On High Order Asymptotic Solutions of the Hopf Equation

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New generalized singular solutions of the "infinitely narrow soliton" type of the Hopf equation are constructed and its interaction are considered.

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1 Introduction

The Hopf equation

$$L(u) \equiv \frac{\partial u}{\partial t}(x,t) + \frac{\partial (u^2)}{\partial x}(x,t) = 0 \qquad (1)$$

is used as one of models describing the motion of waves. Not only smooth functions but shock like wave functions or other generalized singular functions must be considered as some solutions of this equation because nonsmooth solutions of nonlinear equations are very important from the physical viewpoint. But these functions are not solutions of (1) in the classical sense. In order to give a sense the notion of solution in such situations, a notion of asymptotic solution is used.

Definition 1 [5]. A family of smooth functions $u_\varepsilon(x,t)$, indexed by a small parameter ε, is called *an (weak) asymptotic solution of order* m of an equation $Lu(x,t) = f(x,t)$, if

$$< L(u) - f, \varphi >:= \int_{-\infty}^{+\infty} [Lu(x,t) - f(x,t)]\varphi(x)dx$$

$$= o(\varepsilon^m) \text{ for all } \varphi \in D(\mathbf{R}),\ t \in \mathbf{R}. \qquad (2)$$

If (2) holds for all m then $u_\varepsilon(x,t)$ is called *an asymptotic solution of the order infinity.*

The important point to note here is the following: an asymptotic solution is not a function nor a distribution. An interplay between an asymptotic solution $u_\varepsilon(x,t)$ and a distribution can be established by association's relation.

Definition 2. An asymptotic solution $u_\varepsilon(x,t)$ is said to admit *an associated distribution-valued* function $v : \mathbf{R} \to D'(\mathbf{R})$, if $u_\varepsilon(x,t) \to v(t)$ in $D'(\mathbf{R})$ $\forall t \in \mathbf{R}$.

If an associated distribution exists, it is called *a generalized solution.*

An asymptotic solution of the "shock wave" type is associated with the distribution of the form $u_0 + BH(x-Vt)$, where $H(x)$ is the Heaviside's function, u_0 is a constant. This solution models a shock wave of the height B moving with the speed V on some fluid of the depth u_0. The distribution $u_0+BH(x-Vt)$ is a generalized solution of equation (1) if the Hugoniot relation $V = 2u_0 + B$ holds.

The case when an associated distribution-valued function does not exist (or it is trivial) is also possible.

For example, a self-similar asymptotic solutions of the "infinitely narrow soliton" type of the form

$$u_\varepsilon(x,t) = \phi(\frac{x-Vt}{\varepsilon}) + u_0, \qquad (3)$$

where u_0 is a constant, ϕ is some smooth function decreasing at infinity were built in [5,6,8,9,11,12].

Such a solution is associated with a constant u_0 and this constant does not save any information about properties of the asymptotic solution. More visual representation may be given by an asymptotic expansion of the asymptotic solution in the space $D'(\mathbf{R})$.

The asymptotic solutions (3) have asymptotic expansions of the form

$$u_\varepsilon \sim u_0 + A\varepsilon\delta(x - Vt) + ...,$$

where δ is Dirac δ-function. The solution (3) models a narrow wave with the effective amplitude A moving with the speed V on some fluid of the depth u_0. That explains the term "infinitely narrow soliton" type solution.

Note that in [5,6,8,9,11,12] more complex problems were considered, in particular some asymptotic solutions of equations with variable coefficients were constructed too. But all these asymptotic solutions have the order 2 or 3. The problem to construct a self-similar asymptotic solutions of higher order of the Hopf equation was posed in [5].

In the present paper we will construct some large set of asymptotic solutions of order m for arbitrary m.

In view of the fact that equation (1) is nonlinear, the problem considered is closely related with the problem of multiplication of distributions. An approach to solve this problem was developed in [4,7,10] where new objects (new generalized functions or mnemofunctions) were introduced instead of distributions, that formed an algebra, i.e. they allow well-defined multiplication, and at the same time save general properties of distributions. Solutions of partial differential equations in spaces of mnemofunctions were considered in number of papers (see, for example [1,2,3,4,7,10]).

Usually the constructions of new generalized functions are based on some approximation of classical generalized functions by families of smooth functions, indexed by a small parameter ε. It was found out, that investigations of the asymptotic solutions can be interpreted as some constructions in the theory of new generalized functions. But we will use the classical terminology in this paper only.

2 Hugoniot type conditions of high orders

Let $AC(\mathbf{R})$ be the set of all absolutely continuous functions on $\mathbf{R}$. Let F be the set of real functions $f \in AC(\mathbf{R})$, such that there exist limits: $\lim f(x) := f(\pm\infty)$ as $x \to \pm\infty$, and the derivative f' has at $\pm\infty$ one of the following representations:

$$f'(x) = a_\pm x^{-\alpha} + o(|x|^{-(\alpha+\delta)}) \quad \text{as} \quad x \to \pm\infty,$$

$$|a_+| + |a_-| \neq 0, \quad \delta > 0$$

or

$$f'(x) = o(|x|^{-p}) \quad \text{as} \quad x \to \pm\infty, \forall p > 0.$$

We consider the family

$$u_\varepsilon(x,t) = f(\frac{x - Vt}{\varepsilon}) \tag{4}$$

and describe the set F_m of all functions $f \in F$ such that (4) is an asymptotic solution of the order m.

Let us denote by

$$M_k(f) := \int_{-\infty}^{+\infty} x^k f(x)dx$$

and

$$M_k^+(f) := \int_0^{+\infty} x^k f(x)dx$$

the ordinary moments and the right-hand moments of a function f.

Theorem 1. *The family (4), $f \in F$, is a self-similar asymptotic solution of the order m of (1) if and only if the following conditions hold*

i) $g(x) := -Vf'(x) + (f^2)'(x) = o(|x|^{-m-1})$ *as* $x \to \infty$;

ii) $M_j(g) = 0, \quad j = 0, 1, ..., m.$

The proof of Theorem 1 uses concrete form of the Hopf equation slightly and an analogous theorem true for a more wide class of equations including equations with a small parameter. For example, the analogous proposition is true for the Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \varepsilon\frac{\partial^2 u}{\partial x^2} = 0$$

and the Korteweg-de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \varepsilon^2\frac{\partial^3 u}{\partial x^3} = 0,$$

where a small parameter ε models a small viscosity and a small dispersion of medium. We note that KdV-equation can be reduced to the form:

$$\frac{\partial u}{\partial t} + \frac{\partial (u^2)}{\partial x} + \varepsilon^2\mu\frac{\partial^3 u}{\partial x^3} = 0 \qquad (5)$$

and this equation may be considered as a singular perturbation of the Hopf equation.

Thus, analogous theorem is carried for a class of nonlinear differential equations of order l with a small parameter of the following form

$$Lu \equiv \frac{\partial u}{\partial t} + P(\varepsilon, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, ..., \frac{\partial^l u}{\partial x^l}) = 0, \qquad (6)$$

where

$$P(\varepsilon, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, ..., \frac{\partial^l u}{\partial x^l})$$
$$= \sum_{j=0}^{N}\varepsilon^j \sum_{|\beta|=j+1} a_{j,\beta}\prod_{i=0}^{l}(\frac{\partial^i u}{\partial x^i})^{\beta_i},$$

$a_{j,\beta}$ are constant coefficients, $\beta = (\beta_0, ..., \beta_l)$, $\beta_j \in \mathbf{N}\cup\{0\}$, $\beta_0 \le N_1$, is a multi-index and

$$|\beta| = \sum_{i=0}^{l} i\beta_i.$$

Theorem 2. *The family (4), $f^{(l-1)} \in F$, is a self-similar asymptotic solution of the order m of (6) if and only if the following conditions hold*
i)

$$g(x) := -Vf'(x) + \sum_{j=0}^{N}\sum_{|\beta|=j+1} a_{j,\beta}\prod_{i=0}^{l}(f^{(i)}(x))^{\beta_i}$$

$= o(|x|^{-m-1})$ *as* $x \to \infty$;
ii) $M_j(g) = 0, \quad j = 0, 1, ..., m.$

Proof. Substituting (4) in (1) we have

$$< L(u), \varphi >= \frac{1}{\varepsilon}\int g((x - vt)/\varepsilon)\varphi(x)dx, \qquad (7)$$

where $\varphi \in D'(\mathbf{R})$ and

$$g(x) = -Vf'(x) + \sum_{j=0}^{N}\sum_{|\beta|=j+1} a_{j,\beta}\prod_{i=0}^{l}(f^{(i)}(x))^{\beta_i}. \qquad (8)$$

The problem is to investigate asymptotic behavior of the integrals (7).

Such research is carried out on the base of the method of consecutive expansion by A.N. Tikhonov and A.A. Samarski [13] and as a result we obtain the following proposition.

Proposition 1. *Let the function g is locally integrated on $\mathbf{R}$ and it admitts the asymptotic expansion as $x \to \pm\infty$ of the following form*

$$g(x) = a_{\pm}x^{-\alpha} + o(|x|^{-(\alpha+\delta)}), \alpha \in \mathbf{N}, \delta > \mathbf{0}.$$

The family $g(x/\varepsilon)$ has the following asymptotic expansion in the space $D'(\mathbf{R})$:

$$\frac{1}{\varepsilon}g(x/\varepsilon) \sim \sum_{k=0}^{\alpha-2}\varepsilon^k\frac{(-1)^k}{k!}\delta^{(k)}M_k(g)$$
$$+\Big(a_- P(x_-^{-\alpha}) + a_+ P(x_+^{-\alpha})\Big)\varepsilon^{\alpha-1} +$$
$$+\varepsilon^{\alpha-1}\frac{(-1)^{\alpha-1}}{(\alpha-1)!}\delta^{(\alpha-1)}\Big(\tilde{M}_{\alpha-1}(g) + (a_- - a_+)\ln\varepsilon\Big)$$
$$+o(\varepsilon^{\alpha+\delta-1}), \qquad (9)$$

where

$$\tilde{M}_{\alpha-1}(g) = \int_{-\infty}^{-1} x^k[g(x) - a_-\ x^{-\alpha}]dx + \int_{-1}^{1} x^k g(x)dx$$
$$+ \int_{1}^{+\infty} x^k[g(x) - a_+\ x^{-\alpha}]dx$$

and

$$\langle P(x^{-k}), \phi\rangle = \int_{|x|>1} x^{-k}\phi(x)dx$$
$$+ \int_{|x|<1} x^{-k}\Big(\phi(x) - \sum_{j=0}^{k-1}\frac{x^j}{j!}\phi^{(j)}(0)\Big)dx.$$

The conditions *i)* and *ii)* of Theorem 2 are equivalent to the request, that all terms in (9) with ε^k vanish as $k \le m$.

If the function g has more terms in its asymptotic expansion or there exists a total asymptotic expansion, then it is possible to obtain the fuller picture of asymptotic behavior of the integrals (7).

Applying Theorem 1 and Proposition 1 to the family (4) we find that there exist only two type of the asymptotic solutions of the Hopf equation.

Theorem 3. *Let $B := f(+\infty) - f(-\infty) \neq 0$. The family (4), $f \in F$, is a self-similar asymptotic solution of the order m of (1) if and only if the following conditions hold*

i) $\alpha > m+1$;

ii) The Hugoniot condition holds

$$V - 2f(-\infty) = B;$$

iii) The high orders Hugoniot type conditions hold

$$\Big(-V + 2f(-\infty)\Big) M_j(sign x \tilde{f}) + M_j((\tilde{f})^2) = 0,$$

$$j = 0, 1, ..., m-1,$$

where $\tilde{f}(x) = f(x) - f(-\infty) - BH(x)$.

The family (4) has the asymptotic expansion in the space $D'(\mathbf{R})$

$$u_\varepsilon = u_0 + BH(x - Vt) + \varepsilon M_0(\tilde{f})\delta(x - Vt) + ...,$$

this solution is the "shock wave" type solution with the depth $u_0 = f(-\infty)$ and the height B.

Theorem 4. *Let $f(+\infty) - f(-\infty) = 0$. The family (4), $f \in F$, is a nontrivial self-similar asymptotic solution of the order m of (1) if and only if the following conditions hold*

i) $\alpha > m+1$;

ii) $\Big(-V + 2f(-\infty)\Big) M_j(\tilde{f}) + M_j((\tilde{f})^2) = 0$,

$j = 0, 1, ..., m-1$;

The family (4) has the asymptotic expansion in the space $D'(\mathbf{R})$

$$u_\varepsilon = u_0 + \varepsilon A\delta(x - Vt) + ...,$$

it is an asymptotic solution of the "infinitely narrow soliton" type with the depth $u_0 = f(-\infty)$ and the amplitude

$$A = M_0(\tilde{f}), \text{ where } A \neq 0, \quad signA = sign(V - 2u_0).$$

Proof of Theorems 3 and 4 follow immediately from Theorem 1 and Proposition 1. We remark only that the condition $M_0(g) = 0$ from Theorem 1 leads to the Hugoniot condition *ii*) in Theorem 3 and the condition $M_0(g) = 0$ holds automatically under the condition of Theorem 4.

Corollary. *The relation $-V + 2f(-\infty) \neq 0$ is necessary condition for a nontrivial asymptotic solution exists.*

3 The construction of asymptotic solutions of high orders

The condition *ii*) of Theorem 4 means that the moments of the function $\tilde{f}$ with the index $k = 0, 1, ..., m$ and the same moments of the function $\tilde{f}^2$ should be directly proportional. Our problem is to construct such functions now.

In [5] one asymptotic solution of the "infinitely narrow soliton" type was found in the form (4) with

$$f = u_0 + \frac{C_1 + C_2 x}{x^2 + 1},$$

where C_1 and C_2 are some indefinite coefficients. This function generates an asymptotic solution of order 2 if C_1 and C_2 satisfy two conditions (generalized Hugoniot conditions).

The first of the conditions is linear one and may be satisfied simply. But the second condition is square and it is necessary to do some calculations for proof the existence of corresponding coefficients C_1 and C_2.

More essential difficulties appear, if we try to construct an asymptotic solution of higher order in a similar way. Let us find the function f in the form $f = u_0 + C_1 f_1 + C_2 f_2 + ... + C_p f_p$, where f_k are some basic functions such that

$$f_k(x) = \sum_{j=1}^{\infty} a_{k,j} x^{-j} \quad \text{at} \quad \text{infinity}.$$

Then for the indefinite coefficients C_k we get some set of square equations from the conditions *ii*) of

Theorem 4

$$(-V+2u_0)\sum_{j=1}^{p} M_l(f_j)C_j+\sum_{j=1}^{p}\sum_{k=1}^{p} M_l(f_jf_k)C_jC_k = 0; \tag{10}$$

$$l = 0,1,...,m-1.$$

The condition $i)$ leads to a set of linear equations

$$\sum_{j=1}^{p} a_{k,j}C_j = 0; k = 1,...,m+1. \tag{11}$$

The properties of the system (10) - (11) depend on the values of the moments of f_k, f_jf_k and the form of expansions of f_k at infinity. Therefore to sort out the basic functions f_k such as to make the system (10) (11) solvable and to build its solutions is not a simple problem.

One more complicated problem appears if we find an asymptotic solution of the order infinity: it is necessary to construct a function f such that all moments of f and moments of f^2 are directly proportional.

Theorem 1 prompts a more simple way to build almost all desired functions f.

Denote by G_m the set of all functions g satisfying the conditions of Theorem 2, i.e.

$$G_m = \{g \in L_1(\mathbf{R}) | g(x) = o(|x|^{-m-1}) \text{ as } x \to \infty;$$

$$M_j(g) = 0, \ j = 0,1,...,m\}.$$

Equality (8) may be considered as a differential equation with a desired function f and known function $g \in G_m$. Then our problem is to find some functions from G_m and to solve this equation. The set G_m may be described as follows.

Lemma 1. *i) If $g \in G_m$ then there exists some function $J \in C^{(m)}(\mathbf{R})$ such that $J(x) \to 0$ as $x \to \infty$, $J^{(m)} \in AC(\mathbf{R})$ and $g(x) = J^{(m+1)}(x)$.*

ii) If $J \in C^{(m)}(\mathbf{R})$ and $J^{(m)} \in AC(\mathbf{R})$, $J(x) = \sum_{k=1}^{\infty} d_kx^{-k}$ at infinity then the function $g(x) = J^{(m+1)}(x)$ belong to G_m.

The most simple function from G_m is $g = 0$. In the case of KdV equation (5) and $g = 0$ equality (8) has the form

$$-Vf' + (f^2)' + \mu f^{(3)} = 0. \tag{12}$$

Nontrivial solutions of this equation exist, if $\mu \neq 0$. For example the function

$$f(x) = V/2 - \tau + 3\tau\cosh^{-2}(\sqrt{\frac{\tau}{2\mu}}x) \tag{13}$$

is some solution of (12) for given V and all τ such that $\tau\mu > 0$ and it generates "infinitely narrow soliton" type solution of KdV-equation with the amplitude

$$A = 3\sqrt{2\tau\mu}\int\frac{dx}{\cosh^2(x)}$$

and $u_0 = V/2 - \tau$. This solutions were constructed in [9]. It is not only an asymptotic solution of the order infinity but it is an exact solution. In particular, this solution is a strong asymptotic solution. We remind, that a family of smooth functions $u_\varepsilon(x,t)$, indexed by a small parameter ε, is called *a strong asymptotic solution of order m* of the equation $Lu(x,t) = f(x,t)$, if

$$L(u_\varepsilon) - f = o(\varepsilon^m).$$

If $u_\varepsilon(x,t)$ is a family of the form (4) then we have $L(u_\varepsilon)(x,t) = \varepsilon^{-1}g(x/\varepsilon)$ for an operator of the form (6). Therefore a family (4) is a strong asymptotic solution of the equation $Lu = 0$ iff $g = 0$, i.e. if (4) is an exact solution.

In the case $\mu = 0$ equation (5) turns into the Hopf equation (1) and (12) changes into $-Vf'+(f^2)' = 0$. However all solutions of the last equation are trivial (constant).

Therefore nontrivial exact solution of the form (4) of the Hoph equation does not exist, nontrivial strong asymptotic solutions of (1) does not exist too and only weak asymptotic solutions can be considered.

For the Hopf equation and nontrivial $g \in G_m$ equation (8) is $Vf' + (f^2)' = g$ and we can build all solutions of the last equation. Let us consider the function

$$f(x) = V/2 \pm \sqrt{B^2/4 + g_0(x)} \tag{14}$$

where

$$g_0(x) = \int_{-\infty}^{x} g(s)ds,$$

B and V are arbitrary constants, $B^2/4 \geq -\min g_0(x) \neq 0$.

Theorem 5. *Let $g \in G_m, g \neq 0$. Then the function f defined by (14) satisfies the conditions of Theorem 4 and generates by (4) a nontrivial asymptotic solution of the "infinitely narrow soliton" type of order m with the depth $u_0 = f(-\infty) = V/2 \pm B/2$ and the effective amplitude*

$$A = \int_{-\infty}^{+\infty} [f(x) - u_0] dx.$$

Proof. The function (14) is a solution of the equation $Vf' + (f^2)' = g$ and it has desirable properties.

We remark that from $g \neq 0$ follows that $B > 0$ and the condition $-V + 2u_0 \neq 0$ is true because $|-V + 2u_0| = B$.

Corollary. *For all V and u_0 such that $-V + 2u_0 \neq 0$ there exists a nontrivial asymptotic solution of the order infinity with the depth u_0 and the velocity V.*

Proof. Let $\psi \in S(\mathbf{R})$ and its the Fourier transform $\hat{\psi}(\xi) = 0$ at a neighborhood of zero. Then $M_j(\psi) = 0, \quad j = 0, 1, ...,$ and $\psi(x) = o(|x|^{-p})$ for all p. Let $C = -\min \psi_0(x), \ B = 2u_0 - V$. If we set $g = \frac{B^2}{4C}\psi$ then the function f defined by (14) gives an asymptotic solution of the order infinity.

Functions J described by the proposition *ii*) of Lemma 1 can be produced easily in an explicit form.

Example 1. Let $J(x) = 1/(1 + x^2)$ and

$$g_0(x) = J^{(m-1)}(x) = \frac{C}{(1 + x^2)^{m/2}} \sin[m \arctan x],$$

where $C = (-1)^{m-1}(m - 1)!$. Then the function

$$f(x) = V/2 \pm \sqrt{B^2/4 + \frac{C_1}{(1 + x^2)^{m/2}} \sin[m \arctan x]}$$

gives an asymptotic solution of order $m - 1$ of the "infinitely narrow soliton" type if C_1 is small enough.

Example 2. The function $J(x) = \exp(-x^2)$ satisfies the condition *ii*) of Lemma 1 and the function $g_0(x) = (\exp(-x^2))^{(m)}$ generates an asymptotic solution by (14). We remark that this function g is up to a multiplier is the Hermite function and has the form

$$H_\nu(x) = const E_\nu(x) \exp(-x^2),$$

where $E_\nu(x)$ is the Hermite polynomial.

Note that some asymptotic solutions with new properties appear if $B^2/4 = -\min g_0$. Let us consider a particulary case. Let $\min g_0(x) = g_0(x_0)$ and $g_0(x) > g_0(x_0)$ for all $x \neq x_0$. Then a solution of the class $AC(\mathbf{R})$ of the equation (1) of the form differ from (14) exists

$$f(x) = V/2 + sign(x - x_0)\sqrt{B^2/4 + g_0(x)}.$$

For this function we have $f(+\infty) - f(-\infty) = B \neq 0$ and the formula (4) gives a solution of the "shock wave" type by Theorem 3. Note that in this case the parameter B, introduced here, have the same sense that the height B in Theorem 3.

If $M_0(g) = M_1(g) = 0$ then $\lim g_0(x) = 0$ as $x \to \infty$ and $M_0(g_0) = 0$. It follows from this that the function g_0 has both positive and negative values. Therefore the set

$$S(g) = \{x \in \mathbf{R} : g_0(x) = \min g_0\}$$

is not empty, closed and bounded. The complement $\tilde{S}(g) = \mathbf{R} \setminus S(g)$ consists of two half-lines $(-\infty, x_1)$ and $(x_2, +\infty)$ and some finite (can be empty) or denumerable set of intervals. We can choose an arbitrary the sign in (14) at every component of $\tilde{S}(g)$. As a result we obtain a large set of exclusive asymptotic solutions. Among them there exist solutions of the "shock wave" type and solution of the "infinitely narrow soliton" type. More accurately the following theorem holds. Let $\omega(x)$ be a function on $\mathbf{R}$ such that $\omega(x) = 0$ if $x \in S(g)$ and $\omega(x) = const = \pm 1$ on each component of $S(g)$. Denote by $\Omega(g)$ the set of all such functions ω and let $\Omega^\pm(g)$ be a subset of $\Omega(g)$ of the form

$$\Omega^\pm(g) = \{\omega \in \Omega | \ \omega(+\infty)\omega(-\infty) = \pm 1\}.$$

Theorem 6. *Let the function g satisfies the conditions of Theorem 5, $B^2/4 = -\min g_0, \ \omega \in \Omega$ and*

$$f(x) = V/2 + \omega(x)\sqrt{B^2/4 + g_0(x)}. \qquad (15)$$

If $\omega \in \Omega^-$ then f satisfies the conditions of Theorem 3 and gives birth to an asymptotic solution of *order m of the "shock wave" type with height B. If $\omega \in \Omega^+$ then the function f satisfies the conditions of Theorem 4 and gives birth to an asymptotic solution of order m of the "infinitely narrow soliton" type.*

4 On the profile of asymptotic solution

The function f describes a profile of the moving wave. It is natural to believe that the asymptotic solution of high order describes the essence of the matter more exactly and therefore an information about the properties of such functions, giving an asymptotic solution of the high order presents an interest.

Theorem 7. *Let f be a function of the form (14) (it gives an asymptotic solution of order m of the "infinitely narrow soliton" type). Then it oscillates near mean value u_0 at least m times, i.e. there exist some points $y_0 < x_1 < y_1 < x_2 < y_2 < ... < x_m < y_m$ such that $f(x_j) = u_0, \Big(f(y_{2l}) - u_0\Big)\Big(f(y_{2l+1}) - u_0\Big) < 0$.*

This proposition means that "infinitely narrow soliton" of the order m is a collection of waves because it has some number of nodes.

A similar proposition is true for exclusive solution described by Theorem 6.

Theorem 8. *Let $g \in G_m$ and f be a function of the form (14) (it gives an asymptotic solution of the order m). Denote $W(x) = V/2 + B/2\omega(x)$. Then f oscillates near the function $W(x)$ at least m times, i.e.there exist points $y_0 < x_1 < y_1 < x_2 < y_2 < ... < x_m < y_m$, such that $f(x_j) = W(x_j), \Big((f(y_{2l}) - W(y_{2l})\Big)\Big(f(y_{2l+1}) - W(y_{2l+1})\Big) < 0$.*

The family (13) is exact solution of (5) and it is not oscillated function. There is not a contradiction with Theorem 7 because this family is an asymptotic solution of the order 1 only of the Hopf equation. The equation (5) takes into account an infinitely small dispersion. The solutions are more smooth by this dispersion.

5 On interaction of "infinitely narrow soliton" type solutions

Now let us consider a problem of interaction. Let we have two asymptotic solutions of the "infinitely narrow soliton" type of the form similar (4) with the same order m and the same value of u_0:

$$u_j(x,t) = u_0 + f_j(\frac{x - V_j(t-t_0)}{\varepsilon}),\ j = 1,2,$$

where $supp f_j$ are compact sets and $t_0 > 0$.

We consider an initial condition for equation (1) of the form

$$u(x,0) = u_0 + f_1(\frac{x+V_1t_0}{\varepsilon}) + f_2(\frac{x+V_2t_0}{\varepsilon}). \quad (16)$$

For $t < t_0$ the expresion

$$u_\varepsilon(x,t) = u_0 + f_1(\frac{x - V_1(t-t_0)}{\varepsilon}) + f_2(\frac{x - V_2(t-t_0)}{\varepsilon}) \quad (17)$$

give us an asymptotic solution of (1) of order m. This solution has asymptotic expansion of the form

$$u \asymp u_0 + A_1\delta(x - V_1(t-t_0)) + A_2\delta(x - V_2(t-t_0))$$

and represent two "infinitely narrow solitons" localized at the point $x = V_1(t-t_0)$ and point $x = V_1(t-t_0)$ respectively. At the moment of times $t = t_0$ these points coinsid, an interaction of two solitons take place and (17) is not an asymptotic solution at this moment of times.

In order to obtain an asymptotic solution of the initial problem (1)(16) for all t we will construct this solution in the form

$$u_\varepsilon(x,t) = u_0 + K_1(t/\varepsilon)f_1(\frac{x - \varepsilon S_1(\frac{t-t_0}{\varepsilon})}{\varepsilon})$$

$$+K_2(t/\varepsilon)f_2(\frac{x - \varepsilon S_2(\frac{t-t_0}{\varepsilon})}{\varepsilon}), \quad (18)$$

where K_1, K_2, S_1 and S_2 are some unknown functions. These functions must discribe change of amplitudes and velocities.

If we require that (18) is an asymptotic solution by definition 1 we have not a new condition for functions K_1, K_2, S_1 and S_2.

In order to obtain such a condition we give a new definition of an asymptotic solution.

Definition 3. The family (18) is called *a microlocal asymptotic solution* of order m of (1), if the equality (2) holds not only for all real values of t but also for infinitesimally close to t moments of times of the form $t+\varepsilon\tau$.

If a family of the form (18) is an asymptotic solution in the sense of definition 3, it is also an asymptotic solution in the sense of definition 1.

The request that (18) is a microlocal asymptotic solution of (1) of the order 4 leads by using of Proposition 1 to a system of four differential equations with respect to K_1, K_2, S_1 and S_2 of the form

$$K_1'(\tau)M_k(f_1(y-S_1(\tau))+K_1M_k(f_1'(y-S_1(\tau))S_1'(\tau)$$
$$+K_1^2(\tau)M_k((f_1'(y-S_1(\tau))^2(\tau))+K_2'(\tau)M_k$$
$$\times(f_2(y-S_2(\tau))+K_2M_k(f_2'(y-S_2(\tau))S_2'(\tau)$$
$$+K_2^2(\tau)M_k((f_2(y-S_2(\tau))^2(\tau))$$
$$=K_1(\tau)K_2(\tau)M_k(f_1(y-S_1(\tau)f_2(y-S_2(\tau)), \quad (19)$$
$$k=0,1,2,3.$$

We remark that $M_0(f_j')=0$, $M_0((f_j^2)')=0$ for $j=1$ and $j=2$ and $M_0(f_1(y-S_1)f_2(y-S_2))=0$.

Let $m\geq 4$. Then by theorem 2 the relations $(2u_0-V_j)M_k(f_j')+M_k((f_j)^2)=0$ hold for $k=0,1,2,3$ and theses relation hold for $f_j'(y-S_j(\tau))$ and $(f_j^2)'(y-S_j(\tau))$. Therefore the system (19) can be transformed to the form

$$K_1'(\tau)M_k(f_1)+[K_1S_1'(\tau)+K_1^2(\tau)(2u_0-V_1)]$$
$$\times M_k((f_1')(\tau))+K_2'(\tau)M_k(f_2)$$
$$+[K_2S_2'(\tau)+K_2^2(\tau)(2u_0-V_2)]M_k((f_2')(\tau))$$
$$=2K_1(\tau)K_2(\tau)M_k((f_1\cdot T_\tau f_2)'),\ k=0,1,2,3.$$

Left hand side of this system is linear with respect to functions $K_1', K_2', [K_1S_1'(\tau)+K_1^2(\tau)(2u_0-V_1)]$ and $[K_2S_2'(\tau)+K_2^2(\tau)(2u_0-V_2)]$. Therefore the system can be written

$$\mathbf{M}(S_1,S_2)W=K_1K_2R(S_1,S_2), \qquad (20)$$

where

$$W=(K_1', K_2', [K_1S_1'(\tau)+K_1^2(\tau)(2u_0-V_1)],$$
$$[K_2S_2'(\tau)+K_2^2(\tau)(2u_0-V_2)]),$$
$$\mathbf{M}=\begin{pmatrix} M_{01} & M_{02} & 0 & 0 \\ M_{11} & M_{12} & M_{11}' & M_{12}' \\ M_{21} & M_{22} & M_{21}' & M_{22}' \\ M_{31} & M_{32} & M_{31}' & M_{32}' \end{pmatrix},$$
$$M_{ij}=M_i(f_j(y-S_j)),\ M_{ij}'=M_i(f_j'(y-S_j)),$$
$$\mathbf{R}=R(\tau)=(0, M_1(f_1(y-S_1)\cdot f_2(y-S_2)),$$
$$M_2(f_1(y-S_1)\cdot f_2(y-S_2)), M_3(f_1(y-S_1)\cdot f_2(y-S_2))).$$

We remark that

$$M_0(f_j(y-S_j))=M_0(f_j(y)),$$
$$M_1(f_j(y-S_j))=M_1(f_j(y))+S_jM_0(f_j),$$
$$M_2(f_j(y-S_j)=M_2(f_j(y))+2S_jM_1(f_j(y))+S_jM_0(f_j(y),$$
$$M_3(f_j(y-S_j)=M_3(f_j(y))+3S_jM_2(f_j(y))$$
$$+3S_j^2M_1(f_j(y))+S_j^3M_0(f_j(y)).$$

It follow from this that

$$\det\mathbf{M}(S_1,S_2)=\det\mathbf{M}(0,0)=const. \qquad (21)$$

We remark as well that $M_k(f')=kM_{k-1}(f)$ and the matrix $M(0,0)$ have the form

$$\mathbf{M}(0,0)=\begin{pmatrix} M_0(f_1) & M_0(f_2) & 0 & 0 \\ M_1(f_1) & M_1(f_2) & M_0(f_1) & M_0(f_2) \\ M_2(f_1) & M_2(f_2) & 2M_1(f_1) & 2M_1(f_2) \\ M_3(f_1) & M_3(f_2) & 3M_2(f_1) & 3M_2(f_2) \end{pmatrix}.$$

Definition 4. Two functions f_1 and f_2 are called $four-independent$ if $det\mathbf{M}(0,0)\neq 0$.

If the functions f_1 and f_2 are four-independent we can transform system (20) to the form

$$W=K_1K_2\mathbf{M}(S_1,S_2)^{-1}R(S_1,S_2).$$

Initial conditions must be

$$K_1(0)=1, K_2(0)=1, S_1(0)=V_1t_0, S_2(0)=V_2t_0,$$

It follows from the (21) that the terms of matrix $\mathbf{M}(S_1,S_2)^{-1}$ are polynomials of the order 3.

The most important property of the system (20) is the following: there exists a constant C such that $R(S_1, S_2) = 0$, if $|S_1 - S_2| > C$. It follow from this property that, if we have some solution of (20) such that $|S_1(t) - S_2(t)| > C$ for some $t_1 > t_0$, then $K_1(t), K_2(t), S_1'(t), S_2'(t)$ are constant for $t > t_1$. This means that in this case we obtain two new infinitely narrow solitons with some new amplitudes and new velocities after interaction of two initial solitons. One more case is possible too $|S_1(t) - S_2(t)| \leq C$ for all $t > t_0$. In this case two initial solitons stick together.

We constructed some set of weak asymptotic solutions of arbitrary order and described their interactions. However it should be pointed out that this set is very extensive. This is an effect of thethe fact that definition of weak asymptotic solution contains few limitations for considered family of smooth functions. We can give an example (the Schrödinger equation with a δ-potential [1]) when there exists an unique exact solution and there are a lot of weak asymptotic solutions which are very different from the exact one. It follows from this that not all weak asymptotic solutions have a physical sense.

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