Национальная академия наук Беларуси
Труды Института математики. 2019. Том 27. № 1–2. С. 108–126

“„Љ 519.1

Topological Indices of Cographs

V. V. Lepin[†], **V. I. Benediktovich**[†], **M. Talmaciu**[‡]

[†]*Institute of Mathematics, National Academy of Sciences of Belarus, e-mail: lepin@im.bas-net.by*
[‡]*"Vasile Alecsandri"University of Bacau, Romania, e-mail: mtalmaciu@ub.ro*

Џ®бвгЇЁ« 30.10.2019

The Wiener index, the Harary index and edge-Wiener index are classic and well-known topological indices for the characterization of molecular graphs. Firstly, we give the formulae and algorithms for these topological indices of the complements of trees and cographs. Secondly, we present some sufficient conditions for a cograph to be Hamiltonian.

Key words: cograph; Wiener index; edge-Wiener index; Harary index; Hamiltonicity; Hamiltonian connectivity

1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The *order* of a graph G is the number of vertices in G; it is denoted by $n(G) = |V(G)|$. The *size* of a graph G is the number of edges in G; it is denoted by $m(G) = |E(G)|$. The complement of G is denoted by $\overline{G} = (V(\overline{G}), E(\overline{G}))$, where $V(\overline{G}) = V(G)$, $E(\overline{G}) = \{xy : x, y \in V(G), xy \notin E(G)\}$.

If $e = xy$ is an edge of a graph G, then x and y are adjacent, while x and e are incident, as are y and e. If $xy \in E$, we also use $x \sim y$, and $x \nsim y$ whenever x, y are not adjacent in G. If $A, B \subseteq V$ are disjoint and $ab \in E$ for every $a \in A$ and $b \in B$, we say that A and B are *totally adjacent* and we denote them by $A \sim B$, while by $A \nsim B$ we mean that no edge of G connects some vertex of A to a vertex from B and, in this case, we say A and B are *totally non-adjacent*.

The *neighborhood* of the vertex $v \in V$ is the set $N_G(v) = \{u \in V : uv \in E\}$, while $N_G[v] = \{v\} \cup N_G(v)$; or we just denote $N(v)$ and $N[v]$, when G appears clearly from the context. The *degree* of v in G is $deg_G(v) = |N_G(v)|$. The neighborhood of the vertex v in the complement $\overline{G}$ of a graph G will be denoted by $\overline{N}(v)$. The neighborhood of $S \subseteq V$ is the set $N(S) = \bigcup_{v \in S} N(v) - S$ and $N[S] = S \cup N(S)$.

If $U \subseteq V$, by $G[U]$ we denote the subgraph of G induced by U. By $G - X$ we mean the subgraph $G[V - X]$, whenever $X \subseteq V$, but we simply write $G - v$, when $X = \{v\}$.

A graph is complete if every pair of distinct vertices is adjacent. By P_n, C_n, K_n we mean a chordless path on $n \geq 3$ vertices, a chordless cycle on $n \geq 3$ vertices, and a complete graph on $n \geq 1$ vertices, respectively.

The *degree sequence* of an undirected graph is the non-decreasing sequence of its vertex degrees. Let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of G, where $d_1 \leq d_2 \leq \cdots \leq d_n$. Then $d_1 = \delta(G)$ is called the minimum degree of G. In other words, the *minimum degree* of G denoted by $\delta(G)$, is the degree of the vertex with the least number of edges incident to it.

The *union* of simple graphs G and H is the graph $G \cup H$ with the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H)$. If G and H are disjoint, we refer to their union as the disjoint union, and denote it by $G \sqcup H$. The disjoint union of k graphs G is denoted by kG. By starting with a disjoint union of G and H and adding edges joining every vertex of G to every vertex of H, one can obtain the *join* of G and H, denoted by $G \vee H$. Formally,

$$G \vee H = (V(G) \cup V(H), E(G) \cup E(H) \cup \{xy \mid x \in V(G) \wedge y \in V(H)).$$

Note that $\overline{G_1 \vee G_2} = \overline{G_1} \cup \overline{G_2}$.

For any two vertices u and v in G, the distance between u and v, denoted by $d_G(u, v)$, is the number of edges in the shortest path joining u and v. The *eccentricity* $\varepsilon(v)$ of a vertex v is the maximum distance from v to any other vertex. The vertices of minimum eccentricity form the *center*. A tree has exactly one or two adjacent center vertices; in this latter case one speaks of a bicenter. The *diameter* $D(G)$ of a graph G is the maximum eccentricity over all vertices in a graph, while the *radius* $r(G)$ is the minimum eccentricity over all $v \in V(G)$. If any vertex $v \in V(G)$ is adjacent to all the other vertices of G then v is called a well-connected vertex. Thus, if $v \in V(G)$ is a well-connected vertex, then $\varepsilon_G(v) = 1$. For example, all the vertices of a complete graph are well-connected.

Let F denote a family of graphs. A graph G is called $F-$free if G contains no graph from F as an induced subgraph.

The first distance-based topological index was the Wiener index introduced in 1947 by H. Wiener [80]. Later, in 1988 H. Hosoya [40] introduced some counting polynomials in chemistry, among them the Wiener polynomial, which is strongly connected to the Wiener index. Nowadays, it is known as the *Hosoya polynomial*. Another distance-based topological index, the Harary index, was introduced in 1993 by Plavšić et al. [65] and by Ivanciuc et al. [47], independently. All these concepts found many applications in different fields, such as chemistry, biology, networks.

The *Wiener index* of a graph is defined as the sum of distances between all pairs of vertices of a connected graph G, i.e.,

$$W(G) = \sum_{u,v \in V(G)} d_G(u, v). \tag{1}$$

The *distance* $d_G(v)$ *of a vertex* v is the sum of all distances between v and all other vertices of G. Thus, one can define the Wiener index also in a slightly different way:

$$W(G) = \frac{1}{2} \sum_{v \in V(G)} d_G(v),$$

where $\frac{1}{2}$ compensates for the fact that each path between u and v is counted in $d_G(u)$ as well as in $d_G(v)$.

The distance matrix $D(G)$ is defined so that (i, j)-entry, d_{ij}, is equal to $d_G(v_i, v_j)$. Let $D_i(G)$ denote the row sum of $D(G)$ corresponding to vertex v_i. Then

$$W(G) = \frac{1}{2} \sum_{i=1}^{n} D_i(G).$$

For disconnected graphs, we set

$$W(G) = \sum_{u,v \in V(G),\, u-v \text{ path exists in } G} d_G(u, v). \tag{2}$$

In other words, we ignore pairs of vertices u and v for which the distance $d(u, v)$ can be considered as "infinite".

Let G be a disconnected graph with components $G_1, G_2, \ldots, G_k$. By (2) we get

$$W(G) = \sum_{i=1}^{k} W(G_i).$$

Subsequently, several other topological indices were deeply studied. We consider here certain of them.

Wiener indices of trees [24] and Wiener indices of hexagonal systems [23] have been studied intensively. A list of some recent work: characterizations of trees with specified order and degree sequence that maximize the Wiener index [69], the maximum Wiener index of unicyclic graphs with fixed maximum degree [26], inverse Wiener index problems that search for trees with a given Wiener index [30], Wiener indices of iterated line graphs of trees [51, 52, 53], Wiener indices of random trees [78].

For most general classes of graphs, there is no closed formula to calculate their Wiener indices, not even a recursive formula. Finding bounds on Wiener indices of a general class of graphs has attracted many researchers' interest. Entringer et al. [28] showed that for any connected graph with a given order, the Wiener index is minimized by that of a complete graph and maximized by that of a path, and the Wiener index of a tree with a given order attains the minimum value when it is a star and the maximum value when it is a path. Walikar et al. [79] gave some bounds on the Wiener index of a graph in terms of the graph order together with one or two other graph parameters such as size, radius, diameter, connectivity, independent number, and chromatic number.

The Harary index of a graph G, denoted by $H(G)$, has been introduced independently by Ivanciuc et al. [47] and Plavšić et al. [65] in 1993 for the characterization of molecular graphs. The *Harary index* $H(G)$ is defined as the sum of reciprocals of distances between all pairs of vertices of a connected graph G, i.e.

$$H(G) = \sum_{u,v \in V(G),\, u \neq v} \frac{1}{d_G(u,v)}. \tag{3}$$

Note that in any disconnected graph G, the distance is infinite between any two vertices from two distinct components. Therefore its reciprocal can be viewed as zero. Thus we can define validly the Harary index of a disconnected graph G as follows:

$$H(G) = \sum_{i=1}^{k} H(G_i), \tag{4}$$

where $G_1, G_2, \ldots, G_k$ are the components of G.

For K_1 we set $W(K_1) = 0$ and $H(K_1) = 0$.

The edge versions of the Wiener index based on the distance between edges in a graph G were introduced in 2009 [17, 45, 48]. The edge-Wiener index of G is denoted by $W_e(G)$ and defined as

$$W_e(G) = \sum_{\{e,f\} \subseteq E(G)} d_e(e,f),$$

where for distinct pair of edges $e = uv$ and $f = zt$ of G, the distance $d_e(e,f)$ is defined as

$$d_e(e,f) = 1 + \min\{d(u,z), d(u,t), d(v,z), d(v,t)\}.$$

It is easy to see that, $d_e(e,f)$ equals to the distance between vertices e and f in the line graph of G. This implies that, $W_e(G) = W(L(G))$, where $L(G)$ is the line graph of G.

A *Hamiltonian cycle* of the graph G is a cycle which contains all vertex of G. A *Hamiltonian path* of the graph G is a path which contains all vertex of G. The graph G is said to be *Hamiltonian* if it contains a Hamiltonian cycle, and is said to be *traceable* if it contains a Hamiltonian path. If every two vertices of G are connected by a Hamiltonian path, it is said to be *Hamilton-connected*. A graph G is *traceable from a vertex x* if it has a Hamiltonian x-path. A graph is *traceable from every vertex* if it contains a Hamilton path from every vertex. All these concepts belong to Hamiltonicity of graphs. The problem of deciding whether a graph has Hamiltonicity is one of the most difficult classical problems in graph theory. Recently, some topological indices have been applied to this problem. Up to now, there are some references on the Wiener index, hyper-wiener index and Harary index conditions for a graph to be traceable, Hamiltonian, Hamilton-connected, traceable from every vertex. We refer readers to see [16, 9, 42, 41, 50, 61, 60, 57, 58, 83].

The paper is organized as follows. In Section 2 we present some preliminary results and explicit formulae for several important classes of graphs.

In Section 3, the formulae and algorithms for some topological indices of the complements of trees are given.

In Section 4, the formulae and algorithms for some topological indices of cographs are given.

In Section 5 we discuss the Hamiltonicity of graphs in terms of Harary index.

2. Preliminaries

Before giving the proof of our theorems, we introduce some fundamental statements and properties in this section.

When the underlying graph is a tree (as is true for instance for the alkanes originally studied by Wiener), the Wiener index may be calculated more efficiently. If the graph is partitioned into two subtrees by removing a single edge e, then its Wiener index is the sum of the Wiener indices of the two subtrees, together with a third term representing the paths that pass through e. This third term may be calculated in linear time by computing the sum of distances of all vertices from e within each subtree and multiplying the two sums [19]. This divide-and-conquer algorithm can be generalized from trees to graphs of a bounded treewidth, and leads to near-linear-time algorithms for such graphs [20].

An alternative method for calculating the Wiener index of a tree, by Bojan Mohar and Tomaz Pisanski, works by generalizing the problem to graphs with weighted vertices, where the weight of a path is the product of its length with the weights of its two endpoints. If v is a leaf vertex of the tree then the Wiener index of the tree may be calculated by merging v with its parent (adding their weights together), computing the index of the resulting smaller tree, and adding a simple correction term for the paths that pass through the edge from v to its parent. By repeatedly removing leaves in this way, the Wiener index may be calculated in a linear time [13].

A *tree* is a connected, acyclic graph.

A recursive definition of *rooted trees*: A graph with a single vertex r (and no edges) is a tree with root r (the sole base graph). Now, let (G, r) denote a tree with root r. Then $(G_1, r_1) \oplus (G_2, r_2)$ is a tree formed by taking the disjoint union of G_1 and G_2 and adding an edge (r_1, r_2). The root of this new tree is $r = r_1$.

Theorem 1 [24]. *Let (T_1, r_1) and (T_2, r_2) be two rooted trees of orders n_1 and n_2, respectively. If $T = (T_1, r_1) \oplus (T_2, r_2)$ then*

$$W(T) = W(T_1) + W(T_2) + n_1 d_{T_2}(r_2) + n_2 d_{T_1}(r_1) + n_1 n_2.$$

The correctness of the following algorithm is justified by Theorem 1.

Algorithm: finding the Wiener index of a tree

WT(T, r)

Input: a root tree (T, r).

Output: the Wiener index of (T, r); distance $d_T(r)$ of the root; the order of T.

begin
if $|V(T)| = 1$ **then**
$W \leftarrow 0;$
$d \leftarrow 0;$
$n \leftarrow 1;$
else
if $(T, r) = (T_1, r_1) \oplus (T_2, r_2)$ **then**
$(W_1, d_1, n_1) \leftarrow \mathbf{WT}(T_1, r_1);$
$(W_2, d_2, n_2) \leftarrow \mathbf{WT}(T_2, r_2);$
$W \leftarrow W_1 + W_2 + n_1 d_2 + n_2 d_1 + n_1 n_2;$
$d \leftarrow d_1 + d_2 + n_2;$
$n \leftarrow n_1 + n_2;$
end if
end if
return (W, d, n).
end.

3. Indexes of the complements of trees

For $a \geq 1$, $b \geq 1$, let $K_{1,a}$ and $K_{1,b}$ be stars on $a+1$ and $b+1$ vertices, respectively. Then the double star $DS_{a,b}$ is the tree obtained by connecting an edge between two centers of $K_{1,a}$ and $K_{1,b}$. For the double star $DS_{a,b}$ of order n $a + b = n + 2$.

Theorem 2 [68]. *Let T be a tree on n vertices, then*

$$W(\overline{T}) = \begin{cases} 0, \text{ if } n < 3; \\ \frac{n(n+1)}{2}, \text{ if } T \text{ is a double star;} \\ \frac{(n-1)(n+2)}{2}, \text{ otherwise.} \end{cases}$$

Theorem 3. *Let T be a tree on n vertices, then*

$$W_e(\overline{T}) = \frac{1}{4}(n^2 - 3n + 2)(n^2 - 3n) - \sum_{v \in V} \binom{n - deg_T(v)}{2}.$$

Proof. Let $x, y \in V$ then

$$d_{\overline{T}}(x, y) = \begin{cases} 1, \text{ if } xy \notin E(T); \\ 3, \text{ if the diameter } d(T) = 3 \text{ and } \{x, y\} \text{ is the bicenter of } T; \\ 2, \text{ otherwise.} \end{cases}$$

Let us prove that for any two edges e, f of $\overline{T}$ $d_e(e, f) \leq 2$. Consider two non-edges $\{a, b\}$ and $\{p, q\}$ of T. If the non-edges share an endpoint then they are adjacent in $L(T)$ and therefore $d_e(\{a, b\}, \{p, q\}) = 1$. Otherwise, since T is a tree, at least one pair of $\{a, p\}$, $\{a, q\}$, $\{b, p\}$ and $\{b, q\}$ is a non-edge in T, otherwise T has a 4-cycle. Therefore $d_e(\{a, b\}, \{p, q\}) \leq 2$. So all distinct edges of $\overline{T}$ are either at distance 1 or 2.

Let Q be the set of all pairs of edges of $\overline{T}$. We partition Q into two disjoint sets as follows

$$Q_1 = \{\{e, f\} : e, f \in E(\overline{T}),\ d_e(e, f) = 1\},$$

$$Q_2 = \{\{e, f\} : e, f \in E(\overline{T}),\ d_e(e, f) = 2\}.$$

The edge-Wiener index of $\overline{T}$ is obtained by summing the contributions of all pairs of edges over those two sets:

$$W_e(\overline{T}) = |Q_1| + 2|Q_2| = 2|Q| - |Q_1|.$$

A pair of edges of $\overline{T}$ belong to Q_1, if it shares an endpoint. Therefore

$$|Q_1| = \sum_{v \in V} \binom{n - deg_T(v)}{2}.$$

The number of edges of $\overline{T}$ is equal to $m = \binom{n}{2} - (n-1) = \frac{n^2-3n+2}{2}$. The total number of all pairs of edges of $\overline{T}$ is equal to $\binom{m}{2}$. Hence

$$W_e(\overline{T}) = 2\binom{m}{2} - |Q_1| = \frac{1}{4}(n^2 - 3n + 2)(n^2 - 3n) - \sum_{v \in V} \binom{n - deg_T(v)}{2}.$$

This completes the proof. $\square$

Theorem 4. *Let (T_1, r_1) and (T_2, r_2) be two rooted trees of orders n_1 and n_2, respectively. If $T = (T_1, r_1) \oplus (T_2, r_2)$ and $n = |V(T)|$, then*

$$W_e(\overline{T}) = W_e(\overline{T_1}) + W_e(\overline{T_2}) +$$

$$+\frac{1}{4}\left((n^2 - 3n + 2)(n^2 - 3n) - (n_1^2 - 3n_1 + 2)(n_1^2 - 3n_1) - (n_2^2 - 3n_2 + 2)(n_2^2 - 3n_2)\right).$$

Proof. Let $Q_1 = \{\{xy, yz\} : xy, yz \in E(\overline{T})\}$ then

$$W_e(\overline{T}) = \frac{1}{4}(n^2 - 3n + 2)(n^2 - 3n) - |Q_1|$$

by Theorem 3. We partition Q_1 into four disjoint sets as follows

$$Q_{11} = \{\{xy, yz\} : xy, yz \in E(\overline{T}) | \{x, y, z\} \subseteq V(\overline{T_1})\},$$

$$Q_{12} = \{\{xy, yz\} : xy, yz \in E(\overline{T}) | \{x, y, z\} \subseteq V(\overline{T_2})\},$$

$$Q_{13} = \{\{xy, yz\} : xy, yz \in E(\overline{T}) | \{x, y\} \subseteq V(\overline{T_1}), z \in V(\overline{T_2})\},$$

$$Q_{14} = \{\{xy, yz\} : xy, yz \in E(\overline{T}) | \{x, y\} \subseteq V(\overline{T_2}), z \in V(\overline{T_1})\}.$$

We have

$$W_e(\overline{T_1}) = \frac{1}{4}(n_1^2 - 3n_1 + 2)(n_1^2 - 3n_1) - |Q_{11}|,$$

$$W_e(\overline{T_2}) = \frac{1}{4}(n_2^2 - 3n_2 + 2)(n_2^2 - 3n_2) - |Q_{12}|$$

by Theorem 3.

$$|Q_{13}| = \sum_{v \in V_1 \setminus \{r_1\}} (n_1 - deg_{T_1} v) n_2 + (n_1 - deg_{T_1} r_1)(n_2 - 1) =$$

$$= \sum_{v \in V_1} (n_1 - deg_{T_1} v) n_2 - (n_1 - deg_{T_1} r_1) =$$

$$= n_1^2 n_2 - 2(n_1 - 1) n_2.$$

By symmetry,

$$|Q_{14}| = n_2^2 n_1 - 2(n_2-1)n_1.$$

We have

$$W_e(\overline{T}) = \frac{1}{4}(n^2-3n+2)(n^2-3n) - |Q_{11}| - |Q_{12}| - |Q_{13}| - |Q_{14}| =$$

$$= \frac{1}{4}(n^2-3n+2)(n^2-3n) + W_e(\overline{T_1}) - \frac{1}{4}(n_1^2-3n_1+2)(n_1^2-3n_1) + W_e(\overline{T_2}) - \frac{1}{4}(n_2^2-3n_2+2)(n_2^2-3n_2).$$

This completes the proof. □

The correctness of the following algorithm is justified by Theorem 4.

Algorithm: finding the edge-Wiener index of the complement of a tree

WE(T, r)
Input: a root tree (T, r).
Output: the edge-Wiener index of $\overline{T}$; the number $A(n)$; the number of vertices of T.
begin
if $|V(T)| = 1$ **then**
  $WE \leftarrow 0$;
  $A \leftarrow 0$;
  $n \leftarrow 1$;
else
  if $(T, r) = (T_1, r_1) \oplus (T_2, r_2)$ **then**
    $(WE_1, A_1, n_1) \leftarrow$ **WE**(T_1, r_1);
    $(WE_2, A_2, n_2) \leftarrow$ **WE**(T_2, r_2);
    $n \leftarrow n_1 + n_2$;
    $A \leftarrow \frac{1}{4}(n^2-3n+2)(n^2-3n)$;
    $WE \leftarrow WE_1 + WE_2 + A - A_1 - A_2$;
  end if
end if
return (WE, A, n).
end.

Theorem 5. *Let T be a tree on n vertices, then*

$$H(\overline{T}) = \begin{cases} \frac{(n-1)(n-2)}{2}, & \text{if } T \text{ is a star;} \\ \frac{(n+1)^2}{2} - \frac{1}{6}, & \text{if } T \text{ is a double star;} \\ \frac{(n+1)^2}{2}, & \text{otherwise.} \end{cases}$$

Proof. Let $V = V(T) = V(\overline{T})$. If T is a star then $\overline{T} \cong K_1 \cup K_{n-1}$ and $H(\overline{T}) = H(K_1) + H(K_{n-1}) = 0 + \frac{(n-1)(n-2)}{2}$.

Let $\gamma(G, k)$ be the number of vertex pairs of the graph G that are at distance k. Then

$$H(G) = \sum_{k \geq 1} \frac{1}{k} \gamma(G, k). \tag{5}$$

Let $x, y \in V$ then

$$d_{\overline{T}}(x, y) = \begin{cases} 1, & \text{if } xy \notin E(T); \\ 3, & \text{if the diameter } d(T) = 3 \text{ and } \{x, y\} \text{ is the bicenter of } T; \\ 2, & \text{otherwise.} \end{cases}$$

If the diameter $d(T) > 3$ then

$$H(\overline{T}) = \gamma(\overline{T},1) + \frac{1}{2}\gamma(\overline{T},2) =$$

$$\frac{n(n-1)}{2} - (n-1) + \frac{1}{2}(n-1) = \frac{(n+1)^2}{2}.$$

If the diameter $d(T) = 3$ then T is a double star $DS_{a,b}$ of order n. In this case

$$\gamma(\overline{T},1) = \frac{n(n-1)}{2} - (n-1),$$

$$\gamma(\overline{T},2) = n-2,$$

$$\gamma(\overline{T},3) = 1.$$

Hence

$$H(\overline{T}) = \gamma(\overline{T},1) + \frac{1}{2}\gamma(\overline{T},2) + \frac{1}{3}\gamma(\overline{T},2) =$$

$$= \frac{n(n-1)}{2} - (n-1) + \frac{1}{2}(n-2) + \frac{1}{3} =$$

$$= \frac{(n+1)^2}{2} - \frac{1}{2} + \frac{1}{3} =$$

$$= \frac{(n+1)^2}{2} - \frac{1}{6}.$$

This completes the proof. □

4. Indexes of cographs

4.1. Join operation

The next fact is useful. Since its proof is simple, we omit it.

Fact 1. *Let G_1 and G_2 be graphs, $G = G_1 \vee G_2$, $x, y \in V(G)$. Then*

(i) $|V(G_1 \vee G_2)| = |V(G_1)| + |V(G_2)|$;

(ii) $|E(G_1 \vee G_2)| = |E(G_1)| + |E(G_2)| + |V(G_1)| \cdot |V(G_2)|$;

(iii) $d_G(x,y) = \begin{cases} 0, \text{ if } x = y; \\ 1, \text{ if } xy \in E(G_1) \text{ or } xy \in E(G_2) \text{ or } (x \in V(G_1) \wedge y \in V(G_2)); \\ 2, \text{ otherwise.} \end{cases}$

Theorem 6. *Let G_1 and G_2 be graphs. Let the graph $G = G_1 \vee G_2$ has n vertices and m edges. Then*

$$W(G) = W(G_1 \vee G_2) = n^2 - n - m.$$

Proof. Let $n_i = |V(G_i)$ and $m_i = |E(G_i)|$. By Fact 1, we have:

$$\begin{aligned} W(G_1 \vee G_2) &= m_1 + m_2 + n_1 n_2 + 2\left(\frac{n_1(n_1-1)}{2} - m_1\right) + 2\left(\frac{n_2(n_2-1)}{2} - m_2\right) \\ &= n^2 - n - m. \end{aligned}$$

□

Let G be a graph and $m = |E(G)|$. Let $p_1(G)$ be the number of the pairs of edges at distance 1 and $p_2(G)$ be the number of the pairs of edges at distance 2 in G. The number of all other pairs of edges we denote by $p_3(G)$. Then $p_3(G) = \binom{m}{2} - p_1 - p_2$.

Theorem 7. *Let* G_1 *and* G_2 *be graphs,* $n_i = |V(G_i)|$ *and* $m_i = |V(G_i)|$*, where* $i \in \{1,2\}$*. Let the graph* $G = G_1 \vee G_2$ *have* n *vertices and* m *edges. Let* P_3 *be the set of all pairs of edges at distance 3 in* G*. Then*
(i) $p_i(G_1 \cup G_2) = p_i(G_1) + p_i(G_2)$, where $i \in \{1,2\}$;
(ii) $W_e(G_1 \cup G_2) = W_e(G_1) + W_e(G_2)$;
(iii) $p_1(G) = p_1(G_1) + p_1(G_2) + 2(m_1n_2 + m_2n_1) + \frac{1}{2}n_1n_2(n_1+n_2-2)$;
(iv) $p_2(G) = \binom{m}{2} - p_1(G) - p_3(G)$;
(v) $p_3(G) = |P_3(G)| = p_3(G_1) + p_3(G_2)$;
(vi) $W_e(G) = p_1 + 2p_2 + 3p_3$.

Proof. The proofs of (i) and (ii) are obvious.

Let $G = G_1 \vee G_2$ and $S = \{\{x,y\} \mid x \in V(G_1) \wedge y \in V(G_2)\}$. Then

$$\begin{aligned} V(G) &= V(G_1) \cup V(G_2), \\ E(G) &= E(G_1) \cup E(G_2) \cup S. \end{aligned}$$

By fact 1 all distinct vertices of G are either at distance 1 or 2. The vertices at distance 2 are precisely those of G_1 that are not adjacent in G_1, and those of G_2 that are not adjacent in G_2. So all distinct edges of G are either at distance 1, 2 or 3. Therefore (iv) and (vi) are true.

Let F be the set of all pairs of edges of G. We partition F into six disjoint sets as follows

$$\begin{aligned} F_1 &= \{\{g,f\} | g,f \in E(G_1)\}, \\ F_2 &= \{\{g,f\} | g,f \in E(G_2)\}, \\ F_3 &= \{\{g,f\} | g \in E(G_1), f \in E(G_2)\}, \\ F_4 &= \{\{g,f\} | g \in E(G_1), f \in S\}, \\ F_5 &= \{\{g,f\} | g \in E(G_2), f \in S\}, \\ F_6 &= \{\{g,f\} | g,f \in S\}. \end{aligned}$$

Let P_i be the set of all pairs of edges at distance i in G. Let $F_j^i = F_j \cap P_i$, where $i \in \{1,2,3\}$ and $j \in \{1,\dots,6\}$. Then $p_1(G) = \sum_{j=1}^{6} |F_j^1|$.

Let $\{g,f\} \in F_1$, where $g = uv$, $f = zt$. Then

$$d_{e|G}(g,f) = \min\{d(u,z), d(u,t), d(v,z), d(v,t)\} + 1.$$

If the edges g and f share a vertex then exactly one of these four distances is equal to 0. So $d_{e|G}(g,f) = d_{e|G_1}(g,f) = 1$. Therefore $|F_1^1| = p_1(G_1)$. By symmetry, $|F_2^1| = p_1(G_2)$.

There are no edges sharing a vertex in F_3. Therefore $|F_3^1| = 0$.

There are $2m_1n_2$ pairs from F_4 sharing a vertex. Therefore $|F_4^1| = 2m_1n_2$. By symmetry, $|F_5^1| = 2m_2n_1$.

There are $n_1\binom{n_2}{2} + n_2\binom{n_1}{2}$ pairs from F_6 sharing a vertex. Therefore $|F_6^1| = n_1\binom{n_2}{2} + n_2\binom{n_1}{2}$.

The total number of pairs of edges at distance 1 is

$$p_1(G) = \sum_{j=1}^{6} |F_j^1| = p_1(G_1) + p_1(G_2) + 2m_1n_2 + 2m_2n_1 + \frac{n_1n_2(n_1+n_2-2)}{2}.$$

Equality (iii) is proved. Let us prove (v).

The vertices at a distance of at least 2 are precisely those of G_1 that are not adjacent in G_1, and those of G_2 that are not adjacent in G_2. Only such vertices are at a distance 3 in the graph G. Therefore $p_3(G) = p_3(G_1) + p_3(G_2)$. This completes the proof. □

Theorem 8 [20]. *Let G_1 and G_2 be graphs. Let the graph $G = G_1 \vee G_2$ have n vertices and m edges. Then*

$$H(G_1 \vee G_2) = \frac{1}{2}(n_1 n_2 + m_1 + m_2)) + \frac{1}{4}(n_1 + n_2)(n_1 + n_2 - 1) = \frac{1}{2}m + \frac{1}{4}(n^2 - n).$$

4.2. Cographs

A graph is a *cograph* if it can be constructed from one-vertex graphs by disjoint union and complement operations.

Cographs can be recursively defined as follows: a graph G is a cograph if and only if

(i) G is an one-vertex graph, or

(ii) G is the union of two cographs G_1 and G_2, or

(iii) G is the join of two cographs G_1 and G_2.

Every cograph has a binary cotree representation which can be computed in a linear time [13]. A binary cotree is a rooted tree in which each leaf corresponds to a vertex in the cograph and each inner node either represents a disjoint union or a join of its children.

Algorithm: finding the edge-Wiener index of a connected cograph

Input: a connected cograph G and its binary cotree $T(G)$.
Output: the first edge-Wiener index of G
Let r be the root of $T(G)$.
begin
if $|V(G)| = 1$ **then**
  $WE(r) \leftarrow 0$;
else
  Mark all nodes of $T(G)$ as unfinished;
  while exists an unfinished node x with all children finished **do**
    if x is a leaf **then**
      $n(x) \leftarrow 1$;
      $m(x) \leftarrow 0$;
      $p1 \leftarrow 0$; $p2 \leftarrow 0$; $p3 \leftarrow 0$;
    else
      Let y, z be the children of the node x in the cotree.
      if x is a union node **then**
        $n(x) \leftarrow n(y) + n(z)$;
        $m(x) \leftarrow m(y) + m(z)$;
        $p1 \leftarrow p1(y) + p1(z)$; $p2 \leftarrow p2(y) + p2(z)$;
        $p3 \leftarrow \frac{1}{2}m(x)(m(x) - 1) - p1(x) - p2(x)$;
        $WE(x) \leftarrow WE(y) + WE(z)$;
      end if
      if x is a join node **then**
        $n(x) \leftarrow n(y) + n(z)$;
        $m(x) \leftarrow m(y) + m(z) + n(y) \cdot n(z)$;
        $p1 \leftarrow p1(y) + p1(z) + 2(m(x)n(y) + m(y)n(x)) + \frac{1}{2}n(y)n(z)(n(y) + n(z) - 2)$;
        $p3 \leftarrow p3(y) + p3(z)$;
        $p2 \leftarrow \frac{1}{2}m(x)(m(x) - 1) - p1(x) - p3(x)$;

$WE(x) \leftarrow p1(x) + 2p2(x) + 3p3(x);$
end if
end if
mark p as finished
end while
end if
return $WE(r)$.
end.

Theorem 9. *Algorithm finds correctly the edge-Wiener index for cographs and can be implemented in $O(m+n)$ time.*

Proof. We can prove the correctness of the Algorithm by using induction on n, the number of vertices. The Algorithm is obviously correct for $n = 1$. Let the Algorithm be correct for all cographs with fewer than n vertices. The correctness of Algorithm for cographs with n vertices follows from Theorem 7.

This completes the proof. □

4.3. The weak decomposition of a graph.

We remind the characterization of the weak decomposition of a graph.

Definition 1 [73, 74]. *A set $A \subseteq V(G)$ is called a weak set of the graph G if $N_G(A) \neq V(G) - A$ and $G(A)$ is connected. If A is a weak set, maximal with respect to set inclusion, then $G(A)$ is called a weak component. For simplicity, the weak component $G(A)$ will be denoted with A.*

Definition 2 [73, 74]. *Let $G = (V, E)$ be a connected and non-complete graph. If A is a weak set, then the partition $\{A, N(A), V - (A \bigcup N(A))\}$ is called a weak decomposition of G with respect to A.*

The name of a *weak component* is justified by the following result.

Theorem 10 [73, 74]. *Every connected and non-complete graph $G = (V, E)$ admits a weak component A such that $G(V - A) = G(N(A)) + G(\overline{N}(A))$.*

Theorem 11 [14, 15]. *Let $G = (V, E)$ be a connected and non-complete graph and $A \subseteq V$. Then A is a weak component of G if and only if $G(A)$ is connected and $N(A) \sim \overline{N}(A)$.*

The next result, that follows from Theorem 11, ensures the existence of a weak decomposition in a connected and non-complete graph.

Corollary 1. *If $G = (V, E)$ is a connected and non-complete graph, then V admits a weak decomposition (A, B, C), such that $G(A)$ is a weak component and $G(V - A) = G(B) + G(C)$.*
Theorem 11 provides an $O(n+m)$ algorithm for building a weak decomposition for a non-complete and connected graph.

Algorithm: finding the weak decomposition of a graph ([73])
Input: a connected graph with at least two nonadjacent vertices, $G = (V, E)$.
Output: a partition $V = (A, N, R)$ such that $G(A)$ is connected, $N = N(A), A \nsim R = \overline{N}(A)$.
begin
$A :=$ any set of vertices such that $A \cup N(A) \neq V$,
$N := N(A)$,
$R := V - (A \cup N(A))$.
while ($\exists n \in N$, $\exists r \in R$ such that $nr \notin E$) **do**
$A := A \cup \{n\}$,
$N := (N - \{n\}) \cup (N(n) \cap R)$,
$R := R - (N(n) \cap R)$
end while
return (A, N, R)

end.

4.4. Wiener polynomial of a cograph

A polynomial of distances has been introduced in [40] as: $H(G,x) = \sum_k d(G,k)x^k$, where $d(G,0) = |V(G)|, d(G,1) = |E(G)|$ and $d(G,k)$ is the number of vertex pairs of G, the distance of which is equal to k. This polynomial was named Wiener, by its author Hosoya in many articles [37, 71]

The Wiener polynomial $H(G,x)$ has the property that its first derivative evaluated at point $x = 1$ equals the Wiener index, i.e. $H'(G,1) = W(G)$.

More information about the calculation of the Wiener polynomial for different types of graphs can be found in [32, 40, 56, 71]

For a given integer k the inverse Wiener index problem is a problem of finding a graph G, such that $W(G) = k$. The problem was proposed in 1995 by Gutman and Yeh [39], where they posed the following conjecture:

For all but finitely many integers w, there exist trees with Wiener index w.

Li and Wang [59] give solutions to the two conjectures on the inverse problem of the Wiener index of peptoids. Wagner et. al [77]: every sufficiently large even integer is the Wiener index of some hexagon type graph. Wagner [76]: prove that every positive integer which is the Wiener index of some graph (every positive integer other than 2 and 5) is Wiener index of a graph whose cyclomatic number is at most 6.

Conjecture 1. *With the exception of a finite set, each positive integer is the Wiener index of a binary tree.*

In [55] the so-called Wiener inverse problem interval has been considered.

Problem 1. *For given n, find all values w that are Wiener indices of graphs with n nodes.*

Regarding the above problem, let $WG(n)$ be the corresponding set of values w for graphs on n vertices.

Conjecture 2. *The cardinality of $WG(n)$ is of order $\frac{1}{6}n^3 - \frac{1}{2}n^2 - \Theta(n)$.*

Next, we determine the Wiener index and Wiener polynomial and find all values w which are Wiener indices of cographs on n vertices.

Theorem 12. *Let $G = (V,E)$ be a connected non-complete cograph. Then its Wiener polynomial is:*

$$H(G,x) = \left[\frac{1}{2}|V|^2 - \frac{1}{2}|V| - |E|\right]x^2 + |E|x + |V|,$$

its Wiener index is equal to

$$W(G) = |V|^2 - |V| - |E|,$$

moreover for any $w \in WG(n)$

$$\frac{|V|^2 + 3|V|}{2} \leq w \leq |V|^2 + 1$$

and its Harari index is equal to

$$H(G) = \frac{|V|^2 - |V|}{4} + \frac{|E|}{2}.$$

Proof. Let us denote the order and the size of the graph G by n and m, respectively, and let us consider a weak decomposition $V(G) = A \cup N \cup R$ where induced graph $G(A)$ is a connected weakly component, $N = N(A)$ is the neighborhood of set $A, \overline{N}(A)$ is the neighborhood of set A in the complement $\overline{G}$ of graph G and $G(V - A) = G(N) \vee G(R)$. Hence $N \sim R$ and $G(A), G(N)$ and $G(R)$ are cographs as subgraphs of cograph G.

Note, that $A \sim N$ as well. Indeed, let there exist $a \in A$ and $b \in N$, such that $ab \notin E(G)$. Since $N = N(A)$ then there exists $c \in A$, such that $cb \in E(G)$. Since $G(A)$ is connected, then there exists a path $P_k = (a = a_1, a_2, ..., a_k = c) \in G(A)$. Obviously, there exists $i, 1 \leq i < k$, such that $a_i b \notin E(G)$, but $a_{i+1} b \in E(G)$. Then the path $P_4 = (a_i, a_{i+1}, b, r), \forall r \in R$ is an induced 4-path in G, contradiction, because G is a P_4-free graph.

Thus, for any $a, a' \in A, b, b' \in N, r, r' \in R$ we have $d(a,b) = d(b,r) = 1$, $d(a,r) = 2$ and $d(a,a') \leq 2$, $d(b,b') \leq 2$, $d(r,r') \leq 2$.

Hence, since the number of pairs of vertices at distance 1 is equal to m, then the number of the remaining pairs of different vertices at distance 2 and is equal to

$$\binom{n}{2} - m = \frac{n(n-1)}{2} - m.$$

So, the Wiener polynomial is:

$$H(G,x) = \left[\frac{1}{2}n^2 - \frac{1}{2}n - m\right] x^2 + mx + n$$

and the Wiener index is:

$$W(G) = \frac{d}{dx} H(G,x)\,|_{x=1} = n^2 - n - m.$$

Since the maximum number of edges of an undirected connected graph G with n vertices is $(n-1)n/2$ and the minimum number of its edges is $n-1$, then for any $w \in WG(n)$

$$n^2 - n - \frac{n\,(n-1)}{2} \leq w \leq n^2 - n - (n-1)\,,$$

i.e.

$$\frac{n^2 + 3n}{2} \leq w \leq n^2 + 1.$$

Finally,

$$H(G) = m + \frac{1}{2}\left(\frac{n(n-1)}{2} - m\right) = \frac{n(n-1)}{4} + \frac{m}{2}.$$

This completes the proof. □

Corollary 12.1. *For any connected non-complete cograph G of the order n and the size m we have*

$$\frac{m(2m+2-n)}{2} - \frac{m^2}{n-1} \leq W_e(G)$$

and equality holds only when G is a star graph.

Proof. For any edges $e = (v,u), f = (s,t)$ we define the distance between them by

$$\widehat{d}(e,f) = \min\{d(v,s), d(v,t), d(u,s), d(u,t)\}.$$

Let

$$\widehat{W}_e(G) = \sum_{\{e,f\} \subseteq E(G)} \widehat{d}(e,f).$$

Let $|E(G(A))| = a$, $|E(G(R))| = r$ and the induced graphs $G(A)$ and $G(R)$ have k and l pairs of edges at distance 2 respectively. Since the number of pairs of adjacent edges in graph G is equal to $\sum_{v \in G} \frac{deg(v)(deg(v)-1)}{2}$ and the number of pairs of edges (e,f) at distance 2

$(\widehat{d}(e,f)=2)$ is equal $ar+k+l$, then the number of pairs of edges (e,f) at distance 1 $(\widehat{d}(e,f)=1)$ is equal to

$$\frac{m(m-1)}{2}-\sum_{v\in G}\frac{deg(v)(deg(v)-1)}{2}-ar-k-l,$$

then

$$\widehat{W}_e(G)=\Big(\frac{m(m-1)}{2}-\sum_{v\in G}\frac{deg(v)(deg(v)-1)}{2}-ar-k-l\Big)+2(ar+k+l)=$$

$$=\frac{m(m-1)}{2}-\frac{1}{2}\sum_{v\in G}deg(v)^2+\frac{1}{2}\sum_{v\in G}deg(v)+ar+k+l=\frac{m(m+1)}{2}-\frac{1}{2}\sum_{v\in G}deg(v)^2+ar+k+l.$$

It remains to note that for the first Zagreb index $\sum_{v\in G}deg(v)^2\le m\Big(\frac{2m}{n-1}+n-2\Big)$ and equality holds only when G is a star graph or a complete graph [19] and $W_e(G)=\widehat{W}_e(G)+\binom{m}{2}$ as well. This completes the proof. □

5. Index Harary and Hamiltonicity of graph

In 1976 Bondy and Chvatal introduced the closure concept, which plays an important role in the cycle theory. For a graph G of order n and integer $k=k(n)>0$, the $k-$ *closure* of G, denoted by $cl_k(G)$ is obtained from G by sequentially joining pairs of non-adjacent vertices whose degree sum is at least k, until no such vertex pairs exist. They proved the following statements as well:

Theorem 13 [7]. A graph G is Hamiltonian if and only if $n-$ closure $cl_n(G)$ is Hamiltonian.

Theorem 14 [7]. A graph G is Hamiltonian connected if and only if $(n+1)-$ closure $cl_{n+1}(G)$ is Hamiltonian connected.

The following Lemmas are well-known.

Lemma 4.1 [12]. Let G be a graph of order $n\ge 3$ with degree sequence $(d_1,d_2,...,d_n)$, where $d_1\le d_2\le ...\le d_n$. If there is no integer $k<n/2$ such that $d_k\le k$ and $d_{n-k}\le n-k-1$, then G is Hamiltonian.

Lemma 4.2 [6]. Let G be a graph of order $n\ge 3$ with degree sequence $(d_1,d_2,...,d_n)$, where $d_1\le d_2\le ...\le d_n$. If there is no integer $k<n/2$ such that $d_{k-1}\le k$ and $d_{n-k}\le n-k$, then G is Hamiltonian connected.

We have obtained sufficient conditions for Hamiltonicity and Hamiltonian connectivity of a graph depending on the law bound of Harary index of a graph G.

We use the following recently established facts.

Theorem 15 [84]. Let G be a graph of order $n\ge 6k^2+4k+2$ with minimal degree $\delta(G)\ge k\ge 1$. If its size $e(G)\ge\dfrac{n^2-(2k+1)n}{2}$, then G is Hamiltonian unless $cl_n(G)=K_1\vee(K_k+K_{n-k-1})$ or $cl_n(G)=K_k\vee(K_{n-2k}+kK_1)$.

Theorem 16 [11]. Let G be a graph of order $n\ge 6k^2-8k+5$ with minimal degree $\delta(G)\ge k\ge 2$. If its size $e(G)\ge\dfrac{n^2-(2k-1)n+2k-2}{2}$, then G is Hamiltonian connected unless $cl_{n+1}(G)=K_2\vee(K_{n-k-1}+K_{k-1})$ or $cl_{n+1}(G)=K_k\vee(K_{n-2k+1}+(k-1)K_1)$.

We establish the following statements.

Theorem 17. Let G be a graph of order $n\ge 6k^2+4k+2$ with minimal degree $\delta(G)\ge k\ge 1$. If its Harary index $H(G)\ge\dfrac{n(n-k-1)}{2}$, then G is Hamiltonian unless $cl_n(G)=K_1\vee(K_k+K_{n-k-1})$ or $cl_n(G)=K_k\vee(K_{n-2k}+kK_1)$.

Proof. Let G not be Hamiltonian. Then by Theorem 1 $\Gamma=cl_n(G)$ is not Hamiltonian as well. Note that $e(G)\le e(\Gamma)$ and for any vertices v,u $d_G(v,u)\ge d_\Gamma(v,u)$ and hence $H(G)\le H(\Gamma)$. Then we have

$$H(\Gamma) = \frac{1}{2} \sum_{v \in V(\Gamma)} D_v(\Gamma) \le \frac{1}{2} \left(\sum_{v \in V(\Gamma)} d_\Gamma(v) + \frac{1}{2}\left(n - 1 - d_\Gamma(v)\right) \right) =$$

$$= \frac{n(n-1)}{4} + \frac{1}{4} \sum_{v \in V(\Gamma)} d_H(v) = \frac{n(n-1)}{4} + \frac{1}{2} e(\Gamma).$$

Thus we have

$$\frac{n(n-k-1)}{2} \le H(G) \le H(\Gamma) \le \frac{n(n-1)}{4} + \frac{1}{2} e(\Gamma).$$

It follows $e(\Gamma) \ge \frac{1}{2}(n^2 - (2k+1)n)$. Then by Theorem 3 $\Gamma = K_1 \vee (K_k + K_{n-k-1})$ or $\Gamma = K_k \vee (K_{n-2k} + kK_1)$.

By a direct computation, for $\Gamma = K_1 \vee (K_k + K_{n-k-1})$ we obtain $H(\Gamma) = \frac{1}{2}n(n-k-1) + \frac{1}{2}k(k+1) > \frac{1}{2}n(n-k-1)$ and for $\Gamma = K_k \vee (K_{n-2k} + kK_1)$ we obtain $H(\Gamma) = \frac{1}{2}n(n-k-1) + \frac{1}{4}k(3k+1) > \frac{1}{2}n(n-k-1)$.

This completes the proof. □

Theorem 18. Let G be a graph of order $n \ge 6k^2 - 8k + 5$ with minimal degree $\delta(G) \ge k \ge 2$. If its Harary index $H(G) \ge \dfrac{n^2 - kn + k - 1}{2}$, then G is Hamiltonian connected unless $cl_{n+1}(G) = K_2 \vee (K_{n-k-1} + K_{k-1})$ or $cl_{n+1}(G) = K_k \vee (K_{n-2k+1} + (k-1)K_1)$.

Proof. Let G not be Hamiltonian connected. Then by Theorem 1 $\Gamma = cl_{n+1}(G)$ is not Hamiltonian connected as well. Note that $e(G) \le e(\Gamma)$ and for any vertices v, u $d_G(v,u) \ge d_\Gamma(v,u)$ and hence $H(G) \le H(\Gamma)$. Then we have

$$H(\Gamma) = \frac{1}{2} \sum_{v \in V(\Gamma)} D_v(\Gamma) \le \frac{1}{2} \left(\sum_{v \in V(\Gamma)} d_\Gamma(v) + \frac{1}{2}(n - 1 - d_\Gamma(v)) \right) =$$

$$= \frac{n(n-1)}{4} + \frac{1}{4} \sum_{v \in V(\Gamma)} d_H(v) = \frac{n(n-1)}{4} + \frac{1}{2} e(\Gamma).$$

Thus we have

$$\frac{n^2 - kn + k - 1}{2} \le H(G) \le H(\Gamma) \le \frac{n(n-1)}{4} + \frac{1}{2} e(\Gamma).$$

It follows $e(\Gamma) \ge \dfrac{n^2 - (2k-1)n + 2k - 2}{2}$. Then by Theorem 4 $\Gamma = K_2 \vee (K_{n-k-1} + K_{k-1})$ or $\Gamma = K_k \vee (K_{n-2k+1} + (k-1)K_1)$.

By a direct computation, for $\Gamma = K_1 \vee (K_k + K_{n-k-1})$ we obtain $H(\Gamma) = \dfrac{n^2 - kn + k^2 - 2}{2} > \dfrac{n^2 - kn + k - 1}{2}$ when $k \ge 2$ and for $\Gamma = K_k \vee (K_{n-2k} + kK_1)$ we obtain $H(\Gamma) = \dfrac{n^2 - kn}{2} + \dfrac{k(3k-1)}{4} > \dfrac{n^2 - kn + k - 1}{2}$ when $k \ge 1$.

This completes the proof. □

Theorem 19. Let G be a cograph of order $n > 8$ with minimal degree $\delta(G) \ge 2$. If $\rho^\star(G) \ge \dfrac{7}{6}n - 2$, then G is Hamiltonian unless $G \in \{5K_1 \vee K_4, K_3 \vee (K_{1,4} + K_1), K_2 \vee (K_{n-4} + 2K_1), K_1 \vee (K_{n-3} + K_2)\}$.

Proof. Note that $RD(G) = A(G) + \frac{1}{2}A(\overline{G})$. Then by Weil inequality [8] we have $\rho^\star(G) \leq \rho(G) + \rho(\frac{1}{2}\overline{G}) = \rho(G) + \frac{1}{2}\rho(\overline{G}) = \frac{1}{2}\rho(G) + \frac{1}{2}(\rho(G) + \rho(\overline{G}))$. By Theorem of Teprai [75] $\rho(G) + \rho(\overline{G}) \leq \dfrac{4}{3}n - 1$. Hence, $\rho(G) \geq 2\rho^\star(G) - \left(\dfrac{4}{3}n - 1\right) \geq n - 3$. Then by Theorem 3 [5] G is Hamiltonian.

This completes the proof. □

The research was supported by F18RA–014 and AR–FRBCF research projects of bilateral cooperation performed by The Institute of Mathematics of the National Academy of Sciences of Belarus and "Vasile Alecsandri" University of Bacau of the Romanian Academy of Sciences respectively.

References

1. *Azari M., A. Iranmanesh, A. Tehranian* A method for calculating an edge version of the Wiener number of a graph operation. // Util. Math. 2012. Vol. 87. P. 151–164.
2. *Azari M., A. Iranmanesh* Computation of the edge Wiener indices of the sum of graphs. // Ars. Combin. 2011. Vol. 100. P. 113–128.
3. *Azari M., A. Iranmanesh* Computing Wiener-like topological invariants for some composite graphs and some nanotubes and nanotori. // In: I. Gutman, (Ed.), Topics in Chemical Graph Theory, Univ. Kragujevac, Kragujevac, 2014. P 69–90.
4. *Azari M., A. Iranmanesh* The second edge-Wiener index of some composite graphs // Miskolc Math. Notes. 2014. Vol. 15. № 2. P. 305–316.
5. *Benediktovich V.I.* Spectral condition for Hamiltonicity of a graph // Linear Algebra Appl. 2016. Vol. 494. P. 70–79.
6. *Berge C.* Graphs and Hypergraphs, North-Holland, Amsterdam, 1973.
7. *Bond J.A., Chvatal V.* A method in graph theory // Discr. Math. 1976. Vol. 15. № 2. P. 111–135.
8. *Brouwer A.E., Haemers W.H.* Spectra of graphs, Springer, 2011.
9. *Cai G.-X, M.-L. Ye, Y. Gui, R. Li* Hyper-Wiener index and Hamiltonicity of graphs // Ars Combinatoria, 2018. Vol. 139. P. 175–184.
10. *Chen L., X. Li and M. Liu* The (revised) Szeged index and the Wiener index of a nonbipartite graph // European J. Combin. 2014. Vol. 36. P. 237–246.
11. *Chen W., Zhang X.* The number of edges, spectral radius and Hamiltonian-connectedness of graphs // J. Comb. Optim., 2018. Vol 35. P. 1104–1127.
12. *Chvatal V.* On Hamiltonian's ideals // J. Comb. Theory B. 1972. Vol. 12. P. 163–168.
13. *Corneil D. G., Y. Perl, and L. K. Stewart* A linear recognition algorithm for cographs. // *SIAM Journal on Computing*. 1985. Vol. 14 № 4. P. 926–934.
14. *Croitoru C., E. Olaru, M. Talmaciu* Confidentially connected graphs. // The annals of the University "Dunarea de Jos"of Galati. Proceedings of the international conference "The risk in contemporany economy 2000. Supplement to Tome XVIII (XXII), P. 17–20.
15. *Croitoru C., M. Talmaciu,* On Confidentially connected graphs. // Bul. Stiint. Univ. Baia Mare, Ser B, Matematica - Informatica. 2000. Vol. XVI, № 1. P. 13–16.
16. *Cui Z., B. Liu* On Harary matrix, Harary index and Harary energy // MATCH Commun. Math. Comput. Chem., 2012. Vol. 68. P. 815–823.
17. *Dankelmann P., I. Gutman, S.Mukwembi and H. C. Swart* The edge Wiener index of a graph // Discrete Math. 2009. Vol. 309. P. 3452–3457.

18. *Darafsheh M. R., M. H. Khalifeh* Calculation of the Wiener, Szeged, and PI indices of a certain nanostar dendrimer // Ars Combin. 2011. Vol. 100. P. 289–298.
19. *Das K. Ch.* Sharp bounds for the sum of the squares of the degrees of a graph // Kragujevac J. Math. 2003. Vpl. 25. P. 31–49.
20. *Das K. C., K. Xu, I. N. Cangul, A. S. Cevik, A. Graovac* On the Harary index of graph operations. // Journal of Inequalities and Applications 2013. Vol. 1. P. 1–16.
21. *Diudea M. V.* Hosoya Polynomial in tori // MATCH Commun. Math. Comput. Chem. 2002. Vol. 45. P. 109–122.
22. *Diudea M. V.* Wiener index of dendrimers // MATCH Commun. Math. Comput. Chem. 1995. Vol. 32. P. 71–83.
23. *Dobrynin A. A., I. Gutman, S. Klav ž ar and P. Ž igert* Wiener index of hexagonal systems // Acta Appl. Math. 2002. Vol. 72. P. 247–294.
24. *Dobrynin A. A., R. Entringer and I. Gutman* Wiener index of trees: theory and applications // Acta Appl. Math. 2001. Vol. 66. P. 211–249.
25. *Dobrynin A., I. Gutman* On a graph invariant related to the sum of all distances in a graph // Publ. Inst. Math. (Beograd). 1994. Vol. 56. P. 18–22.
26. *Dong H., B. Zhou* Maximum Wiener index of unicyclic graphs with fixed maximum degree // Ars Combin. 2012. Vol. 103. P. 407–416.
27. *Do š li.c T.* Vertex-Weighted Wiener polynomials for composite graphs // ArsMath. Contemp. 2008. Vol. 1. P. 66–80.
28. *Entringer R. C., D. E. Jackson and D. A. Snyder* Distance in graphs // Czech. Math. J. 1976. Vol. 26. P. 283–296.
29. *Entringer R. C.* Distance in graphs: Trees // J. Combin. Math. Combin. Comput. 1997. Vol. 24. P. 65–84.
30. *Fink J., B. Lužar and R. Škrekovski* Some remarks on inverse Wiener index problem // Discrete Appl. Math. 2012. Vol. 160. P. 1851–1858.
31. *Gutman I.* Calculating the Wiener number: the Doyle-Graver method // J. Serb. Chem. Soc. 1993. Vol. 58. P. 745–750.
32. *Gutman I.* Hosoya polynomial and the distance of the total graph of a tree // Univ. Beograd. Publ. Elektrotehn. Fak. Ser.Mat. 1999. Vol. 10. P. 53–58.
33. *Gutman I.* A new method for the calculation of the Wiener number of acyclic molecules // J. Mol. Struct. (Theochem). 1993. Vol. 285. P. 137–142.
34. *Gutman I., J. Lu and M. A. Bautiche* On distance in complements of graphs // Ser. A: appl. math. Inform and mech. 2016. Vol. 8. № 1. P. 35–42.
35. *Gutman I., O. E. Polansky* Mathematical Concepts in Organic Chemistry. Springer, Berlin, 1986.
36. *Gutman I., R. Cruz and J. Rada* Wiener index of Eulerian graphs // Discrete Appl. Math. 2014. Vol. 162. P/ 247–250.
37. *Gutman I., S. Klavzar, M. Petkovsek and P. Zigert* On Hosoya polynomials of benzenoid graphs // MATCH Commun. Math. Comput. Che. 2001. Vol.43. P. 49–66.
38. *Gutman I., Y. N. Yeh, S. L. Lee and Y. L. Luo* Some recent results in the theory of the Wiener number // Indian J. Chem. 1993. Vol. 32A. P. 651–661.
39. *Gutman I., Y. Yeh* The sum of all distances in bipartite graphs // Math. Slovaca. 1995. Vol. 45. P. 327–334.
40. *Hosoya H.* On some counting polynomials in Chemistry, Discrete Appl.Math., 1988. Vol. 19. P. 239–257.
41. *Hua H.-B., B. Ning* Wiener index, Harary index and Hamiltonicity of graphs // Match Communications in Mathematical and in Computer Chemistry. 2016. Vol. 78, № 1. P. 153–162.
42. *Hua H.-B., M. Wang* On Harary index and traceable graphs // MATCH Commun. Math. Comput. Chem. 2013. Vol. 70. P. 297–300.
43. *Iranmanesh A., A. S. Kafrani* Computation of the first edge-Wiener index of TUC4C8(S) nanotube // MATCH Commun. Math. Comput. Chem. 2011. Vol.62, № 2. P. 311–352.
44. *Iranmanesh A., A. S. Kafrani, O. Khormali* A New Version of Hyper-Wiener Index // MATCH Commun. Math. Comput. Chem. 2011. Vol. 65, № 1. P. 113–122.
45. *Iranmanesh A., I. Gutman, O. Khormali and A. Mahmiani* The edge versions of Wiener index // MATCH Commun. Math. Compute. Chem. 2009. Vol. 61. P. 663–672.

46. *Iranmanesh A., M. Azari* Edge-Wiener descriptors in chemical graph theory: A survey // Curr. Org. Chem. 2015. Vol. 19. № 3. P. 219–239.
47. *Ivanciuc O., T. S. Balaban, A. T. Balaban* Reciprocal distance matrix, related local vertex invariants and topological indices // J. Math. Chem. 1993. Vol. 12. P. 309–318.
48. *Khalifeh M. H., H. Yousefi-Azari, A. R. Ashrafi and S. G. Wagner* Some new results on distance-based graph invariants // European J. Combin. 2009. Vol. 30. P. 1149–1163.
49. *Klavžar S.. M. J. Nadja-Arani* Wiener index in weighted graphs via unication of $\Theta*$-classes // European J. Combin. 2014. Vol. 36. P. 71–76.
50. *Klein D.-J., I. Lukovits and I. Gutman* On the defination of the hyper-Wiener index for cycle-containing structures // J. Chem. Inf. Comput. Sci. 1995. Vol. 35. P. 50–52.
51. *Knor M., P. Potočnik and R. Škrekovski* On a conjecture about Wiener index in iterated line graphs of trees // Discrete Math. 2012. Vol. 312. P. 1094–1105.
52. *Knor M., P. Potočnik and R. Škrekovski* Wiener index of iterated line graphs of trees homeomorphic to H // Discrete Math. 2013. Vol. 313. P. 1104–1111.
53. *Knor M., P. Potočnik and R. Škrekovski* Wiener index of iterated line graphs of trees homeomorphic to the claw $K_{1,3}$ // Ars Math. Contemp. 2013. Vol. 6. P. 211–219.
54. *Knor M., R. Škrekovski* Wiener index of generalized 4-stars and of their quadratic line graphs // Australas. J. Comb. 2014. Vol. 58. P. 119–126.
55. *Krnc M., R. Skrekovski* On Wiener Inverse Interval Problem // MATCH Commun. Math. Comput. Chem. 2016. Vol. 75. P. 71–80.
56. *Lepovic M., I. Gutman* A collective property of trees and chemical trees // Journal of Chemical Information and Modeling 1998. Vol. 38. № 5. P. 823–826.
57. *Li R.* Harary index and some Hamiltonian properties of graphs // AKCE International Journal of Graphs and Combinatorics 2015. Vol. 12. P. 64–69.
58. *Li R.* Wiener Index and Some Hamiltonian Properties of Graphs // International Journal of Mathematics and Soft Computing, 2015. Vol. 5, № 1. P. 11–16.
59. *Li X., L. Wang* Solutions for two conjectures on the inverse problem of the Wiener index of peptoids // Siam J. Discrete Math. 2003. Vol. 17. P. 210–218.
60. *Liu R.-F., X. Du, H.-C. Jia* Some observations on Harary index and traceable graphs // MATCH Commun. Math. Comput. Chem. 2017. Vol. 77, № 1. P. 195–208.
61. *Liu R.-F., X. Du, H.-C. Jia* Wiener index on traceable and Hamiltonian graph // Bull. Aust. Math. Soc. 2016. Vol. 94. P. 362–372.
62. *Nadjafi-Arani M. J., H. Khodashenas and A. R. Ashrafi* Relationship between edge Szeged and edge Wiener indices of graphs // Glas.Mat. Ser. III, 2012. Vol. 47. P. 21–29.
63. *Nasiri R., H. Yousefi-Azari, M. R.Darafsheh and A. R. Ashrafi* Remarks on the Wiener index of unicyclic graphs // J. Appl.Math. Comput. 2013. Vol. 41, № 1-2. P. 49–59.
64. *Nikolić S., N. Trinajstić and Z. Mihalić* The Wiener index: Development and applications // Croat. Chem. Acta, 1995. Vol. 68. P. 105–129.
65. *Plavšić D., S. Nikolić, N. Trinajstić, Z. Mihalić* On the Harary index for the characterization of chemical graphs // J. Math. Chem. 1993. Vol. 12. P. 235–250.
66. *Quadras J., K.Arputha Christy and A.Nelson* Edge-Wiener Indices of n-circumscribed Peri-condensed Benzenoid Graphs // International Journal of Mathematics And its Applications 2016. Vol. 4, № 4. P. 117–123.
67. *Sagan B. E., Y. Yeh and P. Zhang* The Wiener polynomial of a graph // Inter. J. Quantum Chem. 1996. Vol. 60. P. 959–969.
68. *Senbagamalar J., J. Baskar Babujee AND I. Gutman* On Wiener index of graph complements // Transactions on Combinatorics. 2014. Vol. 3 № 2 P. 11–15.
69. *Sills A. V., H. Wang* On the maximal Wiener index and related questions // Discrete Appl. Math. 2012. Vol. 160. P. 1615–1623.
70. *Soltani A., A. Iranmanesh* On the edge Wiener index // Filomat. 2014. Vol. 28, № 3. P. 541–549.
71. *Stevanovic D.* Hosoya polynomial of composite graphs // Discrete Math. 2001. Vol. 235. P. 237–244.
72. *Szeider S.* Generalizations of Matched CNF Formulas // Annals of Mathematics and Artificial Intelligence. 2005. Vol. 43, № 1-4. P. 223–238.

73. *Talmaciu M.* Decomposition Problems in the Graph Theory with Applications in Combinatorial Optimization. *Ph. D. Thesis, University "Al. I. Cuza"Iasi, Romania. 2002.*
74. *Talmaciu M., E. Nechita* Recognition Algorithm for diamond-free graphs. // Informatica. 2007. Vol. 18, № 3. P. 457–462.
75. *Terpai T.* Proof of a conjecture of V. Nikiforov // Combinatorica, 2011. Vol. 31. P. 739–754.
76. *Wagner S.* A note on the inverse problem for the Wiener index // MATCH Commun. Math. Comput. Chem. 2010. Vol. 64. P. 639–646.
77. *Wagner S., H. Wang, G. Yu,* Molecular graphs and the inverse Wiener index problem. // Discrete Appl. Math. 2009. Vol. 157. P. 1544–1554.
78. *Wagner S.* On the Wiener index of random trees // Discrete Math. 2012. Vol. 312. P. 1502–1511.
79. *Walikar H. B., V. S. Shigehalli and H. S. Ramane* Bounds on the Wiener number of a graph // Communications in Mathematical and in Computer Chemistry/MATCH, 2004. Vol. 50. P. 117–132.
80. *Wiener H.* Structural determination of paraffin boiling points // J. Am. Chem. Soc. 1947. Vol. 69. P. 17–20.
81. *Xu K., M. Liu, K. C. Das, I. Gutman and B. Furtula* A survey on graphs extremal with respect to distance-based topological indices // MATCH Commun. Math. Comput. Chem., 2014. Vol. 71, № 3. P. 461–508.
82. *Yang and B. Y., Y. N. Yeh* Wiener polynomials of some chemically interesting graphs // Int. J. Quantum Chem., 2004. Vol. 99. P. 80–91.
83. *Zeng T.* Harary index and Hamiltonian property of graphs // MATCH Commun. Math. Comput. Chem., 2013. Vol. 70. P. 645–649.
84. *Zhou Q., Broersma H., Wang L., Lu Y.* On sufficient spectral radius conditions for hamiltonicity // Proceedings of the 17th Cologne-Twente Workshop on Graphs and Combinatorial Optimization, 2019. P. 143–146.

V. V. Lepin, V. I. Benediktovich, M. Talmaciu
Topological Indices of Cographs

Summary

The Wiener index, the Harary index and edge-Wiener index are classic and well-known topological indices for the characterization of molecular graphs. Firstly, we give the formulae and algorithms for these topological indices of the complements of trees and cographs. Secondly, we present some sufficient conditions for a cograph to be Hamiltonian.

Список литературы

[1] Azari M., A. Iranmanesh, A. Tehranian, A method for calculating an edge version of the Wiener number of a graph operation, Util. Math. 87 (2012) 151–164.

[2] Azari M., A. Iranmanesh, Computation of the edge Wiener indices of the sum of graphs, Ars. Combin. 100 (2011) 113–128.

[3] Azari M., A. Iranmanesh, Computing Wiener-like topological invariants for some composite graphs and some nanotubes and nanotori, In: I. Gutman, (Ed.), Topics in Chemical Graph Theory, Univ. Kragujevac, Kragujevac, 2014, 69–90.

[4] Azari M., A. Iranmanesh, The second edge-Wiener index of some composite graphs, Miskolc Math. Notes. 15(2) (2014), 305–316.

[5] Benediktovich V.I. Spectral condition for Hamiltonicity of a graph // Linear Algebra Appl. 494 (2016) 70–79.

[6] Graphs and Hypergraphs, North-Holland, Amsterdam, 1973.

[7] Bond J.A., Chvatal V., A method in graph theory // Discr. Math. 15 (2) (1976), 111–135.

[8] Brouwer A.E., Haemers W.H. Spectra of graphs, Springer, 2011.

[9] Cai G.-X, M.-L. Ye, Y. Gui, R. Li. Hyper-Wiener index and Hamiltonicity of graphs, Ars Combinatoria, 2018, 139: 175–184.

[10] Chen L., X. Li and M. Liu, The (revised) Szeged index and the Wiener index of a nonbipartite graph, European J. Combin., 36 (2014) 237–246.

[11] Chen W., Zhang X., The number of edges, spectral radius and Hamiltonian-connectedness of graphs // J. Comb. Optim., 35 (2018), 1104–1127.

[12] Chvatal V., On Hamiltonian's ideals // J. Comb. Theory B. 12 (1972), 163–168.

[13] Corneil D. G., Y. Perl, and L. K. Stewart, A linear recognition algorithm for cographs. *SIAM Journal on Computing*, 14(4):926–934, 1985.

[14] Croitoru C., E. Olaru, M. Talmaciu, Confidentially connected graphs. *The annals of the University "Dunarea de Jos"of Galati. Proceedings of the international conference "The risk in contemporany economy Supplement to Tome XVIII (XXII)*, 17-20, (2000).

[15] C. Croitoru, M. Talmaciu,. On Confidentially connected graphs. *Bul. Stiint. Univ. Baia Mare, Ser B, Matematica - Informatica*, Vol.**XVI**, Nr. 1, 13-16, (2000).

[16] Cui Z., B. Liu, On Harary matrix, Harary index and Harary energy, MATCH Commun. Math. Comput. Chem., 68(2012): 815–823.

[17] Dankelmann P., I. Gutman, S.Mukwembi and H. C. Swart, The edge Wiener index of a graph // Discrete Math., 309 (2009), 3452–3457.

[18] Darafsheh M. R., M. H. Khalifeh, Calculation of the Wiener, Szeged, and PI indices of a certain nanostar dendrimer, Ars Combin., 100 (2011), 289–298.

[19] Das K. Ch., Sharp bounds for the sum of the squares of the degrees of a graph // Kragujevac J. Math.25 (2003) 31–49.

[20] Das K. C., K. Xu, I. N. Cangul, A. S. Cevik, A. Graovac On the Harary index of graph operations. // Journal of Inequalities and Applications 2013, 2013:339.

[21] Diudea M. V., Hosoya polynomial in tori, MATCH Commun.Math. Comput. Chem., 45 (2002), 109–122.

[22] Diudea M. V., Wiener index of dendrimers, MATCH Commun.Math. Comput. Chem., 32 (1995), 71–83.

[23] Dobrynin A. A., I. Gutman, S. Klav *ž* ar and P. *Ž* igert, Wiener index of hexagonal systems, Acta Appl. Math., 72 (2002) 247–294.

[24] Dobrynin A. A., R. Entringer and I. Gutman, Wiener index of trees: theory and applications, Acta Appl. Math., 66 (2001) 211–249.

[25] Dobrynin A., I. Gutman. On a graph invariant related to the sum of all distances in a graph. Publ. Inst. Math. (Beograd), 56:18–22, 1994.

[26] Dong H., B. Zhou, *Maximum Wiener index of unicyclic graphs with fixed maximum degree.* Ars Combin., **103** (2012), 407–416.

[27] Do š li.c T., Vertex-Weighted Wiener polynomials for composite graphs, ArsMath. Contemp., 1 (2008), 66-80.

[28] Entringer R. C., D. E. Jackson and D. A. Snyder, *Distance in graphs*, Czech. Math. J., **26** (1976), 283–296.

[29] Entringer R. C., Distance in graphs: Trees, J. Combin. Math. Combin. Comput., 24 (1997) 65–84.

[30] Fink J., B. Lužar and R. Škrekovski, *Some remarks on inverse Wiener index problem.* Discrete Appl. Math., **160** (2012), 1851–1858.

[31] Gutman I., Calculating the Wiener number: the Doyle-Graver method, J. Serb. Chem. Soc., 58 (1993), 745–750.

[32] Gutman I., Hosoya polynomial and the distance of the total graph of a tree, Univ. Beograd. Publ. Elektrotehn. Fak. Ser.Mat., 10 (1999), 53–58.

[33] Gutman I., A new method for the calculation of the Wiener number of acyclic molecules, J. Mol. Struct. (Theochem). 285 (1993), 137–142.

[34] Gutman I., J. Lu and M. A. Bautiche, On distance in complements of graphs, Ser. A: appl. math. Inform and mech., 8(1) (2016), 35–42.

[35] Gutman I., O. E. Polansky, Mathematical Concepts in Organic Chemistry, Springer, Berlin, 1986.

[36] Gutman I., R. Cruz and J. Rada, Wiener index of Eulerian graphs, Discrete Appl. Math., 162 (2014) 247–250.

[37] Gutman I., S. Klavzar, M. Petkovsek and P. Zigert, On Hosoya polynomials of benzenoid graphs. *MATCH Commun. Math. Comput. Che.*, **43**, 49–66, (2001).

[38] Gutman I., Y. N. Yeh, S. L. Lee and Y. L. Luo, Some recent results in the theory of the Wiener number, Indian J. Chem., 32A (1993) 651–661.

[39] Gutman I., Y. Yeh, The sum of all distances in bipartite graphs, Math. *Slovaca*, **45**, 327–334, (1995).

[40] Hosoya H., On some counting polynomials in Chemistry, Discrete Appl.Math., 19 (1988), 239–257.

[41] Hua H.-B., B. Ning, Wiener index, Harary index and Hamiltonicity of graphs, Match Communications in Mathematical and in Computer Chemistry, 78(1)(2016): 153–162.

[42] Hua H.-B., M. Wang, On Harary index and traceable graphs, MATCH Commun. Math. Comput. Chem., 70 (2013): 297–300.

[43] Iranmanesh A., A. S. Kafrani, Computation of the first edge-Wiener index of TUC4C8(S) nanotube, MATCH Commun. Math. Comput. Chem. 62(2) (2011) 311–352.

[44] Iranmanesh A., A. S. Kafrani, O. Khormali, A New Version of Hyper-Wiener Index, MATCH Commun. Math. Comput. Chem. 65(1) (2011) 113–122.

[45] Iranmanesh A., I. Gutman, O. Khormali and A. Mahmiani, The edge versions of Wiener index, MATCH Commun. Math. Compute. Chem., 61 (2009), 663–672.

[46] Iranmanesh A., M. Azari, Edge-Wiener descriptors in chemical graph theory: A survey, Curr. Org. Chem., 19 (3) (2015), 219–239.

[47] Ivanciuc O., T. S. Balaban, A. T. Balaban, Reciprocal distance matrix, related local vertex invariants and topological indices // J. Math. Chem., 12(1993): 309–318.

[48] Khalifeh M. H., H. Yousefi-Azari, A. R. Ashrafi and S. G. Wagner, Some new results on distance-based graph invariants, European J. Combin., 30 (2009), 1149–1163.

[49] Klavžar S.. M. J. Nadja-Arani, Wiener index in weighted graphs via unication of $\Theta*$-classes, European J. Combin., 36 (2014) 71–76.

[50] Klein D.-J., I. Lukovits and I. Gutman, On the defination of the hyper-Wiener index for cycle-containing structures, J. Chem. Inf. Comput. Sci., 35(1995): 50–52.

[51] Knor M., P. Potočnik and R. Škrekovski, *On a conjecture about Wiener index in iterated line graphs of trees*, Discrete Math., **312** (2012), 1094–1105.

[52] Knor M., P. Potočnik and R. Škrekovski, *Wiener index of iterated line graphs of trees homeomorphic to H*, Discrete Math., **313** (2013), 1104–1111.

[53] Knor M., P. Potočnik and R. Škrekovski, *Wiener index of iterated line graphs of trees homeomorphic to the claw* $K_{1,3}$, Ars Math. Contemp., **6** (2013), 211–219.

[54] Knor M., R. Škrekovski, Wiener index of generalized 4-stars and of their quadratic line graphs, Australas. J. Comb., 58 (2014) 119–126.

[55] Krnc M., R. Skrekovski, On Wiener Inverse Interval Problem, to appear in *MATCH Commun. Math. Comput. Chem.*

[56] Lepovic M., I. Gutman, A collective property of trees and chemical trees, *Journal of Chemical Information and Modeling,* vol. **38**, no. 5, 823–826, (1998).

[57] Li R., Harary index and some Hamiltonian properties of graphs, AKCE International Journal of Graphs and Combinatorics 12 (2015): 64–69.

[58] Li R., Wiener Index and Some Hamiltonian Properties of Graphs, International Journal of Mathematics and Soft Computing, 5(1)(2015): 11–16.

[59] Li X., L. Wang, Solutions for two conjectures on the inverse problem of the Wiener index of peptoids, *Siam J. Discrete Math.*, **17**, 210–218, (2003).

[60] Liu R.-F., X. Du, H.-C. Jia, Some observations on Harary index and traceable graphs, MATCH Commun. Math. Comput. Chem., 77(1)(2017): 195–208.

[61] Liu R.-F., X. Du, H.-C. Jia, Wiener index on traceable and Hamiltonian graph, Bull. Aust. Math. Soc., 94(2016): 362–372

[62] Nadjafi-Arani M. J., H. Khodashenas and A. R. Ashrafi, Relationship between edge Szeged and edge Wiener indices of graphs, Glas.Mat. Ser. III, 47(67) (2012), 21–29.

[63] Nasiri R., H. Yousefi-Azari, M. R.Darafsheh and A. R. Ashrafi, Remarks on the Wiener index of unicyclic graphs, J. Appl.Math. Comput., 41 (1-2) (2013), 49–59.

[64] Nikolić S., N. Trinajstić and Z. Mihalić, The Wiener index: Development and applications, Croat. Chem. Acta, 68 (1995) 105–129.

[65] Plavšić D., S. Nikolić, N. Trinajstić, Z. Mihalić, On the Harary index for the characterization of chemical graphs, J. Math. Chem., 12(1993): 235-250.

[66] Quadras J., K.Arputha Christy and A.Nelson Edge-Wiener Indices of n-circumscribed Peri-condensed Benzenoid Graphs International Journal of Mathematics And its Applications Volume 4, Issue 4 (2016), 117–123.

[67] Sagan B. E., Y. Yeh and P. Zhang, The Wiener polynomial of a graph, Inter. J. Quantum Chem., 60 (1996), 959–969.

[68] Senbagamalar J., J. Baskar Babujee AND I. Gutman, On Wiener index of graph complements, *Transactions on Combinatorics* Vol. 3 No. 2 (2014), pp. 11–15.

[69] Sills A. V., H. Wang, *On the maximal Wiener index and related questions*, Discrete Appl. Math., **160** (2012), 1615–1623.

[70] Soltani A., A. Iranmanesh, On the edge Wiener index, Filomat, 28(3) (2014), 541–549.

[71] Stevanovic D., Hosoya polynomial of composite graphs. *Discrete Math.*, **235**, 237–244, (2001).

[72] Szeider S., Generalizations of Matched CNF Formulas. *Annals of Mathematics and Artificial Intelligence.*, vol. **43**, No. 1-4, 223–238, (2005).

[73] Talmaciu M., (2002). Decomposition Problems in the Graph Theory with Applications in Combinatorial Optimization. *Ph. D. Thesis, University "Al. I. Cuza"Iasi, Romania.*

[74] Talmaciu M., E. Nechita, Recognition Algorithm for diamond-free graphs. *Informatica.*, Vol. **18**, No. 3, 457–462, (2007).

[75] Terpai T. , Proof of a conjecture of V. Nikiforov // Combinatorica, 31, (2011), 739 – 754.

[76] Wagner S., A note on the inverse problem for the Wiener index, *MATCH Commun. Math. Comput. Chem.*, **64**, 639–646, (2010).

[77] Wagner S., H. Wang, G. Yu, Molecular graphs and the inverse Wiener index problem, *Discrete Appl. Math.*, **157**, 1544–1554, (2009).

[78] Wagner S., *On the Wiener index of random trees*, Discrete Math., **312** (2012), 1502–1511.

[79] Walikar H. B., V. S. Shigehalli and H. S. Ramane, *Bounds on the Wiener number of a graph*, Communications in Mathematical and in Computer Chemistry/MATCH, **50** (2004), 117–132.

[80] Wiener H., Structural determination of paraffin boiling points, *J. Am. Chem. Soc.*, **69**, 17–20,(1947).

[81] Xu K., M. Liu, K. C. Das, I. Gutman and B. Furtula, A survey on graphs extremal with respect to distance-based topological indices, MATCH Commun. Math. Comput. Chem., Vol. 71, 3 (2014), 461–508.

[82] Yang and B. Y., Y. N. Yeh, Wiener polynomials of some chemically interesting graphs, Int. J. Quantum Chem., 99 (2004), 80–91.

[83] Zeng T., Harary index and Hamiltonian property of graphs, MATCH Commun. Math. Comput. Chem., 70(2013): 645–649.

[84] Zhou Q., Broersma H., Wang L., Lu Y., On sufficient spectral radius conditions for hamiltonicity // Proceedings of the 17th Cologne-Twente Workshop on Graphs and Combinatorial Optimization, 143–146.