties of the weighted translation operator and the behavior of orbits (i.e., the dynamical properties) of the mapping $\alpha$ generating the operator.

The task is to describe the spectrum of the operator $B$, i.e., to obtain invertibility conditions for operators of the kind $B - \lambda I$, but fine properties of the operator $B - \lambda I$ (if $\lambda$ belongs to the spectrum) are interesting as well. In the present paper, the following properties are considered: the closedness of the range of the operator, the dimensions of the kernel and cokernel, the existence of a one-sided inverse operator. Those properties are essential for applications: one can use them to find conditions of normal, Fredholm, or semi-Fredholm solvability of the corresponding functional or functional-differential equations.

The main problem is to find dynamical properties of the mapping $\alpha$, affecting fine properties of the operator $B - \lambda I$. Fine properties of a general weighted translation operator depend on the particular nature of the mapping $\alpha$; this is an open area for the general case.

In Secs. 2–4, we review the known main results about spectral properties of weighted translation operators in spaces $L_2(X, \mu)$. The main result is given in Sec. 5; this is the first description of fine spectral properties for a particular class of weighted translation operators in the space $L_2(X, \mu)$ for the case where the generating mapping $\alpha$ has a saddle point. That result shows that the existence of saddle points of the mapping $\alpha$ leads to a substantial change of fine spectral properties of the operator.

## 2. Spectra of Weighted Translation Operators

If $\alpha$ is an invertible operator in a classical space, then the spectrum of the weighted translation operator is described (as a set) for a sufficiently general case. In such a case, it is convenient to represent the operator $B$ as $B = aT_\alpha$, introducing an auxiliary operator $T_\alpha u(x) = \varrho(x)u(\alpha(x))$, where the *normalizing function* $\varrho$ depends on the space $F(X)$: this function is selected in order for the operator $T_\alpha$ to be isometric.

The main concern of the present paper is a model class of weighted translation operators in a particular space of $L_2(X, \mu)$. To compare, we provide known results only for those spaces; no generalizations for other classes of spaces are considered.

For spaces $L_2(X, \mu)$, a normalizing function $\varrho$ exists if and only if the mapping $\alpha$ preserves the class of the measure $\mu$ (i.e., for any measurable set $E$, we have $\mu(\alpha^{-1}(E)) = 0$ if and only if $\mu(E) = 0$). If $X$ is a domain in $\mathbb{R}^n$, the mapping $\alpha$ is a diffeomorphism, and $F(X) = L_2(X)$, then we assign $\varrho(x) = |J_\alpha(x)|^{1/2}$, where $J_\alpha$ is the Jacobian of the diffeomorphism $\alpha$. It is easier to express the properties of the operator $B$ using the obtained coefficient $a(x) = \varrho(x)^{-1}a_0(x)$ called a *reduced coefficient.*

The description of the spectrum in spaces of scalar functions is provided by the following assertion.

**Theorem 2.1.** *Let $X$ be a compact space. Let $\mu$ be a measure on $X$ such that its support coincides with the whole space. Let $\alpha : X \to X$ be an invertible continuous mapping preserving the class of the measure $\mu$. Let $a \in C(X)$. Let $R(B)$ be the spectral radius of the operator $B = aT_\alpha$. Then the following relation holds in $L_2(X, \mu)$:*

$$R(B) = \max_{\nu \in M_\alpha(X)} \exp\left[\int_X \ln|a(x)|\,d\nu\right],$$

*where $a$ is the reduced coefficient, while $M_\alpha(X)$ is the set of probability measures on $X$ that are invariant and ergodic with respect to the mapping $\alpha$.*

*If $a(x) \neq 0$, then the spectrum $\sigma(B)$ is a subset of the ring*

$$K = \{\lambda \in \mathbb{C} : r(B) \le |\lambda| \le R(B)\},$$

*where*

$$r(B) = \min_{\nu \in M_\alpha(X)} \exp\left[\int_X \ln|a(x)|\,d\nu\right];$$

*$K$ is the least ring centered at the origin and containing the spectrum.*

*If the set of nonperiodic points of the mapping $\alpha$ is dense in $X$, then the spectrum coincides with the ring $K$.*

The most simple application of Theorem 2.1 is the case where the set $\mathrm{Fix}(\alpha)$ of fixed points of the mapping is finite and each orbit tends to a fixed point. Then the set $M_\alpha(X)$ consists of a finite number of measures such that each of those measures is concentrated at a fixed point. The following assertion is valid in such a case.

**Corollary 2.1.** *Let the set $\mathrm{Fix}(\alpha)$ of fixed points of a mapping $\alpha$ be finite and let any orbit tend to a fixed point. Then the spectrum of the weighted translation operator $B$ is the ring*

$$\sigma(B) = \{\lambda \in \mathbb{C} : r(B) \le |\lambda| \le R(B)\},$$

*where*

$$R(B) = \max_{x \in \mathrm{Fix}(\alpha)} |a(x)|, \quad r(B) = \min_{x \in \mathrm{Fix}(\alpha)} |a(x)|.$$

The description of the spectrum is more complicated for vector-function spaces. There are two approaches to description of properties of such operators: hyperbolic and orbital.

1. The hyperbolic approach is based on the link between the operator $B$ and properties of an auxiliary mapping called a *linear extension associated with the operator*. It turns out that the most important (within the framework of our concern) property of the linear extension is the hyperbolicity. The following notions are used by that approach.

The *associated linear extension* $\beta$ of the mapping $\alpha$ is constructed as follows. Let us treat the vector product $E = X \times \mathbf{C}^m$ as a vector fibering over $X$ with a natural projection; the set

$$E_x = \{(x, \xi) : \xi \in C^m\} = \{x\} \times \mathbf{C}^m$$

is the layer $E_x$ over the point $x \in X$.

The mapping $\beta : E \to E$ corresponding to the operator $B$ is constructed as follows:

$$\beta(x, \xi) = \left(\alpha(x), \frac{1}{a(x)}\xi\right), \quad x \in X, \quad \xi \in \mathbf{C}^m, \tag{2.1}$$

where $a$ is the reduced coefficient.

The mapping $\beta$ is continuous and it linearly maps the layer $E_x$ to the layer $E_{\alpha(x)}$. This means that $\beta$ is a linear extension of the mapping $\alpha$.

For any vector $\xi \in \mathbf{C}^m$, its norm is given by the standard scalar product $\langle \xi, \eta \rangle$ and is denoted by $|\xi|$. Introduce the notation $\|(x, \xi)\| = |\xi|$.

Following [8], we call a subset $K$ of a fibering $E$ a *vectorial subset* if the set $K_x := K \cap E_x$ is a vector subspace in $E_x$ for any $x$; if subspaces $K_x$ of a vectorial subset continuously depend on $x$, then this vectorial subset is called a *subfibering.*

We say that a subfibering $E^s$ of a fibering $E$ is *stable towards the positive direction* (*instable towards the negative direction*) for the linear extension $\beta$ if it is invariant under the action of $\beta$ and there exist constants $\gamma_s$, $0 < \gamma_s < 1$ ($\gamma_u$, $0 < \gamma_u < 1$) and $C > 0$ such that the following inequality holds:

$$\|\beta_n(x, \xi)\| \le C\gamma_s^n |\xi|, \quad x \in [0, 1], \quad (x, \xi) \in E^s, \quad n = 1, 2, \ldots$$

$$(\|\beta_n(x, \xi)\| \ge C\gamma_u^{-n} |\xi|, \quad x \in [0, 1], \quad (x, \xi) \in E^u, \quad n = 1, 2, \ldots).$$

A linear extension is called *hyperbolic* if it is expanded into a direct Whitney sum of stable and instable subfiberings (see [8]).

2. The orbital approach is based on the following. Let $l_2(\mathbb{Z}; \mathbb{C}^m)$ be the space of two-sided sequences of vectors from $\mathbf{C}^m$: $u = (u(k)), k \in \mathbf{Z}$, $u(k) \in \mathbf{C}^m$, where the norm $\|u\|$ is defined as $\left(\sum_k |u(k)|^2\right)^{1/2}$.

If $m = 1$, then the space is denoted by $l_2(\mathbb{Z})$. Let $W$ denote the translation operator in the space of sequences acting as follows: $(Wu)(k) = u(k+1)$. Let $B = aT_\alpha$ be an operator of kind (1.1). To any point $\tau \in X$ assign the matrix sequence $a_\tau(k) = a(\alpha_k(\tau))$ and the discrete weighted translation operator $B(\tau) = a_\tau W$ acting in $l_2(\mathbb{Z}; \mathbb{C}^m)$ as follows:

$$(B(\tau)u)(k) = a(\alpha_k(\tau))u(k+1). \tag{2.2}$$

To construct the operator $B(\tau)$, we restrict the coefficient $a$ to the orbit of the point $\tau$; that is why the term "orbital approach" is used.

Operators $B(\tau)$ are discrete weighted translation operators. Therefore, it is easier to study them. The orbital approach is based on the link between properties of the operator $B$ and operators $B(\tau)$. Operators $B(\tau)$ can be treated as local representatives of the operator $B$. This refers to so-called "local research methods," while the corresponding assertions are "local principles." From another point of view, operators $B(\tau)$ are images of the operator $B$ under irreducible presentations of a $C^*$-algebra generated by weighted translation operators with the given mapping $\alpha$.

The main invertibility result is as follows.

**Theorem 2.2.** *Let $X$ be a compact space, $\mu$ be a measure over $X$ such that its support coincides with the whole space, $\alpha : X \to X$ be an invertible continuous mapping preserving the class of the measure $\mu$, and $a$ be a continuous matrix-valued function. If the set of nonperiodic points of the mapping $\alpha$ is dense in $X$, then the following conditions for the weighted translation operator $B$ are equivalent to each other in the space $L_2(X, \mu)$:*

(1) *the operator $B - I$ is invertible*;
(2) *all operators $B(\tau) - I$, $\tau \in X$, are invertible*;
(3) *the linear extension $\beta$ associated with the operator $B$ is hyperbolic.*

In [1, 3, 5, 7, 8], the history of the problem and the development of the above approaches (including the case of operators in other function spaces) are provided.

## 3. Spectra of Weighted Translation Operators at Segments

It turns out that the class of weighted translation operators at a segment, given below, is the easiest to study.

Let $X$ be the segment [0,1] and $\alpha : X \to X$ be a diffeomorphism having only two fixed points: 0 and 1. For definiteness, we assume that $x < \alpha(x)$ for $0 < x < 1$. In the space $L_2([0,1], \mathbb{C}^m)$ of vector-functions defined on $[0,1]$ and valued in $\mathbb{C}^m$, consider weighted translation operators of the kind

$$Bu(x) = a_0(x)u(\alpha(x)) = a(x)[\alpha'(x)]^{1/2}u(\alpha(x)) = (aT_\alpha u)(x), \quad x \in X, \tag{3.1}$$

where $a_0(x)$ is a nondegenerate continuous scalar or matrix-valued function defined on $X$. Here the normalizing function $\varrho$ is given by the expression $\varrho(x) = [\alpha'(x)]^{1/2}$, while the reduced coefficient is given by the expression $a(x) = [\alpha'(x)]^{-1/2}a_0(x)$.

Many results were first obtained for WTO of kind (3.1) and extended for more general cases afterwards. Also, note that the above mappings arise in various applications. In particular, they are very important for the singular integrodifferential equations theory (see [7]).

The mapping $\alpha$ has a simple dynamics in the above example: the point 1 is *attracting*, i.e., the orbit of any point $x \in X, x \neq 0$, tends to the point 1, the point 0 is *repelling*, i.e., the orbit of any point $x \neq 0$ from the neighborhood of the origin leaves its neighborhood, and the set $M_\alpha(X)$ consists of two measures (a measure concentrated at the point 0 and a measure concentrated at the point 1). If the functions in the above example are scalar (i.e., if $m = 1$), then Theorem 2.1 implies the following assertion (see [7]).

**Proposition 3.1.** *If $m = 1$ and $a(x) \neq 0$, then the spectrum of operator* (3.1) *in the space $L_2[0,1]$ is the ring*

$$\sigma(B) = \{\lambda \in \mathbb{C} : r \leq |\lambda| \leq R\},$$

*where*

$$R = \max\{|a(0)|,\ |a(1)|\},\ r = \min\{|a(0)|,\ |a(1)|\}.$$

If $m > 1$, i.e., if the operator acts in the space of vector-functions, then Theorem 2.2 implies the following assertion for the considered model class.

**Proposition 3.2.** *If $m > 1$ and $\det a(x) \neq 0$, then the spectrum of operator* (3.1) *in the space $L_2([0,1], \mathbb{C}^m)$ is a union of at most $m$ rings centered at the origin and the radii of those rings are absolute values of eigenvalues of the matrix $a(0)$ or the matrix $a(1)$.*

## 4. Fine Spectral Properties of Weighted Translation Operators at Segments

Fine properties of a weighted translation operator depend on the particular nature of the mapping $\alpha$; the general case is still open. The simplest case of weighted translation operators are discrete weighted translation operators in spaces of sequences of the kind $aW$, where $a$ is a bounded sequence. However, the investigation of fine spectral properties is not a trivial task even for those operators with coefficients of general type (see, e.g., [7]).

In a more general case, properties of the operator $B - \lambda I$ for spectral values of $\lambda$ are investigated only when the mapping of the segment belongs to the model class introduced above, but has two fixed points (as well as other operators such that their investigation is reduced to the said case). The first result was obtained for spaces of scalar functions (see [6, 9]). It is formulated as follows.

**Theorem 4.1.** *Let $X = [0,1]$ and $\alpha : X \to X$ be a $C^1$-diffeomorphism such that $\alpha(0) = 0$, $\alpha(1) = 1$, and $x < \alpha(x)$ for $0 < x < 1$. Let $B$ be an operator of kind* (3.1) *in the space $L_p[0,1]$ of scalar functions, where $a(x) \neq 0$. Then the following assertions are valid*:

*If $|a(1)| < |\lambda| < |a(0)|$, then the operator $B - \lambda I$ is invertible from the left and its kernel is infinite-dimensional.*

*If $|a(0)| < |\lambda| < |a(1)|$, then the operator $B - \lambda I$ is invertible from the right and its cokernel is infinite-dimensional.*

This shows that the attracting point 1 and the repelling point 0 are not of equal status from the point of view of the one-sided invertibility.

In [10], the fine structure of the spectrum of the operator $B$ for a mapping of a segment was described for the case of vector-function spaces. It turns out that this case is more complicated, which is caused by new dynamical properties of the associated linear extension. The main notion for this case is the property of *coherent local hyperbolicity* of the associated linear extension, which is weaker than the hyperbolicity property.

A subfibering $E^{s,0}$ of a fibering $E$ defined in $[0,\delta]$ is called (*locally*) *stable towards the positive direction* (*instable towards the positive direction*) at the point 1 for a linear extension $\beta$ if it is invariant under the action of $\beta$ and there exist a positive $\delta$ and constants $\gamma_s \in (0,1)$ ($\gamma_u \in (0,1)$) and $C > 0$ such that the following inequality is valid for any $x \in [1-\delta, 1]$:

$$\|\beta_n(x,\xi)\| \leq C_s \gamma_s^n |\xi|, \quad (x,\xi) \in E^{s,0}, \qquad n = 1, 2, \ldots$$

$$(\|\beta_n(x,\xi)\| \geq C_u \gamma_u^{-n} |\xi|, \quad x \in [0,\delta], \quad (x,\xi) \in E^{u,0}, \quad n = 1, 2, \ldots).$$

We say that the linear extension $\beta$ is *locally hyperbolic* at the point 1 if there exists a neighborhood of the point 1 such that the fibering $E$ is the direct sum of stable and instable subfiberings in the specified neighborhood.

Subfiberings stable and instable towards the negative direction in a neighborhood of the repelling point 0 are defined analogously (i.e., as subfiberings stable and instable towards the positive direction for the inverse mapping $\beta^{-1}$); the local hyperbolicity at the origin is defined analogously as well.

Let the linear extension $\beta$ be locally hyperbolic at the points 0 and 1. Let $\theta \in (0,1)$ and $\Theta = [\theta, \alpha(\theta))$. For any subfibering $E^{s,0}$ defined over a neighborhood of the point 0, there exists an extension to the set $[0, \alpha(\theta)]$; we will denote that extension by $E^{s,0}$ as well. For any subfibering $E^{s,1}$ defined over a neighborhood of the point 1, there exists an extension to the set $[\theta, 1]$. Both subfiberings are defined over the set $\Theta$. Their mutual location can be characterized by the function

$$d(\tau) = \dim V_\tau, \text{ where } V_\tau := E^{s,0}_\tau \cap E^{s,1}_\tau, \ \tau \in \Theta.$$

We say that the linear extension $\beta$ is *coherently locally hyperbolic* if the function $d(\tau)$ is constant.

We say that the linear extension $\beta$ is *plus-semihyperbolic* (*minus-semihyperbolic*) if it is locally hyperbolic and $d(\tau) = 0$ ($\dim E^{s,0}_\tau + \dim E^{s,1}_\tau = m + d(\tau)$).

The main results of [10] are as follows.

**Theorem 4.2.** *Let $X = [0,1]$. Let $\alpha : X \to X$ be a $C^1$-diffeomorphism such that $\alpha(0) = 0$, $\alpha(1) = 1$, and $\alpha(x) \neq x$ for $0 < x < 1$. Let $B$ be an operator of kind* (3.1) *in the vector-function space $L_p([0,1], \mathbb{C}^m)$, where the coefficient $a$ is a nondegenerate continuous matrix-function. Then the following properties are equivalent to each other*:

(1) *the range of the operator $B - I$ is closed*;
(2) *the linear extension $\beta$ associated with the operator $B$ is coherently locally hyperbolic*;
(3) *all operators $B(\tau) - I$, $\tau \in \Theta$, are Fredholm operators and the dimensions of their kernels are equal to each other.*

**Theorem 4.3.** *If the conditions of Theorem* 4.2 *are satisfied, then the following properties are equivalent to each other*:

(1) *the operator $B - I$ is invertible from the right*;
(2) *the linear extension $\beta$ is minus-semihyperbolic*;
(3) *all operators $B(\tau) - I$, $\tau \in \Theta$, are invertible from the right.*

Note that Theorem 4.2 imposes a condition on the operators $B(\tau)$ only for $\tau \in \Theta$, while the general invertibility theorem (i.e., Theorem 2.2) imposes a condition on the operators $B(\tau)$ for all $\tau \in X$. This distinction is substantial. In the considered case, the discrete operators $B(0)$ and $B(1)$ corresponding to the points 0 and 1 are invertible. Therefore, the dimensions of kernels are equal only for the operators corresponding to inner points of the segment; this assumption is not satisfied for the operators $B(0)$ and $B(1)$.

As we note above, the spectrum of the operator $B$ is the union of a finite number of rings and the properties of operators $B - \lambda I$ are the same for all $\lambda$ that are inner points of one ring. For $m > 1$, there are examples of operators $B - \lambda I$ that are not invertible from one side. This means that there are principal distinctions from the scalar case described by Theorem 4.1. In particular, the following situation might occur: there exist rings from the spectrum of the operator such that the range of the operator $B - \lambda I$ is closed for no points of those rings; it is also possible that the kernel and cokernel of such an operator are infinite-dimensional.

## 5. Fine Spectral Properties: Saddle Point Effect

Mappings of a segment considered above have the simplest dynamical properties. The next step is the class of mappings such that any orbit of them converges to a fixed point and the number of those fixed points is finite. This is a more general case: there exists one more class of fixed points (apart from attracting and repelling ones); this is a class of so-called saddle points.

Considering the description of the spectrum of the weighted translation operator generated by a mapping from the specified class (see Corollary 2.1), we see that the values of the coefficient are of equal status for all fixed points. However, if we have at least two fixed points, then the dependence of fine spectral properties on the coefficient at the repelling point is different from their dependence on the coefficient at the attracting one (see Theorem 4.1). Therefore, the following question is reasonable: *how* *does the existence of a saddle point of the mapping $\alpha$ affect fine spectral properties of the operator $B$?* This question was not investigated earlier even for particular cases. In the present paper, this is done for the model example of a weighted translation operator in the space of scalar functions.

The simplest example of a mapping with saddle points is given below. Let $X$ be the following rectangular triangle in the plane:

$$X = \{x = (x_1, x_2) : 0 \le x_2 \le x_1 \le 1\}. \tag{5.1}$$

Let $\alpha : X \to X$ be the following diffeomorphism:

$$\alpha(x_1, x_2) = (\gamma(x_1), \gamma(x_2)); \tag{5.2}$$

here $\gamma\colon [0, 1] \to [0, 1]$ is a diffeomorphism of the segment from the above class of segment mapping (i.e., the class of mappings having only two fixed points: 0 and 1). For definiteness, we assume that $\gamma(x) > x$ if $0 < x < 1$.

The mapping $\alpha$ has three fixed points:

$$F_1 = (0, 0),\ F_2 = (1, 0),\ F_3 = (1, 1).$$

The orbit of any point converges to one of them. The point $F_1$ is repelling: for any $x \ne F_1$ belonging to a neighborhood of $F_1$, $x$ leaves that neighborhood. The point $F_3$ is attracting: the orbit of any point from a neighborhood of $F_3$ tends to $F_3$. The point $F_2$ is a saddle point: there exist orbits entering its neighborhood and orbits leaving its neighborhood.

The operator $T_\alpha u(x) = \varrho(x)u(\alpha(x))$, where $\varrho(x) = \sqrt{J(x)}$ and $J(x) = \gamma'(x_1)\gamma'(x_2)$ is the Jacobian of the mapping $\alpha$, is a unitary operator in the space $L_2(X)$ of scalar functions defined in $X$. Let $B$ be a weighted translation operator with a reduced coefficient $a \in C(X)$, i.e., an operator of the kind

$$Bu(x) = a(x)[\gamma'(x_1)\gamma'(x_2)]^{-1/2}u(\alpha(x)). \tag{5.3}$$

Assume that $a(x) \ne 0$ for all $x$ and the numbers $|a(0, 0)|$, $|a(1, 0)|$, and $|a(1, 1)|$ are different from each other. By virtue of Corollary 2.1, the spectrum of the operator $B$ is the ring

$$\sigma(B) = \{\lambda : r \le |\lambda| \le R\},$$

where

$$R = \max\{(|a(0, 0)|, |a(1, 0)|, |a(1, 1)|\}, \quad r = \min\{(|a(0, 0)|, |a(1, 0)|, |a(1, 1)|\}.$$

Thus, we have three circles centered at the origin with radii $|a(0, 0)|, |a(1, 0)|$, and $|a(1, 1)|$. Two of those circles form the boundary of the ring that is the spectrum of the operator, while the third circle divides the spectrum into two adjacent rings.

Six different cases are possible (this depends on the kind of the inequality for the numbers $|a(0, 0)|$, $|a(1, 0)|$, and $|a(1, 1)|$). In each of those cases, two adjacent rings forming the spectrum of the operator should be considered. The theorem provided below shows that fine spectral properties of the operator $B$ are different for any of those six cases. Moreover, the values of the coefficient at any of the three fixed points (the attractive, the repelling, and the saddle one) cause different effects for the spectral properties of the operator.

The main result is as follows.

**Theorem 5.1** (the main theorem)**.** *Let $X$ be a triangle of kind* (5.1)*, the mapping $\alpha : X \to X$ be defined by relation* (5.2)*, $a \in C(X)$, $a(x) \ne 0$ for all $x \in X$, and $B$ be weighted translation operator* (5.3) *generated by mapping $\alpha$.*

*If $|a(0, 0)| < |a(1, 1)|$, then properties of the operator $B - \lambda I$ depend on the relations between $|a(0, 0)|$, $|a(1, 0)|$, $|a(1, 1)|$, and $|\lambda|$ as follows.*

(1) *Let $|a(0, 0)| < |a(1, 0)| < |a(1, 1)|$.*

*For any $\lambda$ such that $|a(0, 0)| < |\lambda| < |a(1, 1)|$ and $|\lambda| \ne |a(1, 0)|$, the operator $B - \lambda I$ is invertible from the right and its kernel is infinite-dimensional.*

*If $|\lambda| = |a(1,0)|$, then the kernel of the operator $B - \lambda I$ is infinite-dimensional, while its range is not closed and is everywhere dense.*

(2) *Let $|a(0,0)| < |a(1,1)| < |a(1,0)|$.*

*If $|a(0,0)| < |\lambda| < |a(1,1)|$, then the operator $B - \lambda I$ is invertible from the right and its kernel is infinite-dimensional.*

*If $|a(1,1)| < |\lambda| \leq |a(1,0)|$, then the kernel of the operator $B - \lambda I$ consists only of zero, while its range is not closed and is everywhere dense.*

(3) *Let $|a(1,0)| < |a(0,0)| < |a(1,1)|$.*

*If $|a(1,0)| \leq |\lambda| < |a(0,0)|$, then the kernel of the operator $B - \lambda I$ consists only of zero, while its range is not closed and is everywhere dense.*

*If $|a(0,0)| < |\lambda| < |a(1,1)|$, then the operator $B - \lambda I$ is invertible from the right and its kernel is infinite-dimensional.*

(4) *If $|\lambda| = |a(0,0)|$ or $|\lambda| = |a(1,1)|$, then the range of the operator $B - \lambda I$ is not closed.*

Note that a weaker case of Theorem 5.1 is provided (without a proof) in [2].

If $|a(0,0)| > |a(1,1)|$, then the assumptions of Theorem 5.1 are satisfied for the adjoint operator. Therefore, a similar assertion (with the corresponding amendments) is valid for $|a(0,0)| > |a(1,1)|$ (e.g., we have the invertibility from the left instead of the invertibility from the right).

The prove of the theorem consists of several stages provided below.

## 6. Representation of Operators as Operator-Valued Functions

First we represent the operator $B$ as an operator-valued function $B(\tau)$ such that its values are weighted translation operators of kind (2.2) acting in the space $l_2(\mathbb{Z})$. This reduces the problem to the investigation of discrete weighted translation operators, which are easier for the study. From a more general point of view, the construction considered below is a particular realization of decomposition of the operator algebra generated by considered operators into a direct integral of operator algebras with respect to its center (see [4]).

Select a fundamental domain for the mapping $\alpha$, i.e., a measurable set $\Theta$ in $X$ such that its images $\Theta_k = \alpha_k(\Theta)$ do not meet for different values of $k$, their union is dense in $X$, and the complement of that union is a zero-measure set. There are mappings without a fundamental domain, but it exists (and even is not unique) in the considered example. For instance, one can select

$$\Theta = \{x = (x_1, x_2) \in X : \theta \leq x_1 < \gamma(\theta); 0 < x_2 \leq x_1\}, \tag{6.1}$$

where $\theta$ is an arbitrary number from $(0,1)$.

Let $L_2(\Theta, l_2(\mathbb{Z}))$ denote the space consisting of equivalence classes of measurable mappings $U : \Theta \to l_2(\mathbb{Z})$ such that

$$\|U\| = \left( \int\limits_{\Theta} \|U(\tau)\|^2 d\tau \right)^{1/2} < +\infty.$$

We say that $U$ is *measurable* if the scalar function $U(\tau)(k)$ is measurable for any $k$ as a function of variable $\tau \in \Theta$.

**Lemma 6.1.** *The mapping $L_2(X) \ni u \to U \in L_2(\Theta, l_2(\mathbb{Z}))$, where $U$ is the value $U(\tau)$ of the function $U$, $U(\tau) = u_\tau$, and $u_\tau$ is the sequence*

$$u_\tau(k) = \left[ \prod_{j=0}^{k-1} \varrho(\alpha_j(\tau)) \right] u(\alpha_k(\tau)) \quad \text{for} \quad k > 0,$$

$$u_\tau(k) = \left[ \prod_{j=k}^{-1} \varrho(\alpha_j(\tau)) \right]^{-1} u(\alpha_k(\tau)) \quad \text{for} \quad k \geq 0,$$

$$u_\tau(0) = u(\tau),$$

*defines an isometric isomorphism of the space $L_2(X)$ and the space $L_2(\Theta, l_2(\mathbb{Z}))$. The image of the operator $B$ under this isomorphism is the operator of multiplying by the operator-valued function $B(\tau)$, $\tau \in \Theta$, where the discrete weighted translation operators $B(\tau)$ are defined by (2.2).*

*Proof.* Let $f$ be a function integrable over $X$. Then

$$\int_X f(x)\, dx = \sum_{k=-\infty}^{+\infty} \int_{\Theta_k} f(x)\, dx = \sum_{k=-\infty}^{+\infty} \int_\Theta J_k(\tau) f(\alpha_k(\tau))\, d\tau, \tag{6.2}$$

where $J_k(x)$ is the Jacobian of the mapping $\alpha_k$. Note that

$$J_k(x) = \prod_{j=0}^{k-1} J(\alpha_j(x)) \text{ for } k > 0,$$

$$J_k(x) = \prod_{k}^{-1} [J(\alpha_j(x))]^{-1} \text{ for } k < 0.$$

In particular, taking into account that $\rho(x)^2 = J(x)$, we obtain from (6.2) that

$$\|u\|^2 = \int_X |u(x)|^2 dx = \sum_{k=-\infty}^{+\infty} \int_\Theta [\rho_k(\tau)|u(\alpha_{-k}(\tau))|]^2 d\tau$$

$$= \int_\Theta \left\{ \sum_{k=-\infty}^{+\infty} [\rho_k(\tau)|u(\alpha_k(\tau))|]^2 \right\} d\tau = \int_\Theta \|U(\tau\|^2 d\tau = \|U\|^2$$

for any $u \in L_2(X)$; this implies that the spaces are isomorphic.

The image of the operator $T$ under the constructed mapping is the translation operator $W$ acting in the space of sequences. The image of the operator of multiplying by the function $a$ is the operator of multiplying by the sequence $a_\tau(k) = a(\alpha_{-k}(\tau))$. □

A similar representation can be obtained for operators of kind (3.1) on a segment for a corresponding fundamental domain $\Theta$. Note that the case of operators on segment differs from the case of operators of kind (5.3) on triangle (5.1). The obtained representation includes not all operators of kind (2.2), but only operators corresponding to points from $\Theta$.

In the case of a segment, the invertibility of operators $B(\tau) - \lambda I$ for all $\tau \in \Theta$ implies the invertibility of the operator $B - \lambda I$. However, this is not valid in the case of a triangle because the boundedness as a whole is not guaranteed for the norms of the inverse operators $(B(\tau) - \lambda I)^{-1}$, $\tau \in \Theta$. By virtue of Theorem 2.2, the inverse operators $(B(\tau) - \lambda I)^{-1}$ are uniformly bounded if and only if $B(\tau) - \lambda I$ is bounded for any $\tau \in X$. This is valid for any segment, but it is an additional condition of the theorem in the case of triangle (5.1).

Proving the existence of a right inverse for the operator $B - \lambda I$, we face similar difficulties: the right-sided invertibility of the operator $B - \lambda I$ might be broken even if the operator $B(\tau) - \lambda I$ has a right inverse for any $\tau \in \Theta$.

**Lemma 6.2.** *Let $A$ be a linear bounded operator in a Hilbert space $H$ such that $A$ has a right inverse $R$. Let $P$ be an orthogonal projector on the kernel of the operator $A$. Then the operator $R_{\min} = (I - P)R$ is a right inverse for $A$ with the least norm.*

*Proof.* Let $L = \ker A$. Then $H = L \bigoplus L^\perp$. The general solution of the equation $Ax = y$ is represented as $Ry + h$, where $h \in L$. Therefore, the solution with the least norm is the projection of the vector $x$ on the subspace $L^\perp$, i.e., the vector $(I - P)x = (I - P)Ry$, and this solution does not depend on the original choice of the right inverse $R$. □

The fact that the operator $B - \lambda I$ is representable as the operator-valued function $B(\tau) - \lambda I$ does not imply that any right inverse operator $R$ is representable as an operator-valued function $R(\tau)$, but such a representation is valid for the operator $R_{\min}$.

This implies the following assertion.

**Lemma 6.3.** *The operator $B-\lambda I$ is invertible from the right if and only if the operator $B(\tau)-\lambda I$ has a right inverse $R(\tau)$ for almost any $\tau \in \Theta$ such that all those operators $R(\tau)$ are measurable functions of $\tau$ and their norms are bounded as a whole.*

Further, we construct right inverse operators for the discrete operators $B(\tau) - \lambda I$ and obtain their norm estimates from above and from below.

## 7. Right Inverse Operators in $l_2(\mathbb{Z})$

Let $aW$ be a weighted translation operator with coefficient $a = a(k)$ in the space $l_2(\mathbb{Z})$. Let $a(k) \neq 0$ for any $k$ and let the sequence $a(k)$ have nonzero limits $a(\pm\infty)$ at $\pm\infty$. The spectrum of $aW$ is the ring $K = \{\lambda : r \le |\lambda| \le R\}$, where

$$R = \max\{|a(+\infty)|, |a(-\infty)|\}, \quad r = \min\{|a(+\infty)|, |a(-\infty)|\}.$$

The operator $aW - \lambda I$ is one-side invertible for $r < |\lambda| < R$. In particular, if $|a(-\infty)| < |a(+\infty)|$, then the operator $aW - \lambda I$ is invertible from the right and the right inverse operator is not unique. Let us find norms of various right inverse operators and find out what do those norms depend on.

**Lemma 7.1.** *Let $|a(-\infty)| < |\lambda| < |a(+\infty)|$, $q \in \mathbb{Z}$, and $R_q$ be the operator defined as follows:*

$$R_q = \sum_{j=1}^{\infty} \lambda (aW)^{-j} P_q^+ \; + \; \sum_{j=0}^{\infty} \frac{1}{\lambda^{j+1}} (aW)^j P_q^-, \tag{7.1}$$

*where $P_q^- = I - P_q^+$ and $P_q^+$ is the operator of multiplying by the sequence*

$$p_q(k) = \begin{cases} 1 & \text{for } \; k \ge q, \\ 0 & \text{for } \; k < q. \end{cases}$$

*Then the operator $R_q$ is a right inverse operator for the operator $aW - \lambda I$ and norm estimate* (7.9) *given below is valid; the values included to* (7.9) *are described in the proof of the lemma.*

*Proof.* Construct a solution of the equation $(aW - \lambda I)x = y$ satisfying the condition $x(q) = 0$. It follows from the relation

$$a(k)x(k+1) - \lambda x(k) = y(k)$$

that

$$x(q+l) = \sum_{j=1}^{l} \left[ \lambda^{j-1} \prod_{i=1}^{j} \frac{1}{a(q+l-i)} \right] y(q+l-j) \tag{7.2}$$

for $l > 0$.

The latter relation has the following operator form:

$$P_q^+ x = \sum_{j=1}^{\infty} \lambda^{j-1} (aW)^{-j} P_q^+ y. \tag{7.3}$$

Now let $k < q$. Since

$$x(k) = -\frac{1}{\lambda} y(k) + \frac{a(k)}{\lambda} x(k+1),$$

we have the following representation for $l > 0$:

$$x(q-l) = -\frac{1}{\lambda}y(q-j) + \frac{a(k-j)}{\lambda}x(k_0-j+1) = -\frac{1}{\lambda}\sum_{j=0}^{l}\left[\prod_{i=0}^{j-1}\frac{a(q-l+i)}{\lambda}\right] y(q-l+j).$$

Its operator form is as follows:

$$P_q^- x = \sum_{j=0}^{\infty}\frac{1}{\lambda^{j+1}}(aW)^j P_q^- y. \tag{7.4}$$

This yields (formally) expression (7.1) for the right inverse operator. Let us prove that constructed operator series (7.3) and (7.4) converge.

The operator $(aW)^j$ acts as follows:

$$((aW)^j x)(k) = \left[\prod_{i=0}^{j-1} a(k+i)\right] x(k+j).$$

We have

$$\|(aW)^j\| = \max_{k\in\mathbb{Z}}\prod_{i=0}^{j-1} |a(k+i)|. \tag{7.5}$$

The operator $(aW)^j P_q^-$ acts as follows:

$$[(aW)^j P_q^- x](k) = \begin{cases}\left[\prod_{i=0}^{j-1} a(k+i)\right] x(k+j) & \text{for } k < q-j,\\ 0 & \text{for } k \geq q-j.\end{cases}$$

This implies that

$$\|(aW)^j P_q^-\| = \max_{k<q-j}\prod_{i=0}^{j-1} |a(k+i)|. \tag{7.6}$$

Note that the norm of the operator $(aW)^j P_q^-$ is less than the norm of the operator $(aW)^j$ here. That is why series (7.4) converges, while a similar series

$$\sum_{j=0}^{\infty}\frac{1}{\lambda^{j+1}}(aW)^j$$

diverges.

In the same way, we obtain that

$$\|(aW)^{-j} P_k^+\| = \max_{k\geq q+j}\prod_{i=0}^{j-1}\frac{1}{|a(k-i)|}. \tag{7.7}$$

Let $t_1$ and $t_2$ be such that

$$\frac{|\lambda|}{|a(-\infty)|} < t_1 < 1, \quad \frac{|a(+\infty)|}{|\lambda|} < t_2 < 1. \tag{7.8}$$

Since the limits at infinity exist, it follows that there exists a number $k^+$ such that $\frac{|a(k)|}{|\lambda|} \leq t_2$ for any $k \geq k^+$ and there exists $k^-$ such that $\frac{|\lambda|}{|a(k)}| \leq t_1$ for any $k \leq k^-$. Let $\nu^+(q)$ denote the number of values of $k$ satisfying the conditions $q < k < k^+$ and $\frac{|a(k)|}{|\lambda|} > t_1$. Let $\nu^-(q)$ denote the number of values of $k$ satisfying the conditions $k^- < k < q$ and $\dfrac{|\lambda|}{|a(k)|} > t_2$. Let $M_1 > 1$ and $M_2 > 1$ be such that

$$M_1 \geq \sup_k \frac{|a(k)|}{|\lambda|}, \ M_2 \geq \inf_k \frac{|\lambda|}{|a(k)|}.$$

Using (7.6) and (7.7), we obtain the estimates

$$\left\| \frac{1}{\lambda^j}(aW)^j P_q^- \right\| \leq M_1^j \ \text{ for } \ j \leq \nu^+(q)$$

and

$$\left\| \frac{1}{\lambda^j}(aW)^j P_q^- \right\| \leq M_1^{\nu^+(q)} t_1^{j-\nu^+(q)} \ \text{for} \ \ j > \nu^+(q).$$

This yields the norm estimate

$$\left\| \sum_{j=0}^{+\infty} \frac{1}{\lambda^j}(aW)^j P_q^- \right\| \leq \sum_{j=0}^{+\infty} \frac{1}{|\lambda|^j} \|(aW)^j P_q^-\| \leq \frac{M_1^{\nu^+(q)}}{M_1 - 1} + \frac{M_1^{\nu^+(q)}}{1 - t_1}$$

and the convergence of series (7.3). In the same way, we have the inequalities

$$\|\lambda^j (aW)^{-j} P_k^+\| \leq M_2^j \quad \text{for} \ \ j \leq \nu^-(q)$$

and

$$\|\lambda^j (aW)^{-j} P_k^+\| \leq M_2^{\nu^-(q)} t_2^{j-\nu^-(q)} \ \ \text{for} \ \ j > \nu^-(q).$$

Therefore, the following inequality holds:

$$\left\| \sum_{j=1}^{+\infty} \lambda^j (aW)^{-j} P_k^+ \right\| \leq \frac{M_2^{\nu^-(q)}}{M_2 - 1} + \frac{M_2^{\nu^-(q)}}{1 - t_2}.$$

Thus, we obtain the sought estimate for the right inverse operator:

$$\|R_q\| \leq \frac{1}{|\lambda|} \left[ \frac{M_1^{\nu^+(q)}}{M_1 - 1} + \frac{M_1^{\nu^+(q)}}{1 - t_1} \right] + \frac{M_2^{\nu^-(q)}}{M_2 - 1} + \frac{M_2^{\nu^-(q)}}{1 - t_2}. \tag{7.9}$$

□

Now we obtain the estimate from below for the right inverse operator.

Any solution of the homogeneous difference equation

$$a(j)u(j+1) - \lambda u(j) = 0 \tag{7.10}$$

is uniquely defined by its "initial condition," which is its value at a given point. Therefore, any solution of that equation is (up to a constant factor) the sequence $w_0$ defined as follows:

$$w_0(j) = \frac{1}{\lambda^{|j|}} \prod_{i=j}^{-1} a(i) \ \text{ for } \ j < 0,$$

$$w_0(j) = \lambda^j \left[ \prod_{i=0}^{j-1} a(i) \right]^{-1} \quad \text{for } j \geq 0.$$

This sequence is the solution of (7.10) satisfying the condition $w_0(0) = 1$. For any $k$, we consider a normalized solution $w_k$ of Eq. (7.10) such that $w_k(k) = 1$. Obviously, $w_k(j) = \dfrac{1}{w_0(k)} w_0(j)$.

Let $w_k^{\pm} = P_{k+1}^{\pm} w_k$, where $P_k^{\pm}$ are the projectors introduced in Lemma 7.1. The sequences $w_k^{\pm}$ can be constructed as follows:

$$w_k^-(j) = \begin{cases} \lambda^{k-j} \prod_{i=j}^{k-1} a(i) & \text{for } j \le k, \\ 0 & \text{for } j > k; \end{cases}$$

$$w_k^+(j) = \begin{cases} \lambda^{j-k} \left[\prod_{i=k}^{j-1} a(i)\right]^{-1} & \text{for } j > k, \\ 0 & \text{for } j \le k. \end{cases}$$

**Lemma 7.2.** *Let $R_{\min}$ be the right inverse operator for $aW - \lambda I$ with the least norm. Then the estimate*

$$\|R_{\min}\|^2 \ge \frac{1}{|\lambda|} \frac{A_k^2 B_k^2}{A_k^2 + B_k^2}, \tag{7.11}$$

*where $A_k = \|w_k^+\|$ and $B_k = \|w_k^-\|$, is valid for any $k$.*

*Proof.* Let $e_k$ be a standard basis vector in $l_2(\mathbb{Z})$, i.e., $e_k(k) = 1$ and $e_k(j) = 0$ for $j \ne k$. First we construct all solutions of the equation

$$(aW - \lambda I)x = \lambda e_k. \tag{7.12}$$

If $j > k$, then $((aW - \lambda I)x)(j) = 0$. Therefore, $x(j) = C_k' w_k^+(j)$ for $j > k$, where $C_k'$ is an arbitrary constant.

In the same way, we have $x(j) = C_k'' w_k^-(j)$ for $j \le k$. Thus, $x = C_k' w_k^+ + C_k'' w_k^-$.

If $j = k$, then

$$[(aW - \lambda I)x](k) = C_k'[a(k) w_k^+(k+1) - \lambda w_k^+(k)] + C_k''[a(k) w_k^-(k+1) - \lambda w_k^-(k)] = \lambda.$$

Since

$$w_k^+(k) = 0, \; w_k^+(k+1) = \frac{\lambda}{a(k)}, \; w_k^-(k) = 1, \; w_k^-(k+1) = 0,$$

it follows that $\lambda(C_k^+ + C_k^-) = \lambda$. Thus, any solution $x$ of Eq. (7.12) can be represented as

$$x = C_k w_k^+ + (C_k - 1) w_k^-,$$

where $C_k = \lambda C_k^+$ is an arbitrary constant. Hence,

$$\|x\|^2 = |C_k|^2 A_k^2 + |C_k - 1|^2 B_k^2.$$

Now we find the solution $\widetilde{x}_k$ of Eq. (7.12) with the least norm. Using Lemma 6.2, we apply the operator $I - P$ to $x$. Here $P$ is the projector on the kernel of the operator given as follows:

$$Px = \frac{1}{\|w_k\|^2}(x, w_k) w_k,$$

where $(x, w_k)$ is the scalar product of $x$ and $w_k$.

We have

$$\|w_k\|^2 = A_k^2 + B_k^2, \quad (x, w_k) = C_k A_k^2 + (C_k - 1) B_k^2.$$

Therefore,

$$Px = \frac{C_k A_k^2 + (C_k - 1) B_k^2}{A_k^2 + B_k^2} w_k,$$

while the solution of (7.12) having the least norm is represented as follows:

$$\widetilde{x}_k = R_{\min}(\lambda e_k) = (I - P)x = \frac{B_k^2}{A_k^2 + B_k^2}\, w_k^+ + \frac{A_k^2}{A_k^2 + B_k^2}\, w_k^-;$$

we see that

$$\|\widetilde{x}_k\|^2 = \left[\frac{B_k^2}{A_k^2+B_k^2}\right]^2 A_k^2 + \left[\frac{A_k^2}{A_k^2+B_k^2}\right]^2 B_k^2 = \frac{A_k^2 B_k^2}{A_k^2+B_k^2}.$$

□

In particular, the following estimate from below is valid for the norm of the least right inverse:

$$\|R_{\min}\| \geq \sup_k \|R_{\min} e_k\| = \sup_k \frac{1}{|\lambda|}\sqrt{\frac{A_k^2 B_k^2}{A_k^2+B_k^2}}.$$

It is easy to check that

$$\frac{A_k^2 B_k^2}{A_k^2+B_k^2} \to 0 \quad \text{as } k \to \pm\infty.$$

Therefore, there exists a finite $k$ providing the supremum of $\|R_{\min} e_k\|$. Below we find a value of $k$ such that the value of $\|R_{\min} e_k\|$ is sufficiently close to the greatest one.

It follows from the proof of the lemma that the solution of Eq. (7.12) has the least norm if $C_k = \dfrac{B_k^2}{A_k^2+B_k^2}$; this implies that $0 \leq C_k \leq 1$. Note that, constructing the operator $R_q$, we use the following rule to select the constant $C_k$:

$$C_k = \begin{cases} 1 & \text{for } k \geq q, \\ 0 & \text{for } k < q. \end{cases}$$

As we see above, such $C_k$ does not guarantee that the constructed right inverse has the least norm, but we see from the proof of the theorem that there exists a value of $q$ such that the norm of the operator $R_q$ is small enough.

## 8. The Proof of the Main Theorem

Consider the operator $B(\tau) - \lambda I$ for $\tau \in \Theta$. Its coefficient is the sequence $a_\tau(k) = a(\alpha_k(\tau))$. The orbit of $\alpha_k(\tau)$ tends to the point $(1,1) \in X$ as $k \to +\infty$ and to the point $(0,0) \in X$ as $k \to -\infty$. Hence, $a_\tau(k) \to a(1,1)$ as $k \to +\infty$ and $a_\tau(k) \to a(0,0)$ as $k \to -\infty$. Therefore, results of Sec. 7 are applicable to the study of the operator $B(\tau) - \lambda I$.

Consecutively consider all cases mentioned at the formulation of the theorem.

Let the following condition be satisfied:

$$|a(0,0)| < |\lambda| < |a(1,1)|. \tag{8.1}$$

Then, by virtue of Lemma 7.1, the operator $R_q(\tau)$ defined by (7.1) is a right inverse operator for the operator $B(\tau) - \lambda I$ for any $\tau$ and any $q$.

Let us show that if $|\lambda| \neq |a(1,0)|$ , then for any $\tau$ there exists a number $q = q(\tau)$ such that the norms of the right inverse operators $R_{q(\tau)}(\tau)$ are bounded as a whole. By virtue of Lemma 6.3, this implies that there exists a right inverse operator for the operator $B - \lambda I$.

First we note that $t_1$ and $t_2$ in estimate (7.9) depend only on $a(\pm\infty)$. Since $a(\pm\infty)$ do not depend on the coefficients $a_\tau$ of the operators $B(\tau)$, it follows that $t_1$ and $t_2$ do not depend on $\tau$. Also, we may select the same numbers $M_1$ and $M_2$ for all values of $\tau$:

$$M_1 = \max_{x \in X} \frac{|a(x)|}{|\lambda|}, \quad M_2 = \max_{x \in X} \frac{|\lambda|}{|a(x)|}.$$

The numbers $\nu^\pm(q)$ from estimate (7.9) depend on $\tau$, i.e., $\nu^\pm(q) = \nu^\pm(q,\tau)$. The rate of those values is not bounded. Since $M_\pm > 1$, it follows that the values $M_\pm^{\nu^\pm(q,\tau)}$ from estimate (7.9) might be too large if the choice of $q$ is wrong.

Apart from (8.1), impose the following condition:

$$|a(0,0)| < |\lambda| < |a(1,0)|. \tag{8.2}$$

For $\delta > 0$ consider neighborhoods of fixed points of the kind

$$V(0,0) = \{(x_1, x_2) \in X : 0 \le x_1 < \delta\},$$
$$V(1,1) = \{(x_1, x_2) \in X : \ 1 - \delta < x_2 \le 1\},$$
$$V(1,0) = \{(x_1, x_2) \in X : 1 - \delta < x_1 \le 1, \ 0 \le x_2 < \delta\}.$$

Select $t_1$ and $t_2$ to satisfy the following inequalities:

$$\frac{|\lambda|}{|a(1,1)|} < t_1 < 1, \quad \frac{|a(0,0)|}{|\lambda|} < t_2 < 1. \tag{8.3}$$

It follows from (8.2) that

$$\frac{|\lambda|}{|a(1,0)|} < 1.$$

Hence, one can select $t_1$ to satisfy the inequality

$$\frac{|\lambda|}{|a(1,0)|} < t_1 < 1.$$

Now we take a positive $\delta$ so small that the inequality

$$\frac{|\lambda|}{|a(x_1, x_2)|} < t_1 < 1 \tag{8.4}$$

is satisfied in $V(1,1)$ and in $V(1,0)$, while the inequality

$$\frac{|a(x_1, x_2)|}{|\lambda|} < t_2 < 1 \tag{8.5}$$

is satisfied in $V(0,0)$.

Iterations of the mapping $\gamma : [0,1] \to [0,1]$ generate an orbit $\delta_k$ of the point $\delta$ such that $\delta_k = \gamma(\delta_{k-1}), \delta_0 = \delta$; that orbit tends, monotonously increasing, to the point 1 as $k \to +\infty$. Let $N$ be the number of values of $k$ satisfying the inequality $\delta \le \delta_k \le 1 - \delta$.

For any $x \in X$, iterations of the mapping $\alpha$ generate an orbit of the kind

$$\alpha_k(x) = (\gamma_k(x_1), \gamma_k(x_2)).$$

Therefore, if $\alpha_k(x)$ lies outside the selected neighborhoods of the fixed points, then either $\delta \le \gamma_k(x_1) \le 1 - \delta$ or $\delta \le \gamma_k(x_2) \le 1 - \delta$. Hence, the orbit of any $x \in X$ contains at most $2N$ points lying outside the selected neighborhoods of the fixed points. Note that $N$ does not depend on the orbit.

For any $\tau$, select a number $q = q(\tau)$ such that $\alpha_q(\tau)$ is the last point belonging to the neighborhood $V(0,0)$, i.e.,

$$q(\tau) = \max\{k : \alpha_k(\tau) \in V(0,0)\}.$$

If the considered fundamental domain $\Theta$ is of kind (6.1), then it is possible to select $q$ independent of $\tau$: it should be such that the inequality $\gamma_q(\theta) < \delta \le \gamma_{q+1}(\theta)$ is satisfied. Then it is evident that $q(\tau)$ is measurable.

Since $\delta$ might be small, it usually turns out that the corresponding value of $q(\tau)$ is negative, but its absolute value is large.

If $q$ is selected for any $\tau$ as above, then inequality (8.5) holds for any $k \le q(\tau)$. Therefore, we have $\nu^-(q(\tau), \tau) = 0$.

Let $\nu^+(q(\tau), \tau)$ be the number of values of $k$ such that $k \ge q(\tau)$ and the inequality (8.4) is not valid. If a point of $\alpha_k(\tau)$ is in $V(1,0)$ or in $V(1,1)$, then inequality (8.4) is valid. Any orbit contains at most $2N$ points lying outside the mentioned neighborhoods. Therefore, $\nu^+((q(\tau), \tau) \le 2N$ for any $\tau$. Hence, Lemma 7.1 implies the following uniform estimate for right inverse operators $R_{q(\tau)}(\tau)$:

$$\|R_{q(\tau)}(\tau)\| \le \frac{1}{|\lambda|}\frac{1}{1 - t_1} + \frac{M_2^{2N}}{M_2 - 1} + \frac{M_2^{2N}}{1 - t_2}.$$

Note that we used inequalities (8.1) and (8.2). Under the assumptions of the theorem, they hold if either

$$|a(0,0)| < |\lambda| < |a(1,0)| < |a(1,1)|$$

or

$$|a(0,1)| < |a(0,0)| < |\lambda| < |a(1,1)|.$$

Thus, the existence of a right inverse operator for $B - \lambda I$ is proved for those two cases.

Now we suppose that condition (8.1) and the inequality

$$|a(0,1)| < |\lambda| < |a(1,1)| \tag{8.6}$$

are satisfied.

Taking into account that $\dfrac{|a(0,1)|}{|\lambda|} < 1$, select $t_2$ to satisfy the inequality $\dfrac{|a(0,1)|}{|\lambda|} < t_2 < 1$. Now, unlike the previous case, we can select $\delta$ to satisfy inequality (8.4) only in the neighborhood $V(1,1)$ and to satisfy inequality (8.5) in the neighborhoods $V(0,0)$ and $V(1,0)$.

Unlike the previous case, select $q(\tau)$ so large that $\alpha_{q(\tau)}(\tau)$ is the first point belonging to the neighborhood $V(1,1)$. Then

$$\nu^+(q(\tau),\tau) = 0, \quad \nu^-(q(\tau),\tau) \le 2N.$$

This implies a similar uniform estimate

$$\|R_{q(\tau)}(\tau)\| \le \frac{1}{|\lambda|}\left[\frac{M_1^{2N}}{M_1 - 1} + \frac{M_1^{2N}}{1 - t_1}\right] + \frac{1}{1 - t_2}$$

and the existence of a right inverse operator.

Note that conditions (8.1) and (8.6) used above are satisfied in two other cases from the formulation of the theorem: the case where

$$|a(0,0)| < |a(1,0)| < |\lambda| < |a(1,1)|$$

and the case where

$$|a(0,1)| < |a(0,01)| < |\lambda| < |a(1,1)|.$$

For all four cases considered above, any operator $B(\tau) - \lambda I$ has a nonzero kernel; this implies that the operator $B - \lambda I$ has an infinite-dimensional kernel.

For all other cases, we have to prove that the range of the operator $B - \lambda I$ is not closed.

We start from the most interesting case where

$$|a(0,0)| < |a(0,1)| = |\lambda| < |a(1,1)|.$$

Then for any $\tau$ there exists a right inverse operator for the operator $B(\tau) - \lambda I$. For any $\tau$, consider the right inverse operator with the least norm (it is denoted by $R_{\min}(\tau)$) and prove that the set of such right inverse operators is essentially unbounded, i.e., the measure of the set

$$\{\tau \in \Theta : \|R_{\min}(\tau)\| \ge C\}$$

is positive for any $C$. Due to Lemma 6.3, this contradicts the existence of a right inverse operator for the operator $B - \lambda I$.

Let $\varepsilon > 0$; select a positive $\delta$ such that the following inequalities are satisfied in the neighborhood $V(1,0)$:

$$\frac{|a(x)|}{|\lambda|} \ge 1 - \varepsilon, \quad \frac{|\lambda|}{|a(x)|} \ge 1 - \varepsilon.$$

Let a positive integer $K$ be so large that $(1-\varepsilon)^{2K} < 1/2$. Let $\Theta$ be defined by (6.1), where $\theta < 1 - \delta$.

Consider the set

$$\Theta_K = \{\tau = (\tau_1, \tau_2) \in \Theta : \delta < \tau_2 < \gamma(\delta), \quad \gamma_{-2K-2}(\delta) < \tau_1 < \gamma_{-2K-1}(\delta)\}. \tag{8.7}$$

The measure of this set is positive and the orbit of each point of this set contains at least $2K + 1$ points lying in $V(1,0)$.

The set of points of an orbit, lying inside $V(1,0)$, form a segment of that orbit, i.e., they are the points $\alpha_j(\tau)$ such that their numbers $j$ satisfy the condition $j_0(\tau) \le j \le j_1(\tau)$, where $j_0(\tau)$ is the least of the numbers of $\alpha_j(\tau)$ belonging to the specified neighborhood, while $j_1(\tau)$ is the greatest of such numbers. By construction, the inequality $j_1(\tau) - j_0(\tau) > 2K + 1$ is valid for $\tau \in \Theta_K$.

Let $k(\tau)$ be the integer part of the number $1/2[j_1(\tau) + j_0(\tau)]$. Estimate the norm of the minimal solution $R_{\min}(\tau)e_{k(\tau)}$.

Applying Lemma 7.2, we obtain the inequality

$$\|R_{\min}(\tau)e_{k(\tau)}\| \ge \frac{1}{|\lambda|}\frac{A_{k(\tau)}(\tau)^2 B_{k(\tau)}(\tau)^2}{A_{k(\tau)}(\tau)^2 + B_{k(\tau)}(\tau)^2}.$$

Let us find estimates from below for $A_k$ and $B_k$. We have

$$w^+_{k(\tau)}(k(\tau)+j) = \prod_{i=k(\tau)}^{k(\tau)+j-1} \frac{\lambda}{a(\alpha_i(\tau))}.$$

By construction, if $k(\tau) \le i < k(\tau) + K$, then $\alpha_i(\tau) \in V(1,0)$. Hence, the following inequality holds for the given values of $i$:

$$\frac{|\lambda|}{|a(\alpha_i(\tau)|} \ge 1 - \varepsilon.$$

Therefore, we have $|w^+_{k(\tau)}(k(\tau)+j)| \ge (1-\varepsilon)^j$ for $0 \le j \le K$. Taking into account the inequality $(1-\varepsilon)^{2K} < 1/2$, we deduce the following estimate from below:

$$[A_{k(\tau)}(\tau)]^2 = \|w^+_{k(\tau)}\|^2 \ge \sum_{j=0}^{K-1}(1-\varepsilon)^{2j} = \frac{1-(1-\varepsilon)^{2K}}{1-(1-\varepsilon)^2} \ge \frac{1-(1-\varepsilon)^{2K}}{2\varepsilon} \ge \frac{1}{4\varepsilon}.$$

For $0 \le j \le N$, the elements of the sequence $w^-_{k(\tau)}$ satisfy a similar estimate:

$$|w^-_{k(\tau)}(k(\tau)-j)| \ge (1-\varepsilon)^j, \quad 1 \le j \le K.$$

Hence,

$$[B_{k(\tau)}(\tau)]^2 = \|w^-_{k(\tau)}\|^2 \ge \frac{1}{4\varepsilon}.$$

Let $D = \left\{(t,s):\ t \ge \frac{1}{4\varepsilon},\ s \ge \frac{1}{4\varepsilon}\right\}$. Consider the function $f(t,s) = \dfrac{ts}{t+s}$ in $D$. The least value of the function $f$ in $D$ is equal to $\dfrac{1}{8\varepsilon}$. As we proved above, if $\tau \in \Theta_K$, then all points $(A_{k(\tau)}(\tau)^2, B_{k(\tau)}(\tau)^2)$ belong to $D$. Therefore,

$$\frac{A_k(\tau)^2 B_k(\tau)^2}{A_k(\tau)^2 + B_k(\tau)^2} = f(A_k^2, B_k^2) \ge \frac{1}{8\varepsilon}.$$

Thus, for any number $C = 1/(8\varepsilon|\lambda|)$, we found a set $\Theta_K$ such that its measure is positive and

$$\|R_{\min}(\tau)\| \ge C$$

for any $\tau \in \Theta_K$.

By virtue of Lemma 6.3, this contradicts the existence of a right inverse operator.

In particular, the range of the operator $B - \lambda I$ does not coincide with the whole space. Since the kernel of the adjoint operator $B^* - \overline{\lambda} I$ consists only of the origin, it follows that the range of the operator $B - \lambda I$ is a dense set that is not closed.

Consider the case where

$$|a(0,0)| < |a(1,1)| < |\lambda| < |a(0,1)|. \tag{8.8}$$

If (8.8) holds, then any operator $B(\tau) - \lambda I$ is invertible. In particular, this implies that the kernel of the operator $B - \lambda I$ consists only of the origin. The inverse operator can be represented as a series

$$(B(\tau) - \lambda I)^{-1} = \sum_{j=0}^{+\infty} \frac{1}{\lambda^j} B(\tau)^j.$$

It is easy to check (this follows from general results on series consisting of weighted translation operators) that the following estimate is valid for any $j \geq 0$:

$$\|(B(\tau) - \lambda I)^{-1}\| \geq \frac{1}{|\lambda|^j} \|B(\tau)^j\|.$$

Select $M$ such that the inequality $1 < M < \dfrac{|a(0,1)|}{|\lambda|}$ holds; select a positive $\delta$ such that the following inequality is valid in $V(1,0)$:

$$\frac{|a(x)|}{|\lambda|} \geq M. \tag{8.9}$$

If $\tau \in \Theta_K$, where $\Theta_K$ is a set of kind (8.7), then $V(0,1)$ contains a segment of orbit, containing at least $2K$ points. Then, using estimate (8.9) and the representation for the norm given by (7.6), we see that

$$\frac{1}{|\lambda|^{2K}} \|B(\tau)^{2K}\| \geq M^{2K}.$$

Since $K$ may be arbitrarily large, while $M > 1$, it follows that the norms of inverse operators $(B(\tau) - \lambda I)^{-1}$ are essentially unbounded as a whole and the operator $B - \lambda I$ is not invertible. Since the kernel of the operator adjoint to $B - \lambda I$ consists only of the origin, it follows that the range of the operator $B - \lambda I$ is everywhere dense, but it does not coincide with the whole space.

The case where

$$|a(0,1)| \leq \lambda| < |a(0,0)| < |a(1,1)|$$

is investigated in the same way.

Note that the fact that the operator $B - \lambda I$ is not invertible in the above cases follows from Theorem 2.2 as well.

If $|\lambda| = |a(0,0)|$ or $|\lambda| = |a(1,1)|$, then the range of any operator $B(\tau) - \lambda I$ is not closed; in particular, those operators are not one-sided invertible. This yields that the range of the operator $B - \lambda I$ is not closed.

This completes the proof of the theorem.

If $|\lambda| = |a(0,0)|$ or $|\lambda| = |a(1,1)|$, then there are various examples: the kernel of the operator $B - \lambda I$ and the kernel of the adjoint operator may consist only of the origin, but those kernels may be infinite-dimensional; this depends on the properties of the coefficient $a$. Therefore, the assertion of the latter theorem referring to the above case cannot be generally strengthened.

## 9. Uniform Minus-Semihyperbolicity

Let us analyze what properties of linear extension (2.1) and discrete operators $B(\tau)$ correspond to the conditions providing the fact that the operator $B - \lambda I$ is invertible from the right, but is not invertible. Such an analysis allows us to formulate the main theorem in a form similar to Theorems 4.2 and 4.3.

Without loss of generality, we assume that $\lambda = 1$.

Then $E = X \times \mathbb{C}$, while the linear extension $\beta$ acts as follows:

$$\beta(x, \xi) = \left(\alpha(x), \frac{1}{a(x)} \xi\right).$$

By virtue of the main theorem, the following conditions are necessary and sufficient for the invertibility from the right of the operator $B - I$:

$$|a(0,0)| < 1 < |a(1,1)|, \tag{9.1}$$

$$|a(1,0)| \neq 1. \tag{9.2}$$

Suppose that condition (9.1) is satisfied. It follows from the inequality $|a(1,1)| > 1$ that there exists a (sufficiently small) neighborhood of the point $(1,1)$ such that the fibering $E$ is locally stable towards the positive direction over the specified neighborhood. It follows from the inequality $|a(0,0)| < 1$ that there exists a neighborhood of the point $(0,0)$ such that the fibering $E$ is locally stable towards the negative direction over the specified neighborhood.

To compare, consider properties of the linear extension for the case of an operator of kind (3.1) on segment if $m = 1$ and $|a(0)| < 1 < |a(1)|$ (i.e., the conditions are similar). Then the fibering $E$ is locally stable towards the positive direction over not only a small neighborhood of the point 1, but over a "large" its neighborhood, i.e., any set of the kind $[\delta, 1]$, as well. Similarly, the fibering $E$ is locally stable towards the negative direction over not only a small neighborhood of the point 0, but over any set of the kind $[0, 1-\delta]$. Thus, any point of the segment belongs either to a set such that the fibering $E$ is locally stable towards the positive direction over that set or to a set such that the fibering $E$ is locally stable towards the negative direction over that set.

If we consider a triangle and mapping (5.2), then a principally different case may occur: it is possible that a similar property of the linear extension does not hold. Let

$$X^+(C,\mu) = \{x \in X : \|\beta_n(x,\xi)\| \le C\mu^n|\xi| \text{ for } \ n \ge 0\}.$$

The fibering $E$ is locally stable towards the positive direction over the above set with the given parameters $C$ and $\mu$. In the same way, the set

$$X^-(C,\mu) = \{x \in X : \|\beta_{-n}(x,\xi)\| \le C\mu^n|\xi| \text{ for } \ n \ge 0\}$$

is such that the fibering $E$ is locally stable towards the negative direction over it with the given parameters $C$ and $\mu$.

A linear extension $\beta$ is called *uniformly minus-semihyperbolic* if there exist $C$ and $\mu < 1$ such that

$$X^+(C,\mu) \cup X^-(C,\mu) = X. \tag{9.3}$$

Thus, for a segment, the local hyperbolicity at the points 0 and 1 implies the uniform minus-semihyperbolicity. We note that, for a triangle, condition (9.2) is equivalent to the uniform minus-semihyperbolicity and the same condition is equivalent to the existence of a right inverse operator.

To describe the corresponding properties, introduce the following notions. Let $\mu \in (0,1)$ be given. For any $x \in X$, define

$$\chi_\mu^+(x) = \sup_{n \ge 0} \frac{\mu^{-n}}{\prod\limits_{j=0}^{n-1} |a(\alpha_j(x))|},$$

$$\chi_\mu^-(x) = \sup_{n \ge 0} \mu^n \prod_{j=0}^{n-1} |a(\alpha_j(x))|.$$

Then

$$X^+(C,\mu) = \{x \in X : \chi_\mu^+(x) \le C\},$$

$$X^-(C,\mu) = \{x \in X : \chi_\mu^-(x) \le C\}.$$

By virtue of (9.1), the function $\chi_\mu^+(x)$ is bounded in a neighborhood of the point $(1,1)$ and unbounded in a neighborhood of the point $(0,0)$, while the function $\chi_\mu^-(x)$ is bounded in a neighborhood of the point $(0,0)$ and unbounded in a neighborhood of the point $(1,1)$.

If $|a(1,0)| = 1$, then both functions $\chi_\mu^\pm(x)$ are unbounded in a neighborhood of the point $(1,0)$, while both sets $X^+(C,\mu)$ and $X^-(C,\mu)$ contain a neighborhood of the point $(1,0)$ for no values of $C$ and $\mu < 1$; hence, (9.3) is not valid.

If $|a(1,0)| > 1$, then the function $\chi_\mu^+(x)$ is bounded in a neighborhood of the point $(1,1)$ and in a neighborhood of the point $(1,0)$ and there exists a (sufficiently large) $C$ such that the set $X^+(C,\mu)$ contains a neighborhood of the point $(1,0)$; hence, (9.3) is valid.

If $|a(1,0)| < 1$, then the function $\chi_\mu^-(x)$ is bounded in a neighborhood of the point $(0,0)$ and in a neighborhood of the point $(1,0)$ and there exists a (sufficiently large) $C$ such that the set $X^-(C,\mu)$ contains a neighborhood of the point $(1,0)$; hence, (9.3) is valid.

Thus, we obtain the following assertion.

**Theorem 9.1.** *Let $B$ be an operator of kind* (5.3)*, the assumptions of Theorem* 5.1 *be satisfied, and inequality* (9.1) *hold. The the following properties are equivalent to each other*:

(1) *the operator $B - I$ is invertible from the right, but is not invertible*;

(2) *the linear extension $\beta$ associated with the operator is uniformly minus-semihyperbolic.*

## 10. Conclusion

The general area of study of weighted translation operators contain two main directions:

(1) having properties of the mapping, to obtain properties of the corresponding weighted translation operators (the direct problem);

(2) having properties of the weighted translation operator, to obtain properties of the mapping and the coefficient of the specified operator (the inverse problem).

Obviously, the obtained results contain a certain progress for the direct problem: we found that the existence of a saddle point of the generating mapping affects fine properties of the weighted translation operator and we describe the character of this effect.

Note that the obtained results contain a certain progress for the inverse problem as well. Indeed, if we know only the spectrum of an operator (5.3), then we can find the absolute value of the coefficient only for two (of three) fixed points. If the studied fine properties of the operator $B$ are known, then (see the main theorem), we can find the absolute value of the coefficient for all three fixed points; moreover, we can find the type of the fixed point (i.e., attracting, repelling, or saddle) corresponding to each of those three values.

Also, note that the introduced notion of uniform semihyperbolicity is useful for a more complicated problem, which is the investigation of fine properties of weighted translation operators (5.3) in spaces of vector-functions. This notion shows the terms to formulate analogs of Theorem 4.2 and 4.3 in the case where the generating mapping has a saddle point.

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