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ПОЛНАЯ СХОДИМОСТЬ ЧАСТИЧНЫХ ВЗВЕШЕННЫХ СУММ ОТРИЦАТЕЛЬНО ОРТАНТНО ЗАВИСИМЫХ СЛУЧАЙНЫХ ВЕЛИЧИН

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Аннотация. Отрицательная ортантная зависимость рассматривается как обобщение независимости случайных величин, которое ввели К. Джоаг-Дев и Ф. Прошан. Многие исследователи изучали неравенства и законы больших чисел для таких последовательностей случайных величин. В частности, понятие полной сходимости, определенное П. Л. Хсу и Г. Роббинсом, привлекло значительное внимание. Устанавливается полная сходимость для частичных взвешенных сумм отрицательно ортантно зависимых случайных величин, над которыми доминирует случайная величина X. Приводятся достаточные условия такой сходимости при выполнении мягких предположений относительно весов и моментов случайной величины X.

Ключевые слова: полная сходимость; отрицательная ортантная зависимость; взвешенные суммы; предельные теоремы; зависимые случайные величины.

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COMPLETE CONVERGENCE FOR PARTIAL WEIGHTED SUMS OF NEGATIVELY ORTHANT DEPENDENT RANDOM VARIABLES

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Abstract. Negatively orthant dependence is regarded as a generalisation of independence for random variables, introduced by K. Joag-Dev and F. Proschan. Numerous researchers have investigated inequalities and laws of large numbers for such sequences of random variables. In particular, the concept of complete convergence, defined by P. L. Hsu and H. Robbins, has attracted significant attention. Complete convergence for partial weighted sums of negatively orthant dependent random variables dominated by a random variable X is established. Sufficient conditions for this type of convergence are provided under mild assumptions on the weights and the moments of random variable X.

Keywords: complete convergence; negatively orthant dependence; weighted sums; limit theorems; dependent random variables.

Introduction and preliminaries

The concept of a complete convergence was introduced by P. L. Hsu and H. Robbins [1] while K. Joag-Dev and F. Proschan introduced the definition of negatively orthant dependent (NOD) sequences [2]. Since then a wide range of limit theorems for NOD random variables have been developed by numerous researchers. H. C. Kim established a Hájek – Rényi type inequality [3] while N. Asadian and his colleagues obtained a Rosenthal-type inequality [4]. In addition, A. Volodin presented the Kolmogorov exponential inequality [5]. Results concerning almost sure convergence appear in the articles [6–8]. Complete convergence has been investigated by the authors of the works [9–15]. This paper aims to establish complete convergence for partial weighted sums of NOD random variables dominated by a random variable.

Definition 1. A sequence of random variables $\{X_n,\ n \geq 1\}$ is said to converge completely to a random variable X if for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P\left(\left|X_n - X\right| > \varepsilon\right) < \infty.$$

Definition 2. A finite collection of random variables $X_1, X_2, \ldots, X_n$ is said to be NOD if

$$P\left(X_1 > x_1,\ X_2 > x_2,\ \ldots,\ X_n > x_n\right) \leq \prod_{i=1}^{n} P\left(X_i > x_i\right)$$

and

$$P\left(X_1 \leq x_1,\ X_2 \leq x_2,\ \ldots,\ X_n \leq x_n\right) \leq \prod_{i=1}^{n} P\left(X_i \leq x_i\right)$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$. An infinite sequence $\{X_n,\ n \geq 1\}$ is said to be NOD if every finite subcollection is NOD.

Definition 3. An array of random variables $\{X_{ni}, 1 \leq i \leq n,\ n \geq 1\}$ is said to be rowwise NOD random variables if for every $n \geq 1$ $\{X_{ni}, 1 \leq i \leq n\}$ is a sequence of NOD random variables.

Definition 4. A sequence of random variables $\{X_n,\ n \geq 1\}$ is said to be stochastically dominated by a random variable X if for any $x \geq 0$

$$\sup_{n \geq 1} P\left(\left|X_n\right| > x\right) \leq P\left(\left|X\right| > x\right).$$

Let $I(A)$ be the indicator function of the set A. The symbol C denotes a positive constant, which may be different in each appearance, and $a_n = O(b_n)$ stands for $a_n \leq C b_n$. The following lemmas are useful for the proof of the main results of the work.

Lemma 1 [16]. *Let random variables $X_1, X_2, \ldots, X_n$ be NOD; $f_1, f_2, \ldots, f_n$ be all non-decreasing (or all non-increasing) functions. Then random variables $f_1(X_1), f_2(X_2), \ldots, f_n(X_n)$ are NOD.*

Lemma 2 [16]. *Let $X_1, X_2, \ldots, X_n$ be non-negative NOD random variables. Then*

$$E\left(\prod_{i=1}^{n} X_i\right) \le \prod_{i=1}^{n} E(X_i).$$

Lemma 3 [4; 12]. *Let $p \ge 2$ and $\{X_n,\ n \ge 1\}$ be a sequence of NOD random variables with $E(X_n) = 0$ and $E\left(|X_n|^p\right) < \infty$ for every $n \ge 1$. Then there exists a positive constant C depending only on p such that for every $n \ge 1$*

$$E\left|\sum_{i=1}^{n} X_i\right|^p \le C\left[\sum_{i=1}^{p} E|X_i|^p + \left(\sum_{i=1}^{n} E|X_i|^2\right)^{\frac{p}{2}}\right],$$

$$E\left(\max_{1 \le j \le n}\left|\sum_{i=1}^{j} X_i\right|^p\right) \le C\log^p(2n)\left[\sum_{i=1}^{p} E|X_i|^p + \left(\sum_{i=1}^{n} E|X_i|^2\right)^{\frac{p}{2}}\right].$$

Lemma 4. *For any $x \in \mathbb{R}$ and $r \in [1, 2]$ the following inequality holds:*

$$e^x \le 1 + x + \frac{1}{r}|x|^r e^{|x|}.$$

P r o o f. Let us consider 2 cases.

1. Case $x \ge 0$. Since $e^x - 1 - x = e^x \int_0^x te^{-t}dt,$ it is sufficient to show

$$f(x) = \frac{1}{r}x^r - \int_0^x te^{-t}dt \ge 0 \text{ for all } x \ge 0.$$

This inequality holds because $f(0) = 0,$

$$x^{2-r}e^{-x} \le \sup_{x>0} x^{2-r}e^{-x} = (2-r)^{2-r}e^{r-2} \le e^{2-r}e^{r-2} = 1,$$

and $f'(x) = x^{r-1}\left(1 - x^{2-r}e^{-x}\right) \ge 0$ for all $x > 0$.

2. Case $x < 0$. From the case 1 we have

$$\frac{1}{r}|x|^r e^{|x|} \ge e^{|x|}\int_0^{|x|} te^{-t}dt = \left(e^x - 1 - x\right) + \left(e^{-x} - e^x + 2x\right),\ x < 0.$$

Since $e^{-x} - e^x + 2x \ge 0$ for all $x < 0$, we obtain

$$\frac{1}{r}|x|^r e^{|x|} \ge e^x - 1 - x,\ x < 0.$$

Lemma is proved.

Lemma 5. *Let $p, q > 0$ and $\{X_n,\ n \ge 1\}$ be a sequence of random variables, which is stochastically dominated by a random variable X with $E\left(|X|^p\right) < \infty$. For all $n \ge 1$ and $x > 0$ the following inequalities hold:*

(i) $E\left(|X_n|^q I\left(|X_n| \le x\right)\right) \le E\left(|X|^q I\left(|X| \le x\right)\right) + x^q P\left(|X| > x\right);$

(ii) $E\left(|X_n|^q I\left(|X_n| > x\right)\right) \le E\left(|X|^q I\left(|X| > x\right)\right),\ q \le p.$

P r o o f. Let us prove the inequality (i). We have

$$E\left(|X_n|^q I\left(|X_n| \le x\right)\right) = \int_0^x t^q dP\left(|X_n| \le t\right) = -\int_0^x t^q dP\left(|X_n| > t\right) =$$

$$= -x^q P\left(|X_n| > x\right) + \int_0^x P\left(|X_n| > t\right)dt^q \le \int_0^x P\left(|X_n| > t\right)dt^q =$$

$$= x^q P(|X| > x) - \int_0^x t^q dP(|X| > t) = x^q P(|X| > x) + \int_0^x t^q dP(|X| \le t) =$$

$$= x^q P(|X| > x) + E\left(|X|^q I(|X| \le x)\right).$$

Let us prove the inequality (ii). From the inequality (i) for $q \le p$ and letting $x \to \infty$ we obtain

$$E|X_n|^q \le E|X|^q < \infty \text{ for all } n \ge 1.$$

Similarly, for $q \le p$ we have

$$E\left(|X_n|^q I(|X_n| > x)\right) = \int_x^{+\infty} t^q dP(|X_n| \le t) = -\int_x^{+\infty} t^q dP(|X_n| > t) =$$

$$= x^q P(|X_n| > x) + \int_x^{+\infty} P(|X_n| > t) dt^q \le$$

$$\le x^q P(|X| > x) + \int_x^{+\infty} P(|X| > t) dt^q = E\left(|X|^q I(|X| > x)\right).$$

Lemma 6. *Let* $\{X_n,\ x \ge 1\}$ *be a sequence of NOD random variables with mean zero and* $0 < B_n^2 = \sum_{i=1}^n EX_i^2 < \infty$. *Then for any* $x > 0$ *and* $y > 0$

$$P(|S_n| \ge x) \le \sum_{k=1}^n P(|X_k| \ge y) + 2\exp\left(\frac{x}{y} - \frac{x}{y}\log\left(1 + \frac{xy}{B_n}\right)\right),$$

where $S_n = \sum_{k=1}^n X_k$.

P r o o f. For a fixed $y > 0$ and each $n \ge 1$ put $X'_n = X_n I(X_n < y)$ and $S'_n = \sum_{i=1}^n X'_i$. Then we have

$$P(S_n \ge x) = P(S_n \ge x,\ S_n \ne S'_n) + P(S_n \ge x,\ S_n = S'_n) =$$

$$= P(S_n \ne S'_n) + P(S'_n \ge x) \le \sum_{k=1}^n P(X_k \ge y) + P(S'_n \ge x).$$

Note that for a fixed $h > 0$ the function $f(u) = \dfrac{e^{hu} - hu - 1}{u^2}$ is increasing for u. Since $E(X_k) = 0$, for any $y > 0$

$$\int_{-\infty}^y t dP(X_k \le t) = -\int_y^{+\infty} t dP(X_k \le t) \le 0,\ k \ge 1.$$

Therefore,

$$Ee^{hX'_k} = \int_{-\infty}^y e^{ht} dP(X_k \le t) + \int_y^{+\infty} dP(X_k \le t) =$$

$$= 1 + h\int_{-\infty}^y t dP(X_k \le t) + \int_{-\infty}^y \left(e^{ht} - 1 - ht\right) dP(X_k \le t) \le$$

$$\le 1 + \int_{-\infty}^y \frac{e^{ht} - ht - 1}{t^2} t^2 dP(X_k \le t) \le 1 + \frac{e^{hy} - hy - 1}{y^2} \int_{-\infty}^y t^2 dP(X_k \le t) \le$$

$$\le \exp\left(\frac{e^{hy} - hy - 1}{y^2} E\left(X_k^2\right)\right),\ k \ge 1.$$

By Markov inequality and lemma 2, for any $h > 0$

$$P(S'_n \ge x) \le e^{-hx} E\left(e^{hS'_n}\right) \le e^{-hx} \prod_{k=1}^n Ee^{hX'_k} \le \exp\left(-hx + \frac{e^{hy} - hy - 1}{y^2} B_n^2\right).$$

Put $h = \frac{1}{y}\log\left(1 + \frac{xy}{B_n^2}\right)$ and get

$$P(S_n' \geq x) \leq \exp\left(\frac{x}{y} - \frac{x}{y}\log\left(1 + \frac{xy}{B_n^2}\right) - \frac{1}{y^2}\log\left(1 + \frac{xy}{B_n^2}\right)\right) \leq \exp\left(\frac{x}{y} - \frac{x}{y}\log\left(1 + \frac{xy}{B_n^2}\right)\right).$$

This implies that

$$P(S_n \geq x) \leq \sum_{k=1}^{n} P(X_k \geq y) + \exp\left(\frac{x}{y} - \frac{x}{y}\log\left(1 + \frac{xy}{B_n^2}\right)\right).$$

Replacing X_k with $-X_k$, we also have

$$P(S_n \leq -x) \leq \sum_{k=1}^{n} P(X_k \leq -y) + \exp\left(\frac{x}{y} - \frac{x}{y}\log\left(1 + \frac{xy}{B_n^2}\right)\right).$$

Combining the last two inequalities, we obtain the desired result. Lemma is proved.

Main results

The following theorem extends the result of lemma 2.3 from the article [11] to the case $r \in (1, 2]$.

Theorem 1. *Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise NOD random variables with $E(X_{ni}) = 0$ and let $\{b_n, n \geq 1\}$ be a sequence of positive real numbers. We assume that there exists a positive constant $r \in (1, 2]$ such that*

$$\max_{1 \leq i \leq n} |X_{ni}| = O(b_n),$$

$$\sum_{i=1}^{n} E|X_{ni}|^r = O(b_n),$$

$$\sum_{n=1}^{\infty} e^{\frac{-\alpha}{b_n^{(r-1)^{-1}}}} < \infty \text{ for some } \alpha > 0.$$

Then $\sum_{i=1}^{n} X_{ni}$ converges completely to zero.

P r o o f. Applying lemma 4 for any $t > 0$, we have

$$E(\exp(tX_{ni})) \leq E\left\{1 + tX_{ni} + \frac{1}{r}|tX_{ni}|^r \exp(|tX_{ni}|)\right\} = 1 + \frac{1}{r}E\left(|tX_{ni}|^r \exp(|tX_{ni}|)\right) \leq$$

$$\leq 1 + \frac{1}{r}E\left(|tX_{ni}|^r \exp(Ctb_n)\right) \leq \exp\left(\frac{t^r}{r}\exp(Ctb_n)E\left(|X_{ni}|^r\right)\right).$$

By Markov inequality and lemma 2, for any $\varepsilon > 0$

$$P\left(\sum_{i=1}^{n} X_{ni} > \varepsilon\right) \leq e^{-t\varepsilon} E\exp\left(t\sum_{i=1}^{n} X_{ni}\right) \leq e^{-t\varepsilon}\prod_{i=1}^{n} E\exp(tX_{ni}) \leq$$

$$\leq e^{-t\varepsilon}\exp\left(\frac{t^r}{r}\exp(Ctb_n)\sum_{i=1}^{n} E\left(|X_{ni}|^r\right)\right) \leq e^{-t\varepsilon}\exp\left(t^r \exp(Ctb_n)Cb_n\right).$$

Put $t = \frac{\alpha + 1}{\varepsilon b_n^{(r-1)^{-1}}}$ for n large enough and $C > 0$ such that $C(\alpha + 1)^r \varepsilon^{-r} > 1$ we obtain

$$P\left(\sum_{i=1}^{n} X_{ni} > \varepsilon\right) \leq \exp\left(\left(C(\alpha + 1)^r \varepsilon^{-r} \exp\left(C(\alpha + 1)\varepsilon^{-1} b_n^{(r-2)(r-1)^{-1}}\right) - \alpha - 1\right) b_n^{-(r-1)^{-1}}\right) \leq$$

$$\leq \exp\left(-C(\alpha + 1)^r \varepsilon^{-r} \frac{\alpha}{b_n^{(r-1)^{-1}}}\right) \leq \exp\left(-\frac{\alpha}{b_n^{(r-1)^{-1}}}\right).$$

Therefore,

$$\sum_{n=1}^{\infty} P\left(\sum_{i=1}^{n} X_{ni} > \varepsilon\right) < \infty.$$

Since $\{-X_{ni}, 1 \le i \le n, n \ge 1\}$ is still an array of rowwise NOD random variables, we can replace X_{ni} with $-X_{ni}$ from the above statement. That is

$$\sum_{n=1}^{\infty} P\left(\sum_{i=1}^{n} X_{ni} < -\varepsilon\right) < \infty.$$

Theorem is proved.

Theorem 2. *Let $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of rowwise NOD random variables such that $E(X_{ni}) = 0$ and $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ are stochastically dominated by a random variable X satisfying $E\left(|X|^{p+1}\log|X|\right) < \infty$ for some $p \in (1, 2)$. We assume that $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of positive real numbers such that*

$$\max_{1 \le i \le n} |a_{ni}| = O\left(n^{\frac{-1}{p}}\right) \text{ and } \sum_{i=1}^{n} a_{ni}^2 = O\left(n^{\frac{-2}{1+p}}\right).$$

Then $\sum_{i=1}^{n} a_{ni} X_{ni}$ converges completely to zero.

P r o o f. For $q \in \left(\frac{2p}{2-p}, 1+p\right)$ put $X_{ni} = X'_{ni} + X''_{ni}$, where

$$X'_{ni} = X_{ni} I\left(|X_{ni}| \le n^{\frac{1}{q}}\right), \quad X''_{ni} = X_{ni} I\left(|X_{ni}| > n^{\frac{1}{q}}\right), \; n \ge 1, 1 \le i \le n.$$

For any $\varepsilon > 0$ we have

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} a_{ni} X_{ni}\right| > 2\varepsilon\right) = \sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} a_{ni}(X_{ni} - EX_{ni})\right| > 2\varepsilon\right) \le$$

$$\le \sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} a_{ni}(X'_{ni} - EX'_{ni})\right| > \varepsilon\right) + \sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} a_{ni}(X''_{ni} - EX''_{ni})\right| > \varepsilon\right).$$

For n large enough and $b_n = (\log n)^{-1}$, $2 \ge r > \frac{pq}{q-p}$ we get

$$|a_{ni}(X'_{ni} - EX'_{ni})| \le 2|a_{ni}| n^{\frac{1}{q}} \le C n^{\frac{1}{q} - \frac{1}{p}} \le C b_n,$$

$$\sum_{i=1}^{n} E|a_{ni}(X'_{ni} - EX'_{ni})|^r \le C \sum_{i=1}^{n} E|a_{ni} X'_{ni}|^r \le C n^{1 + \frac{r}{q} - \frac{r}{p}} \le C b_n,$$

$$\sum_{n=1}^{\infty} e^{\frac{-\alpha}{b_n^{(r-1)^{-1}}}} < \infty \text{ for any } \alpha > 1.$$

Therefore, by theorem 1, we obtain

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} a_{ni}(X'_{ni} - EX'_{ni})\right| > \varepsilon\right) < \infty.$$

Additionally, by Markov inequality, C_r-inequality and lemma 3, we have

$$P\left(\left|\sum_{i=1}^{n} a_{ni}(X''_{ni} - EX''_{ni})\right| > \varepsilon\right) \le \frac{1}{\varepsilon^2 E}\left|\sum_{i=1}^{n} a_{ni}(X''_{ni} - EX''_{ni})\right|^2 \le$$

$$\le C \sum_{i=1}^{n} a_{ni}^2 E|X''_{ni} - EX''_{ni}|^2 \le C \sum_{i=1}^{n} a_{ni}^2 E|X''_{ni}|^2 = C \sum_{i=1}^{n} a_{ni}^2 E\left(|X_{ni}|^2 I\left(|X_{ni}| > n^{\frac{1}{q}}\right)\right) \le$$

$$\leq C\sum_{i=1}^{n} a_{ni}^2 E\left(|X|^2 I\left(|X| > n^{\frac{1}{q}}\right)\right) \leq C\sum_{i=1}^{n} a_{ni}^2 E\left(|X|^2 I\left(|X| > n^{\frac{1}{1+p}}\right)\right) \leq$$

$$\leq Cn^{-1} E\left(|X|^{p+1} I\left(|X| > n^{\frac{1}{1+p}}\right)\right).$$

Consequently,

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} a_{ni}\left(X''_{ni} - EX''_{ni}\right)\right| > \varepsilon\right) \leq \sum_{n=1}^{\infty} n^{-1} E\left(|X|^{p+1} I\left(|X| > n^{\frac{1}{1+p}}\right)\right) =$$

$$= C\sum_{n=1}^{\infty} n^{-1} \sum_{k=n}^{\infty} E\left(|X|^{p+1} I\left(k^{\frac{1}{1+p}} < |X| \leq (k+1)^{\frac{1}{1+p}}\right)\right) =$$

$$= C\sum_{k=1}^{\infty}\sum_{n=1}^{k} n^{-1} E\left(|X|^{p+1} I\left(k^{\frac{1}{1+p}} < |X| \leq (k+1)^{\frac{1}{1+p}}\right)\right) \leq$$

$$\leq C\sum_{k=1}^{\infty} \log(k) E\left(|X|^{p+1} I\left(k^{\frac{1}{1+p}} < |X| \leq (k+1)^{\frac{1}{1+p}}\right)\right) \leq$$

$$\leq C\sum_{k=1}^{\infty} E\left(|X|^{p+1} \log(|X|) I\left(k^{\frac{1}{1+p}} < |X| \leq (k+1)^{\frac{1}{1+p}}\right)\right) =$$

$$= CE\left(|X|^{p+1} \log(|X|)\right) < \infty.$$

Theorem is proved.

Theorem 3. *Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise NOD random variables with $E(X_{ni}) = 0$, $1 \leq i \leq n$, $n \geq 1$, which is stochastically dominated by a random variable X; $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of positive real numbers; $\{b_n, n \geq 1\}$ be a sequence of positive real numbers, $b_n \uparrow \infty$. Let $\{\psi_n(t), n \geq 1\}$ be a sequence of non-negative even functions satisfying the following conditions:*

$$\frac{\psi_n(t)}{|t|} \uparrow \textit{ and } \frac{\psi_n(t)}{|t|^p} \downarrow \textit{ as } |t| \uparrow$$

for some real number $p \in (1, 2]$. We assume that

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{E\psi_i(X)}{\psi_i(b_n)} < \infty \textit{ and } \sum_{n=1}^{\infty}\sum_{i=1}^{n} a_{ni}^2 \frac{E\psi_i(X)}{\psi_i(b_n)} < \infty.$$

Then

$$\frac{1}{b_n}\sum_{i=1}^{n} a_{ni} X_{ni} \to 0$$

completely.

P r o o f. Note that for any $r \in [0, 2]$

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} a_{ni}^r \frac{E\psi_i(X)}{\psi_i(b_n)} \leq \sum_{n=1}^{\infty}\sum_{i=1}^{n} \left(1 + a_{ni}^2\right) \frac{E\psi_i(X)}{\psi_i(b_n)} < \infty.$$

For a fixed $n \geq 1$ define

$$X_i^{(n)} = -b_n I\left(X_{ni} < -b_n\right) + X_{ni} I\left(|X_{ni}| \leq b_n\right) + b_n I\left(X_{ni} > b_n\right), 1 \leq i \leq n,$$

$$T_j^{(n)} = \sum_{i=1}^{j} a_{ni}\left(X_i^{(n)} - EX_i^{(n)}\right), 1 \leq j \leq n.$$

For any $\varepsilon > 0$

$$P\left(\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| > b_n\varepsilon\right) =$$

$$= P\left(\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| > b_n\varepsilon,\ \max_{1\le i\le n}\left|X_{ni}\right| \le b_n\right) + P\left(\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| > b_n\varepsilon,\ \max_{1\le i\le n}\left|X_{ni}\right| > b_n\right) \le$$

$$\le P\left(\left|\sum_{i=1}^{n} a_{ni}X_i^{(n)}\right| > b_n\varepsilon\right) + P\left(\max_{1\le i\le n}\left|X_{ni}\right| > b_n\right) \le$$

$$\le P\left(\left|T_n^{(n)}\right| > b_n\varepsilon - \left|\sum_{i=1}^{n} a_{ni}EX_i^{(n)}\right|\right) + P\left(\max_{1\le i\le n}\left|X_{ni}\right| > b_n\right).$$

We will show that

$$\frac{1}{b_n}\sum_{i=1}^{n} a_{ni}EX_i^{(n)} \to 0 \text{ as } n\to\infty.$$

Indeed, it is clear that $\left|X_i^{(n)}\right| \le \left|X_{ni}\right| I\left(\left|X_{ni}\right| \le b_n\right)$, $1\le i\le n$, $n\ge 1$. Since $E\left(X_{ni}\right) = 0$ and $\frac{\psi_n(t)}{|t|}\uparrow$ as $t\uparrow$, we have

$$\left|b_n^{-1}\sum_{i=1}^{n} a_{ni}EX_i^{(n)}\right| \le$$

$$\le \left|b_n^{-1}\sum_{i=1}^{n} a_{ni}E\left(X_i^{(n)}I\left(\left|X_{ni}\right|\le b_n\right)\right)\right| + \left|b_n^{-1}\sum_{i=1}^{n} a_{ni}E\left(X_i^{(n)}I\left(\left|X_{ni}\right| > b_n\right)\right)\right| \le$$

$$\le \left|b_n^{-1}\sum_{i=1}^{n} a_{ni}E\left(X_{ni}I\left(\left|X_{ni}\right|\le b_n\right)\right)\right| + b_n^{-1}\sum_{i=1}^{n} a_{ni}E\left(\left|X_i^{(n)}\right|I\left(\left|X_{ni}\right| > b_n\right)\right) =$$

$$= \left|b_n^{-1}\sum_{i=1}^{n} a_{ni}E\left(X_{ni}I\left(\left|X_{ni}\right| > b_n\right)\right)\right| + \sum_{i=1}^{n} a_{ni}P\left(\left|X_{ni}\right| > b_n\right) \le$$

$$\le b_n^{-1}\sum_{i=1}^{n} a_{ni}E\left(|X|I\left(|X| > b_n\right)\right) + \sum_{i=1}^{n} a_{ni}P\left(|X| > b_n\right) \le$$

$$\le \sum_{i=1}^{n} a_{ni}\frac{E\left(\psi_i(X)I\left(|X| > b_n\right)\right)}{\psi_i\left(b_n\right)} + \sum_{i=1}^{n} a_{ni}\frac{E\psi_i(X)}{\psi_i\left(b_n\right)} \le$$

$$\le 2\sum_{i=1}^{n} a_{ni}\frac{E\psi_i(X)}{\psi_i\left(b_n\right)} \to 0 \text{ as } n\to\infty.$$

Thus, for n large enough

$$P\left(\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| > b_n\varepsilon\right) \le P\left(\left|T_n^{(n)}\right| > \frac{b_n\varepsilon}{2}\right) + P\left(\max_{1\le i\le n}\left|X_{ni}\right| > b_n\right).$$

To complete the proof, it is sufficient to show:

$$I_1 = \sum_{i=1}^{\infty} P\left(\left|T_n^{(n)}\right| > \frac{b_n\varepsilon}{2}\right) < \infty$$

and

$$I_2 = \sum_{i=1}^{\infty} P\left(\max_{1\le i\le n}\left|X_{ni}\right| > b_n\right) < \infty.$$

Put $B_n^2 = \sum_{i=1}^{n} a_{ni}^2 E\left(X_i^{(n)} - EX_i^{(n)}\right)^2$, $n\ge 1$. Take $x = \frac{\varepsilon b_n}{2}$, $y = \frac{\varepsilon b_n}{2\eta}$ and $\eta > 1$. On the one hand, by lemma 6, we have

$$I_1 \le \sum_{n=1}^{\infty}\sum_{i=1}^{n} P\left(a_{ni}\left|X_i^{(n)} - EX_i^{(n)}\right| > \frac{\varepsilon b_n}{2}\right) + C\sum_{n=1}^{\infty}\left(\frac{B_n^2}{B_n^2 + \frac{\varepsilon^2 b_n^2}{2\eta}}\right)^{\eta} := I_{11} + I_{12}.$$

By Markov inequality, C_r-inequality, lemma 5 and the facts that $\frac{\psi_i(|t|)}{|t|}\uparrow$ and $\frac{\psi_i(|t|)}{|t|^p}\downarrow$ as $|t|\uparrow$, we obtain

$$I_{11} \le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} a_{ni}^p \frac{E\left|X_i^{(n)} - EX_i^{(n)}\right|^p}{b_n^p} \le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} a_{ni}^p \frac{E\left|X_i^{(n)}\right|^p}{b_n^p} \le$$

$$\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} a_{ni}^p \frac{E\left(|X_{ni}|^p I\left(|X_{ni}| \le b_n\right)\right)}{b_n^p} \le$$

$$\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} a_{ni}^p \frac{E\left(|X|^p I\left(|X| \le b_n\right)\right)}{b_n^p} + C\sum_{n=1}^{\infty}\sum_{i=1}^{n} a_{ni}^p P\left(|X| > b_n\right) \le$$

$$\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} a_{ni}^p \frac{E\left(\psi_i(X) I\left(|X| \le b_n\right)\right)}{\psi_i(b_n)} + C\sum_{n=1}^{\infty}\sum_{i=1}^{n} a_{ni}^p \frac{E\psi_i(X)}{\psi_i(b_n)} \le$$

$$\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} a_{ni}^p \frac{E\psi_i(X)}{\psi_i(b_n)} < \infty$$

and

$$I_{12} \le C\sum_{n=1}^{\infty}\left(b_n^{-2}\sum_{i=1}^{n} a_{ni}^2 E\left(X_i^{(n)} - EX_i^{(n)}\right)^2\right)^{\eta} \le C\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} a_{ni}^2 \frac{E\left|X_i^{(n)}\right|^2}{b_n^2}\right)^{\eta} \le$$

$$\le C\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} a_{ni}^2 \frac{E\left(|X_{ni}|^2 I\left(|X_{ni}| \le b_n\right)\right)}{b_n^2}\right)^{\eta} \le C\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} a_{ni}^2 \frac{E\left(|X_{ni}|^p I\left(|X_{ni}| \le b_n\right)\right)}{b_n^p}\right)^{\eta} \le$$

$$\le C\left(\sum_{n=1}^{\infty}\sum_{i=1}^{n} a_{ni}^2 \frac{E\left(|X_{ni}|^p I\left(|X_{ni}| \le b_n\right)\right)}{b_n^p}\right)^{\eta} \le$$

$$\le C\left(\sum_{n=1}^{\infty}\sum_{i=1}^{n} a_{ni}^2 \frac{E\left(|X|^p I\left(|X| \le b_n\right)\right)}{b_n^p} + \sum_{n=1}^{\infty}\sum_{i=1}^{n} a_{ni}^2 P\left(|X| > b_n\right)\right)^{\eta} \le$$

$$\le C\left(\sum_{n=1}^{\infty}\sum_{i=1}^{n} a_{ni}^2 \frac{E\psi_i(X)}{\psi_i(b_n)}\right)^{\eta} < \infty.$$

On the other hand,

$$I_2 = \sum_{n=1}^{\infty} P\left(\max_{1\le i\le n}|X_{ni}| > b_n\right) \le \sum_{n=1}^{\infty}\sum_{i=1}^{n} P\left(|X_{ni}| > b_n\right) \le$$

$$\le \sum_{n=1}^{\infty}\sum_{i=1}^{n} P\left(|X| > b_n\right) \le \sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{E\psi_i(X)}{\psi_i(b_n)} < \infty.$$

Theorem is proved.

Theorem 4. *Let* $\{X_{ni}, 1 \le i \le n,\ n \ge 1\}$ *be an array of rowwise NOD random variables with* $E(X_{ni}) = 0$, $1 \le i \le n,\ n \ge 1$, *which is stochastically dominated by a random variable X;* $\{a_{ni}, 1 \le i \le n,\ n \ge 1\}$ *be an array of positive real numbers;* $\{b_n,\ n \ge 1\}$ *be a sequence of positive real numbers,* $b_n \uparrow \infty$. *Let* $\psi(t)$ *be a non-negative even function satisfying the following conditions:*

$$\frac{\psi(|t|)}{|t|}\uparrow \ \text{and}\ \frac{\psi(|t|)}{|t|^p}\downarrow \ \text{as}\ |t|\uparrow$$

for some real number $p \in (1, 2]$. *Put* $A(n, t) = \sum_{i=1}^{n} a_{ni}^t,\ n \ge 1,\ t \ge 0$. *If*

$$\sum_{n=1}^{\infty} A(n, t)\frac{E\psi(X)}{\psi(b_n)} < \infty,\ t \in \{0, 1\},\ \text{and}\ \sum_{n=1}^{\infty} A^p(n, 2)\log^{2p}(n)\frac{E\psi(X)}{\psi(b_n)} < \infty,$$

then

$$\frac{1}{b_n}\max_{1 \le j \le n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| \to 0$$

completely.

P r o o f. We continue to use the notations $X_i^{(n)}$ and $T_i^{(n)}$ introduced in the proof of theorem 3. For any $\varepsilon > 0$

$$P\left(\max_{1 \le j \le n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > b_n\varepsilon\right) \le$$

$$\le P\left(\max_{1 \le j \le n}\left|\sum_{i=1}^{j} a_{ni}X_i^{(n)}\right| > b_n\varepsilon\right) + P\left(\max_{1 \le i \le n}|X_{ni}| > b_n\right) \le$$

$$\le P\left(\max_{1 \le j \le n}\left|T_j^{(n)}\right| > b_n\varepsilon - \max_{1 \le j \le n}\left|\sum_{i=1}^{j} a_{ni}EX_i^{(n)}\right|\right) + P\left(\max_{1 \le i \le n}|X_{ni}| > b_n\right).$$

Since $\frac{\psi(t)}{|t|}\uparrow$, it follows that

$$b_n^{-1}\max_{1 \le j \le n}\left|\sum_{i=1}^{j} a_{ni}EX_i^{(n)}\right| \le$$

$$\le b_n^{-1}\max_{1 \le j \le n}\left|\sum_{i=1}^{j} a_{ni}E\left(X_{ni}I\left(|X_{ni}| > b_n\right)\right)\right| + \max_{1 \le j \le n}\sum_{i=1}^{j} a_{ni}P\left(|X_{ni}| > b_n\right) \le$$

$$\le b_n^{-1}\sum_{i=1}^{n} a_{ni}E\left(|X|I\left(|X| > b_n\right)\right) + \sum_{i=1}^{n} a_{ni}P\left(|X| > b_n\right) \le$$

$$\le A(n, 1)E\left(\frac{|X|I\left(|X| > b_n\right)}{b_n}\right) + A(n, 1)\frac{E\psi(X)}{\psi(b_n)} \le$$

$$\le 2A(n, 1)\frac{E\psi(X)}{\psi(b_n)} \to 0\ \text{ as } n \to \infty.$$

Therefore, for n large enough

$$P\left(\max_{1 \le j \le n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > b_n\varepsilon\right) \le P\left(\max_{1 \le j \le n}\left|T_j^{(n)}\right| > \frac{b_n\varepsilon}{2}\right) + P\left(\max_{1 \le i \le n}|X_{ni}| > b_n\right).$$

It is sufficient to show that

$$J_1 = \sum_{n=1}^{\infty} P\left(\max_{1 \le j \le n}\left|T_j^{(n)}\right| > \frac{b_n\varepsilon}{2}\right) < \infty$$

and

$$J_2 = \sum_{n=1}^{\infty} P\left(\max_{1 \le i \le n} |X_{ni}| > b_n\right) < \infty.$$

On the one hand, by Markov inequality, lemma 3 and C_r-inequality, we have

$$J_1 \le C \sum_{n=1}^{\infty} b_n^{-2p} E\left(\max_{1 \le j \le n} \left|T_j^{(n)}\right|^{2p}\right) \le$$

$$\le C \sum_{n=1}^{\infty} b_n^{-2p} \log^{2p}(n) \left[\sum_{i=1}^{n} a_{ni}^{2p} E\left|X_i^{(n)} - EX_i^{(n)}\right|^{2p} + \left(\sum_{i=1}^{n} a_{ni}^{2} E\left|X_i^{(n)} - EX_i^{(n)}\right|^{2}\right)^{p}\right] \le$$

$$\le C \sum_{n=1}^{\infty} b_n^{-2p} \log^{2p}(n) \left[\sum_{i=1}^{n} a_{ni}^{2p} E\left|X_i^{(n)}\right|^{2p} + \left(\sum_{i=1}^{n} a_{ni}^{2} E\left|X_i^{(n)}\right|^{2}\right)^{p}\right] \le$$

$$\le C \sum_{n=1}^{\infty} b_n^{-2p} \log^{2p}(n) \left[\sum_{i=1}^{n} a_{ni}^{2p} E\left(|X_{ni}|^{2p} I(|X_{ni}| \le b_n)\right) + \left(\sum_{i=1}^{n} a_{ni}^{2} E\left(|X_{ni}|^{2} I(|X_{ni}| \le b_n)\right)\right)^{p}\right].$$

On the other hand, by lemma 5 and the facts that $\frac{\psi(t)}{|t|} \uparrow$ and $\frac{\psi(t)}{|t|^p} \downarrow$ as $|t| \uparrow$, we obtain

$$b_n^{-2p} \sum_{i=1}^{n} a_{ni}^{2p} E\left(|X_{ni}|^{2p} I(|X_{ni}| \le b_n)\right) \le b_n^{-p} \sum_{i=1}^{n} a_{ni}^{2p} E\left(|X_{ni}|^{p} I(|X_{ni}| \le b_n)\right) \le$$

$$\le b_n^{-p} \sum_{i=1}^{n} a_{ni}^{2p} \left[E\left(|X|^{p} I(|X| \le b_n)\right) + b_n^{p} P(|X| > b_n)\right] \le$$

$$\le A(n, 2p)\left[\frac{E\left(\psi(X) I(|X| \le b_n)\right)}{\psi(b_n)} + \frac{E\psi(X)}{\psi(b_n)}\right] \le CA(n, 2p)\frac{E\psi(X)}{\psi(b_n)}$$

and

$$b_n^{-2} \sum_{i=1}^{n} a_{ni}^{2} E\left(|X_{ni}|^{2} I(|X_{ni}| \le b_n)\right) \le$$

$$\le b_n^{-2} A(n, 2)\left[E\left(|X|^{2} I(|X| \le b_n)\right) + b_n^{2} P(|X| > b_n)\right] \le$$

$$\le A(n, 2)\left[E\left(\frac{|X|^{2} I(|X| \le b_n)}{b_n^2}\right) + P\left(\psi^{\frac{1}{p}}(X) > \psi^{\frac{1}{p}}(b_n)\right)\right] \le$$

$$\le A(n, 2)\left[E\left(\frac{|X| I(|X| \le b_n)}{b_n}\right) + \frac{E\psi^{\frac{1}{p}}(X)}{\psi^{\frac{1}{p}}(b_n)}\right] \le$$

$$\le A(n, 2)\left[\frac{E\left(\psi^{\frac{1}{p}}(X) I(|X| \le b_n)\right)}{\psi^{\frac{1}{p}}(b_n)} + \frac{E\psi^{\frac{1}{p}}(X)}{\psi^{\frac{1}{p}}(b_n)}\right] \le CA(n, 2)\frac{E\psi^{\frac{1}{p}}(X)}{\psi^{\frac{1}{p}}(b_n)}.$$

Therefore,

$$J_1 \le C \sum_{n=1}^{\infty} \log^{2p}(n)\left[A(n, 2p) + A^p(n, 2)\right]\frac{E\psi(X)}{\psi(b_n)} \le$$

$$\le C \sum_{n=1}^{\infty} A^p(n, 2)\log^{2p}(n)\frac{E\psi(X)}{\psi(b_n)} < \infty.$$

For J_2 we have

$$J_2 \le \sum_{n=1}^{\infty}\sum_{i=1}^{n} P(|X_{ni}| > b_n) \le \sum_{n=1}^{\infty} nP(|X| > b_n) \le \sum_{n=1}^{\infty} n\frac{E\psi(X)}{\psi(b_n)} < \infty.$$

Theorem is proved.

Without the assumption of stochastic domination of $\{X_{ni}, 1 \le i \le n,\ n \ge 1\}$ the proof of theorem 3 immediately yields the following result.

Theorem 5. *Let* $\{X_{ni}, 1 \le i \le n,\ n \ge 1\}$ *be an array of rowwise NOD random variables with* $E(X_{ni}) = 0, 1 \le i \le n,\ n \ge 1$; $\{a_{ni}, 1 \le i \le n,\ n \ge 1\}$ *be an array of positive real numbers;* $\{b_n,\ n \ge 1\}$ *be a sequence of positive real numbers,* $b_n \uparrow \infty$. *Let* $\{\psi_n(t),\ n \ge 1\}$ *be a sequence of non-negative even functions satisfying the following conditions:*

$$\frac{\psi_n(|t|)}{|t|}\uparrow \text{ and } \frac{\psi_n(|t|)}{|t|^p}\downarrow \text{ as } |t|\uparrow$$

for some real number $p \in (1, 2]$. *We assume that*

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\psi_i(X_{ni})}{\psi_i(b_n)} < \infty \text{ and } \sum_{n=1}^{\infty}\sum_{i=1}^{n} a_{ni}^2\frac{E\psi_i(X_{ni})}{\psi_i(b_n)} < \infty.$$

Then

$$\frac{1}{b_n}\sum_{i=1}^{n} a_{ni}X_{ni} \to 0$$

completely.

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