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СЛАБОЕ РЕШЕНИЕ СМЕШАННОЙ ЗАДАЧИ ДЛЯ ПОЛУЛИНЕЙНОГО ГИПЕРБОЛИЧЕСКОГО УРАВНЕНИЯ ТРЕТЬЕГО ПОРЯДКА С ВОЛНОВЫМ ОПЕРАТОРОМ

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Аннотация. Для полулинейного гиперболического уравнения третьего порядка, заданного в первом квадранте, изучается слабое решение смешанной задачи, в которой начальные условия заданы на пространственной полупрямой, а смешанные условия – на временной полупрямой. Оператор в уравнении представляет собой композицию из волнового оператора и оператора переноса. Слабое решение определяется как решение системы связанных интегральных уравнений, которым удовлетворяет классическое решение. Показывается, что при некоторых условиях гладкости начальных и граничных данных рассматриваемая задача допускает существование и единственность слабого решения. Устанавливается, что дважды непрерывно дифференцируемое слабое решение является пределом классических решений изучаемой задачи.

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Ключевые слова: нелинейное гиперболическое уравнение третьего порядка; смешанная задача; обобщенное решение; слабое решение.

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MILD SOLUTION OF A MIXED PROBLEM FOR A THIRD-ORDER SEMILINEAR HYPERBOLIC EQUATION WITH THE WAVE OPERATOR

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Abstract. For a semilinear hyperbolic equation of the third-order given in the first quadrant we study a mild solution of a mixed problem in which the initial conditions are specified on the spatial half-line and the mixed conditions are specified on the time half-line. The operator in the equation is a composition of the wave operator and the transport operator. Mild solution is defined as a solution to coupled integral equations that are satisfied by a classical solution. It is shown that under some smoothness conditions on the initial and boundary data the problem under consideration admits the existence and uniqueness of the mild solution. It is established that the twice continuously differentiable mild solution is the limit of classical solutions to the problem under study.

Keywords: non-linear hyperbolic equation of the third-order; mixed problem; generalised solution; mild solution.

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Introduction and statement of the problem

Third-order differential equations arise in the description of certain physical phenomena. For example, we use such hyperbolic equations of the indicated order in modelling the propagation of linear acoustic waves in a medium with dispersion [1]. Mathematical models of such problems were considered in papers [2–4].

In the domain $Q = (0, \infty) \times (0, \infty)$ of two independent variables (t, x) we consider one dimensional nonlinear equation of the third-order

$$\left(\frac{\partial}{\partial t} + b\frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial t^2} - a^2\frac{\partial^2}{\partial x^2}\right)u(t, x) = f\left(t, x, u(t, x)\right),\ (t, x) \in Q, \tag{1}$$

where a and b are positive (to be definite) real numbers, $a \neq b$. On the lower part $t = 0$ of the boundary ∂Q we supplement equation (1) with the initial conditions

$$u(0, x) = \varphi_0(x),\ \frac{\partial u}{\partial t}(0, x) = \varphi_1(x),\ \frac{\partial^2 u}{\partial t^2}(0, x) = \varphi_2(x),\ x \in [0, \infty), \tag{2}$$

and on the lateral part of the boundary ∂Q we supplement it with the mixed conditions

$$u(t, 0) = \mu_0(t),\ \frac{\partial u}{\partial x}(t, 0) = \mu_1(t),\ t \in [0, \infty). \tag{3}$$

In the paper [5] a mild solution to the Cauchy problem for equation (1) is considered. The authors of the work [6] obtained a mild solution to a similar equation

$$\left(\tau\frac{\partial^3}{\partial t^3} + \alpha\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2} - b\frac{\partial^3}{\partial x^2 \partial t}\right)u(t, x) = f\left(u(t, x),\ \frac{\partial u}{\partial t}(t, x)\right)$$

in a Hilbert space $H^2(\Omega) \cap H_0^1(\Omega)$, where $\Omega \subset \mathbb{R}$. As a result of the work [7] it was constructed a generalised solution to the Jordan – Moore – Gibson – Thompson equation

$$\left(\tau\frac{\partial^3}{\partial t^3}+\frac{\partial^2}{\partial t^2}-\left(\delta+\tau c^2\right)\frac{\partial^3}{\partial x^2\partial t}-c^2\frac{\partial^2}{\partial x^2}\right)u(t,x)=\frac{1}{2}\frac{\partial}{\partial t}\left(\kappa\left(\frac{\partial u}{\partial t}(t,x)\right)^2+\sigma\left(\frac{\partial u}{\partial x}(t,x)\right)^2\right).$$

In the article [8] a weak solution for an equation similar to equation (1) with power non-linearity is considered.

Mild solution

In our paper [9] it was proved the following statement.

Theorem 1. *Let the conditions* $\varphi_0\in C^3([0,\infty))$, $\varphi_1\in C^2([0,\infty))$, $\varphi_2\in C^1([0,\infty))$, $\mu_0\in C^3([0,\infty))$, $\mu_1\in C^2([0,\infty))$ *and* $f\in C^1(\overline{Q}\times\mathbb{R})$ *be satisfied, and let the function f satisfy the Lipschitz condition with a function* $L\in C(\overline{Q})$ *with respect to the third variable, i. e. for any pair* $(t,x)\in\overline{Q}$ *and any real numbers* z_1 *and* z_2 *the inequality* $\left|f(t,x,z_1)-f(t,x,z_2)\right|\le L(t,x)|z_1-z_2|$ *holds. Problem (1) – (3) has a unique solution* $u:\overline{Q}\mapsto\mathbb{R}$ *in the class* $C^3(\overline{Q})$ *if and only if the matching conditions*

$$\mu_0(0)=\varphi_0(0), \tag{4}$$

$$\mu_0'(0)=\varphi_1(0),\ \mu_1(0)=\varphi_0'(0), \tag{5}$$

$$\mu_0''(0)=\varphi_2(0),\ \mu_1'(0)=\varphi_1'(0), \tag{6}$$

$$\mu_0'''(0)=f(0,0,\varphi_0(0))-b\varphi_2'(0)+a^2\varphi_1''(0)+a^2b\varphi_0'''(0),\ \mu_1''(0)=\varphi_2'(0) \tag{7}$$

are satisfied. This solution is determined by the formulas

$$u(t,x)=u^{(j)}(t,x),\ (t,x)\in\overline{Q^{(j)}},\ j=1,2,3, \tag{8}$$

$$\begin{aligned}u^{(1)}(t,x)=v(t,x)+\int_0^t\Bigg\{&\int_0^{a(t-\tau)+x}\frac{f\left(\tau,z,u^{(1)}(\tau,z)\right)\left(a(t-\tau)+x-z\right)}{2a(a+b)}dz+\\&+\int_0^{x-b(t-\tau)}\frac{f\left(\tau,z,u^{(1)}(\tau,z)\right)\left(b(t-\tau)-x+z\right)}{(a-b)(a+b)}dz+\\&+\int_0^{x-a(t-\tau)}\frac{f\left(\tau,z,u^{(1)}(\tau,z)\right)\left(a(\tau-t)+x-z\right)}{2a(a-b)}dz\Bigg\}d\tau,\ (t,x)\in\overline{Q^{(1)}},\end{aligned} \tag{9}$$

$$\begin{aligned}u^{(2)}(t,x)=v(t,x)+\int_{t-\frac{x}{b}}^{t}\Bigg\{&\int_0^{x+a(t-\tau)}\frac{\left(x-z+a(t-\tau)\right)f\left(\tau,z,u^{(1)}(\tau,z)\right)}{2a(a+b)}dz+\\+\int_0^{x-b(t-\tau)}\frac{\left(z-x+b(t-\tau)\right)f\left(\tau,z,u^{(1)}(\tau,z)\right)}{(a-b)(a+b)}dz&+\int_0^{x-a(t-\tau)}\frac{\left(x-z+a(\tau-t)\right)f\left(\tau,z,u^{(1)}(\tau,z)\right)}{2a(a-b)}dz\Bigg\}d\tau+\\+\int_0^{t-\frac{x}{b}}\Bigg\{\int_0^{x+a(t-\tau)}\frac{\left(x-z+a(t-\tau)\right)f\left(\tau,z,u^{(2)}(\tau,z)\right)}{2a(a+b)}dz&+\int_0^{x-a(t-\tau)}\frac{\left(x-z+a(\tau-t)\right)f\left(\tau,z,u^{(2)}(\tau,z)\right)}{2a(a-b)}dz-\\-\int_0^{a\left(t-\tau-\frac{x}{b}\right)}&\frac{\left(bz+a\left(x+b(\tau-t)\right)\right)f\left(\tau,z,u^{(2)}(\tau,z)\right)}{a(a-b)(a+b)}dz\Bigg\}d\tau,\ a<b,\ (t,x)\in\overline{Q^{(2)}},\end{aligned} \tag{10}$$

$$u^{(3)}(t,x) = v(t,x) + \int\limits_{t-\frac{x}{b}}^{t} \left\{ \int\limits_{0}^{x+a(t-\tau)} \frac{\left(x - z + a(t-\tau)\right) f\left(\tau, z, u^{(1)}(\tau, z)\right)}{2a(a+b)} dz + \right.$$

$$\left. + \int\limits_{0}^{x-b(t-\tau)} \frac{\left(z - x + b(t-\tau)\right) f\left(\tau, z, u^{(1)}(\tau, z)\right)}{(a-b)(a+b)} dz + \int\limits_{0}^{x-a(t-\tau)} \frac{\left(x - z + a(\tau-t)\right) f\left(\tau, z, u^{(1)}(\tau, z)\right)}{2a(a-b)} dz \right\} d\tau +$$

$$+ \int\limits_{t-\frac{x}{a}}^{t-\frac{x}{b}} \left\{ \int\limits_{0}^{x+a(t-\tau)} \frac{\left(x - z + a(t-\tau)\right) f\left(\tau, z, u^{(2)}(\tau, z)\right)}{2a(a+b)} dz + \int\limits_{0}^{x-a(t-\tau)} \frac{\left(x - z + a(\tau-t)\right) f\left(\tau, z, u^{(2)}(\tau, z)\right)}{2a(a-b)} dz - \right.$$

$$\left. - \int\limits_{0}^{a\left(t-\frac{x}{b}-\tau\right)} \frac{\left(bz + a\left(x + b(\tau-t)\right)\right) f\left(\tau, z, u^{(2)}(\tau, z)\right)}{a(a-b)(a+b)} dz \right\} d\tau +$$

$$+ \int\limits_{0}^{t-\frac{x}{a}} \left\{ \int\limits_{0}^{x+a(t-\tau)} \frac{\left(x - z + a(t-\tau)\right) f\left(\tau, z, u^{(3)}(\tau, z)\right)}{2a(a+b)} dz + \int\limits_{0}^{a(t-\tau)-x} \frac{\left(x + z + a(\tau-t)\right) f\left(\tau, z, u^{(3)}(\tau, z)\right)}{2a(a-b)} dz - \right.$$

$$\left. - \int\limits_{0}^{a\left(t-\frac{x}{b}-\tau\right)} \frac{\left(bz + a\left(x + b(\tau-t)\right)\right) f\left(\tau, z, u^{(3)}(\tau, z)\right)}{a(a-b)(a+b)} dz \right\} d\tau,\ a < b,\ (t,x) \in \overline{Q^{(3)}}, \qquad (11)$$

$$u^{(2)}(t,x) = v(t,x) + \int\limits_{0}^{t-\frac{x}{a}} \left\{ \int\limits_{0}^{x+a(t-\tau)} \frac{\left(x - z + a(t-\tau)\right) f\left(\tau, z, u^{(2)}(\tau, z)\right)}{2a(a+b)} dz + \right.$$

$$\left. + \int\limits_{0}^{x-b(t-\tau)} \frac{\left(-x + z + b(t-\tau)\right) f\left(\tau, z, u^{(2)}(\tau, z)\right)}{a^2 - b^2} dz + \int\limits_{0}^{a(t-\tau)-x} \frac{\left(x + z + a(\tau-t)\right) f\left(\tau, z, u^{(2)}(\tau, z)\right)}{2a(a-b)} dz \right\} d\tau +$$

$$+ \int\limits_{t-\frac{x}{a}}^{t} \left\{ \int\limits_{0}^{x+a(t-\tau)} \frac{\left(x - z + a(t-\tau)\right) f\left(\tau, z, u^{(1)}(\tau, z)\right)}{2a(a+b)} dz + \int\limits_{0}^{x-b(t-\tau)} \frac{\left(-x + z + b(t-\tau)\right) f\left(\tau, z, u^{(1)}(\tau, z)\right)}{(a-b)(a+b)} dz + \right.$$

$$\left. + \int\limits_{0}^{x-a(t-\tau)} \frac{\left(x - z + a(\tau-t)\right) f\left(\tau, z, u^{(1)}(\tau, z)\right)}{2a(a-b)} dz \right\} d\tau,\ b < a,\ (t,x) \in \overline{Q^{(2)}}, \qquad (12)$$

$$u^{(3)}(t,x) = v(t,x) + \int\limits_{t-\frac{x}{b}}^{t-\frac{x}{a}} \left\{ \int\limits_{0}^{x+a(t-\tau)} \frac{\left(x - z + a(t-\tau)\right) f\left(\tau, z, u^{(2)}(\tau, z)\right)}{2a(a+b)} dz + \right.$$

$$\left. + \int\limits_{0}^{x-b(t-\tau)} \frac{\left(-x + z + b(t-\tau)\right) f\left(\tau, z, u^{(2)}(\tau, z)\right)}{a^2 - b^2} dz + \int\limits_{0}^{a(t-\tau)-x} \frac{\left(x + z + a(\tau-t)\right) f\left(\tau, z, u^{(2)}(\tau, z)\right)}{2a(a-b)} dz \right\} d\tau +$$

$$+\int\limits_{t-\frac{x}{a}}^{t}\left\{\int\limits_{0}^{x+a(t-\tau)}\frac{\left(x-z+a(t-\tau)\right)f\left(\tau,z,u^{(1)}(\tau,z)\right)}{2a(a+b)}dz+\int\limits_{0}^{x-b(t-\tau)}\frac{\left(-x+z+b(t-\tau)\right)f\left(\tau,z,u^{(1)}(\tau,z)\right)}{(a-b)(a+b)}dz+\right.$$

$$\left.+\int\limits_{0}^{x-a(t-\tau)}\frac{\left(x-z+a(\tau-t)\right)f\left(\tau,z,u^{(1)}(\tau,z)\right)}{2a(a-b)}dz\right\}d\tau+$$

$$+\int\limits_{0}^{t-\frac{x}{b}}\left\{\int\limits_{0}^{x+a(t-\tau)}\frac{\left(x-z+a(t-\tau)\right)f\left(\tau,z,u^{(3)}(\tau,z)\right)}{2a(a+b)}dz+\int\limits_{0}^{a(t-\tau)-x}\frac{\left(x+z+a(\tau-t)\right)f\left(\tau,z,u^{(3)}(\tau,z)\right)}{2a(a-b)}dz-\right.$$

$$\left.-\int\limits_{0}^{a\left(t-\frac{x}{b}-\tau\right)}\frac{\left(bz+a\left(x+b(\tau-t)\right)\right)f\left(\tau,z,u^{(3)}(\tau,z)\right)}{a(a-b)(a+b)}dz\right\}d\tau,\ b<a,\ (t,x)\in\overline{Q^{(3)}},\tag{13}$$

where v is a function defined by the formula

$$v(t,x)=\sum_{k=1}^{2}\frac{(-1)^k\left(\left(a+(-1)^kb\right)\Im[\varphi_1]+\Im^2[\varphi_2]+ab\varphi_0\right)\left(x+(-1)^kat\right)}{2a\left(a+(-1)^kb\right)}+\frac{\left(a^2\varphi_0-\Im^2[\varphi_2]\right)(x-bt)}{a^2-b^2},\ (t,x)\in\overline{Q^{(1)}},$$

$$v(t,x)=\sum_{k=1}^{2}\frac{(-1)^k\left(\left(a+(-1)^kb\right)\Im[\varphi_1]+\Im^2[\varphi_2]+ab\varphi_0\right)\left(x+(-1)^kat\right)}{2a\left(a+(-1)^kb\right)}+\frac{1}{a-b}\left(\frac{b}{a}\Im[\varphi_1]\left(at-\frac{ax}{b}\right)-\right.$$

$$\left.-ab\Im[\mu_1]\left(t-\frac{x}{b}\right)+\frac{b\Im^2[\varphi_2]\left(at-\frac{ax}{b}\right)}{a(a+b)}+\frac{b^2\varphi_0\left(at-\frac{ax}{b}\right)}{a+b}+a\varphi(0)-b\mu_0\left(t-\frac{x}{b}\right)\right),\ a<b,\ (t,x)\in\overline{Q^{(2)}},$$

$$v(t,x)=\frac{\left((a+b)\Im[\varphi_1]+\Im^2[\varphi_2]+ab\varphi_0\right)(x+at)}{2a(a+b)}+\frac{\left(a^2\varphi_0-\Im^2[\varphi_2]\right)(x-bt)}{a^2-b^2}-\frac{1}{2a(a-b)}\times$$

$$\times\left(\left((a+b)\Im[\varphi_1]+\Im^2[\varphi_2]+ab\varphi_0\right)(at-x)-2a^2b\Im[\mu_1]\left(t-\frac{x}{a}\right)+\right.$$

$$\left.+2a^2\varphi_0(0)-2a^2\mu_0\left(t-\frac{x}{a}\right)\right),\ b<a,\ (t,x)\in\overline{Q^{(2)}},$$

$$v(t,x)=\frac{1}{a-b}\left(\frac{b}{a}\Im[\varphi_1]\left(at-\frac{ax}{b}\right)-ab\Im[\mu_1]\left(t-\frac{x}{b}\right)+\frac{b\Im^2[\varphi_2]\left(at-\frac{ax}{b}\right)}{a(a+b)}+\frac{b^2\varphi_0\left(at-\frac{ax}{b}\right)}{a+b}-\right.$$

$$\left.-b\mu_0\left(t-\frac{x}{b}\right)\right)+\frac{\left((a+b)\Im[\varphi_1]+\Im^2[\varphi_2]+ab\varphi_0\right)(x+at)}{2a(a+b)}-$$

$$-\frac{1}{2a(a-b)}\left(\left((a+b)\Im[\varphi_1]+\Im^2[\varphi_2]+ab\varphi_0\right)(at-x)-2a^2b\Im[\mu_1]\left(t-\frac{x}{a}\right)-\right.$$

$$\left.-2a^2\mu_0\left(t-\frac{x}{a}\right)\right),\ (t,x)\in\overline{Q^{(3)}},$$

where $\Im$ *is the integration operator acting by the formula* $\Im[h](x)=\int\limits_0^x h(z)dz$, *and*

$$Q^{(1)}=\{(t,x)|x-at>0\wedge x-bt>0\},$$

$$Q^{(2)}=Q\setminus\overline{Q^{(1)}\cup Q^{(3)}},$$

$$Q^{(3)}=\{(t,x)|x-at<0\wedge x-bt<0\}.$$

Representation of the function u in the form of equations (8)–(12) is well defined even if the functions $u^{(j)}(t,x)$, $j=1,2,3$, are not differentiable, prompting a generalisation of the classical solution using a weak type of solution that has the same form as the classical solution and will be one, when certain smoothness conditions and comparability conditions are fulfilled. This is roughly the same motivation that encourages looking for solutions in various Sobolev spaces [10; 11], in the form of power series (often formal or convergent in the sense of Cesàro) [12], or in the sense of generalised functions [13; 14]. Moreover, in many cases of linear abstract Cauchy problems, the concepts of mild and weak solutions (in the dual sense, in the sense of distributions) are equivalent[1].

Based on sources [15; 16], we introduce the following definition.

Definition. We call the function u representable in the form of equations (8)–(13) a mild solution of problem (1)–(3).

Definition introduces the notion of mild solution [16]. However, in the literature such solutions are also called solutions in the broad sense [17; 18], weak solutions [19] and generalised solutions [20].

The results of the work [9] yield the following statements.

Remark 1. Any classical solution of problem (1)–(3) is also a mild solution of this problem.

Remark 2. If the additional smoothness conditions $\varphi_0\in C^3([0,\infty))$, $\varphi_1\in C^2([0,\infty))$, $\varphi_2\in C^1([0,\infty))$, $\mu_0\in C^3([0,\infty))$, $\mu_1\in C^2([0,\infty))$ and $f\in C^1(\overline{Q}\times\mathbb{R})$, and the matching conditions (4)–(7) are satisfied, then the mild solution of problem (1)–(3) is classical.

Denote $\tilde{Q}=\overline{Q}\setminus\{(t,x)|x-at=0\vee x-bt=0\}$. The following statement holds.

Theorem 2. *Let the conditions* $\varphi_0\in C([0,\infty))$, $\varphi_1\in L^1_{\text{loc}}([0,\infty))$, $\varphi_2\in W^{-1,1}_{\text{loc}}([0,\infty))$, $\mu_0\in C([0,\infty))$, $\mu_1\in L^1_{\text{loc}}([0,\infty))$ *and* $f\in C(\overline{Q}\times\mathbb{R})$ *be satisfied, and let the function f satisfy the Lipschitz condition with a function* $L\in C(\overline{Q})$ *with respect to the third variable, i. e. for any pair* $(t,x)\in\overline{Q}$ *and any real numbers* z_1 *and* z_2 *the inequality* $|f(t,x,z_1)-f(t,x,z_2)|\le L(t,x)|z_1-z_2|$ *holds. Problem (1)–(3) has a unique mild solution* $u:\overline{Q}\mapsto\mathbb{R}$ *in the class* $C(\tilde{Q})$.

Proof. The solvability of equations (9)–(13) in the class of continuous functions follows from the article [9].

For a mild solution, smoothness can be increased if the matching conditions (4)–(7) are partially satisfied, as it is done in the following theorem.

Theorem 3. *Let the conditions* $\varphi_0\in C([0,\infty))$, $\varphi_1\in L^1_{\text{loc}}([0,\infty))$, $\varphi_2\in W^{-1,1}_{\text{loc}}([0,\infty))$, $\mu_0\in C([0,\infty))$, $\mu_1\in L^1_{\text{loc}}([0,\infty))$ *and* $f\in C(\overline{Q}\times\mathbb{R})$ *be satisfied, and let the function f satisfy the Lipschitz condition with a function* $L\in C(\overline{Q})$ *with respect to the third variable, i. e. for any pair* $(t,x)\in\overline{Q}$ *and any real numbers* z_1 *and* z_2 *the inequality* $|f(t,x,z_1)-f(t,x,z_2)|\le L(t,x)|z_1-z_2|$ *holds. Problem (1)–(3) has a unique mild solution* $u:\overline{Q}\mapsto\mathbb{R}$ *in the class* $C(\overline{Q})$ *if and only if the condition (4) is satisfied.*

Proof. If $\mu_0(0)=\varphi_0(0)$, then a mild solution u of problem (1)–(3) is continuous on the set $\{(t,x)|x-at=0\vee x-bt=0\}$ due to the relations

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[1] What «mild solution» means, and how to find it? // Mathoverflow : website. URL: https://mathoverflow.net/questions/320300/what-mild-solution-means-and-how-to-find-it (date of access: 21.07.2025) ; Is a mild solution the same thing as a weak solution? // Mathematics : website. URL: https://math.stackexchange.com/questions/708407/is-a-mild-solution-the-same-thing-as-a-weak-solution (date of access: 21.07.2025).

$$\left[(u)^{+}-(u)^{-}\right](t, x=bt)=\frac{b\left(\varphi_0(0)-\mu_0(0)\right)}{a-b},\ \left[(u)^{+}-(u)^{-}\right](t, x=bt)=$$

$$=\frac{a\left(\varphi_0(0)-\mu_0(0)\right)}{a-b},\ a<b, \tag{14}$$

$$\left[(u)^{+}-(u)^{-}\right](t, x=bt)=\frac{b\left(\mu_0(0)-\varphi_0(0)\right)}{a-b},\ \left[(u)^{+}-(u)^{-}\right](t, x=bt)=$$

$$=\frac{a\left(\mu_0(0)-\varphi_0(0)\right)}{a-b},\ b<a, \tag{15}$$

which follows from the equations (8)–(13). Here by $(u)^{\pm}$ we have denoted the limit values of the function u calculated on different sides of the curve $x=r(t)$, where r is a real-valued function, i. e. $(u)^{\pm}(t, x=r(t))= =\lim\limits_{\delta\to 0+} u(t, r(t)\pm\delta)$. Thus, the solution not only satisfies $u\in C(\tilde{Q})$ but is also a continuous function on the closure $\bar{Q}$, $u\in C(\bar{Q})$.

However, if $\mu_0(0)\neq\varphi_0(0)$, then a mild solution u of problem (1)–(3) cannot be continuous on the set $\bar{Q}$ by the relations (14) and (15). The proof of the theorem is complete.

Following the work [21] and assuming that $\varphi_0\in C^2([0,\infty))$, $\varphi_1\in C^1([0,\infty))$, $\varphi_2\in C([0,\infty))$, $\mu_0\in C^2([0,\infty))$, $\mu_1\in C^1([0,\infty))$, $f\in C(\bar{Q}\times\mathbb{R})$, we prove that a mild solution u of problem (1)–(3) from the class $C^2(\bar{Q})$ is a limit of the classical solutions of problem (1)–(3).

Since the spaces $C^6(\bar{Q})$ and $C^1(\bar{Q}\times\mathbb{R})$ are dense in $C^2(\bar{Q})$ and $C(\bar{Q}\times\mathbb{R})$, respectively [22], there are sequences $w^{(n)}\in C^6(\bar{Q})$ and $f^{(n)}\in C^1(\bar{Q}\times\mathbb{R})$ such that $\lim\limits_{n\to\infty}\rho\left(w^{(n)}, u\right)_{C^2(\bar{Q})}=0$ and $\lim\limits_{n\to\infty}\rho\left(f^{(n)}, f\right)_{C(\bar{Q}\times\mathbb{R})}=0$, where $\rho(x_1, x_2)_X$ is the distance between elements $x_1\in X$ and $x_2\in X$ in a metric space X. Additionally, we want to choose the sequence $w^{(n)}$ so that the condition

$$\frac{\partial^3 w^{(n)}}{\partial t^3}(t,x)=f^{(n)}\left(t,x,w^{(n)}(t,x)\right)-b\frac{\partial^3 w^{(n)}}{\partial t^2\partial x}(t,x)+a^2\frac{\partial^3 w^{(n)}}{\partial t\partial x^2}(t,x)+a^2 b\frac{\partial^3 w^{(n)}}{\partial x^3}(t,x)$$

is also satisfied.

Let us consider on the closed ray $[0,\infty)$ a continuous function $h_\delta(x)$ with the parameter $\delta\in(0,\infty)$:

$$h_\delta(x)=\begin{cases}\dfrac{x}{\delta},\ x<\delta,\\ 1,\ x\geq\delta.\end{cases}$$

There are obvious equalities

$$w^{(n)}(t,x)=w^{(n)}(0,x)+t\frac{\partial w^{(n)}}{\partial t}(0,x)+\frac{t^2}{2}\frac{\partial^2 w^{(n)}}{\partial t^2}(0,x)+\frac{1}{2}\int_0^t (t-\tau)^2\frac{\partial^3 w^{(n)}}{\partial t^3}(\tau,x)d\tau,$$

$$w^{(n)}(t,x)=w^{(n)}(t,0)+x\frac{\partial w^{(n)}}{\partial x}(t,0)+\frac{x^2}{2}\frac{\partial^2 w^{(n)}}{\partial x^2}(t,0)+\frac{1}{2}\int_0^x (x-\xi)^2\frac{\partial^3 w^{(n)}}{\partial x^3}(t,\xi)d\xi,$$

which imply

$$w^{(n)}(t,x)=w^{(n)}(0,0)+x\frac{\partial w^{(n)}}{\partial x}(0,0)+\frac{x^2}{2}\frac{\partial^2 w^{(n)}}{\partial x^2}(0,0)+\frac{1}{2}\int_0^x (x-\xi)^2\frac{\partial^3 w^{(n)}}{\partial x^3}(0,\xi)d\xi+$$

$$+t\frac{\partial w^{(n)}}{\partial t}(0,x)+\frac{t^2}{2}\frac{\partial^2 w^{(n)}}{\partial t^2}(0,x)+\frac{1}{2}\int_0^t (t-\tau)^2\frac{\partial^3 w^{(n)}}{\partial t^3}(\tau,x)d\tau.$$

Let

$$z^{(n)}(t, x) = w^{(n)}(0, 0) + x\frac{\partial w^{(n)}}{\partial x}(0, 0) + \frac{x^2}{2}\frac{\partial^2 w^{(n)}}{\partial x^2}(0, 0) + t\frac{\partial w^{(n)}}{\partial t}(0, x) + \frac{t^2}{2}\frac{\partial^2 w^{(n)}}{\partial t^2}(0, x) + \\ + \frac{1}{2}\int_0^x (x-\xi)^2\left[h_\delta(\xi)\frac{\partial^3 w^{(n)}}{\partial x^3}(0, \xi) + \alpha\left(1 - h_\delta(\xi)\right)\right]d\xi + \frac{1}{2}\int_0^t (t-\tau)^2\frac{\partial^3 w^{(n)}}{\partial t^3}(\tau, x)d\tau, \tag{16}$$

where

$$\alpha = \frac{1}{a^2 b}\left(4\frac{\partial^3 w^{(n)}}{\partial t^3}(0, 0) + b\frac{\partial^3 w^{(n)}}{\partial t^2\partial x}(0, 0) - a^2\frac{\partial^3 w^{(n)}}{\partial t\partial x^2}(0, 0) - f^{(n)}\left(0, 0, w^{(n)}(0, 0)\right)\right).$$

It is easy to see that the function $z^{(n)}(t, x)$ belongs to the class $C^3\left(\bar{Q}\right)$. Let us denote

$$\tilde{\varphi}_i^{(n)}(x) = \frac{\partial^i z^{(n)}}{\partial t^i}(0, x),\ i = 0, 1, 2,\ x \in [0, \infty), \tag{17}$$

$$\tilde{\mu}_i^{(n)}(t) = \frac{\partial^i z^{(n)}}{\partial x^i}(t, 0),\ i = 0, 1,\ t \in [0, \infty). \tag{18}$$

Similarly to theorem 1 from the article [9], we derive the following matching conditions by differentiating «conditions» (17) and (18) and taking into account the formula (16):

$$\begin{gathered}\tilde{\mu}_0^{(n)}(0) = \tilde{\varphi}_0^{(n)}(0),\ D\tilde{\mu}_0^{(n)}(0) = \tilde{\varphi}_1^{(n)}(0),\ \tilde{\mu}_1^{(n)}(0) = D\tilde{\varphi}_0^{(n)}(0), \\ D^2\tilde{\mu}_0^{(n)}(0) = \tilde{\varphi}_2^{(n)}(0),\ D\tilde{\mu}_1^{(n)}(0) = D\tilde{\varphi}_1^{(n)}(0),\ D^2\tilde{\mu}_1^{(n)}(0) = D\tilde{\varphi}_2^{(n)}(0), \\ D^3\tilde{\mu}_0^{(n)}(0) = f^{(n)}\left(0, 0, \tilde{\varphi}_0(0)\right) - bD\tilde{\varphi}_2^{(n)}(0) + a^2 D^2\tilde{\varphi}_1^{(n)}(0) + a^2 bD^3\tilde{\varphi}_0^{(n)}(0),\end{gathered} \tag{19}$$

where D is the Newton – Leibnitz operator.

We consider the difference

$$z^{(n)}(t, x) - w^{(n)}(t, x) = w^{(n)}(0, 0) - w^{(n)}(t, x) + x\frac{\partial w^{(n)}}{\partial x}(0, 0) +$$

$$+ \frac{x^2}{2}\frac{\partial^2 w^{(n)}}{\partial x^2}(0, 0) + t\frac{\partial w^{(n)}}{\partial t}(0, x) + \frac{t^2}{2}\frac{\partial^2 w^{(n)}}{\partial t^2}(0, x) +$$

$$+ \frac{1}{2}\int_0^x (x-\xi)^2\left[h_\delta(\xi)\frac{\partial^3 w^{(n)}}{\partial x^3}(0, \xi) + \alpha\left(1 - h_\delta(\xi)\right)\right]d\xi + \frac{1}{2}\int_0^t (t-\tau)^2\frac{\partial^3 w^{(n)}}{\partial t^3}(\tau, x)d\tau =$$

$$= \frac{1}{2}\int_0^x (x-\xi)^2\left[\left(h_\delta(\xi) - 1\right)\frac{\partial^3 w^{(n)}}{\partial x^3}(0, \xi) + \alpha\left(1 - h_\delta(\xi)\right)\right]d\xi =$$

$$= \frac{\alpha\delta\left(\delta^2 + 6x^2 - 4\delta x\right)}{24} + \frac{1}{2}\int_0^x (x-\xi)^2\left(h_\delta(\xi) - 1\right)\frac{\partial^3 w^{(n)}}{\partial x^3}(0, \xi)d\xi.$$

Therefore, the value $\left|z^{(n)}(t, x) - w^{(n)}(t, x)\right|$ can be estimated as

$$\left|z^{(n)}(t, x) - w^{(n)}(t, x)\right| \le \left|\frac{\delta\left(\delta^2 + 6x^2 - 4\delta x\right)\left(\alpha - \max\limits_{\xi \in [0, \delta]}\frac{\partial^3 w^{(n)}}{\partial x^3}(0, \xi)\right)}{24}\right|. \tag{20}$$

Let us introduce in the Fréchet space $C\left(\bar{Q}\right)$ a countable system of seminorms

$$\mathfrak{p}_m\left(g\right)=\max_{(t,\, x)\in[0,\, m]\times[0,\, m]}\left|g\left(t,\, x\right)\right|\exp\left(-x\right),\ m\in\mathbb{N}.$$

If δ is small enough, then estimate of the expression (20) yields the inequality

$$\mathfrak{p}_m\left(z^{(n)}-w^{(n)}\right)\le\frac{1}{n}. \tag{21}$$

Now let us consider the inequality

$$0\le\mathfrak{p}_m\left(z^{(n)}-u\right)\le\mathfrak{p}_m\left(z^{(n)}-w^{(n)}\right)+\mathfrak{p}_m\left(w^{(n)}-u\right).$$

From estimate of the inequality (21) it follows that $\mathfrak{p}_m\left(z^{(n)}-w^{(n)}\right)\to 0$ as $n\to\infty$. And since $w^{(n)}:\bar{Q}\mapsto\mathbb{R}$ converges to $u:\bar{Q}\mapsto\mathbb{R}$ in the space $C\left(\bar{Q}\right)$, which obviously implies $\lim\limits_{n\to\infty}\mathfrak{p}_m\left(w^{(n)}-u\right)$ for any $m\in\mathbb{N}$. Then by the squeeze theorem, we have that $\lim\limits_{n\to\infty}z^{(n)}=u.$ Thus, we constructed a sequence $z^{(n)}\in C^3\left(\bar{Q}\right)$, which converges to the function $u:\bar{Q}\mapsto\mathbb{R}$ in the space $C\left(\bar{Q}\right)$, and matching conditions (19) are satisfied.

In addition, we note that

$$\tilde{\varphi}_i=\frac{\partial^i u}{\partial t^i}\left(0,\,\cdot\right)=\lim_{n\to\infty}\frac{\partial^i z^{(n)}}{\partial t^i}\left(0,\,\cdot\right),\ i=0,\,1,\,2,$$

and

$$\tilde{\mu}_i=\frac{\partial^i u}{\partial x^i}\left(\cdot,\,0\right)=\lim_{n\to\infty}\frac{\partial^i z^{(n)}}{\partial x^i}\left(\cdot,\,0\right),\ i=0,\,1,$$

where the limits are understood in the spaces $C^{2-i}\left(\left[0,\,\infty\right)\right)$.

Let us introduce the function $v^{(n)}\left(t,\,x\right)$ and the operator $K_n:C\left(\bar{Q}\right)\mapsto C\left(\bar{Q}\right)$:

$$v^{(n)}\left(t,\,x\right)=\sum_{k=1}^{2}\frac{(-1)^k\left(\left(a+(-1)^k b\right)\Im\left[\tilde{\varphi}_1^{(n)}\right]+\Im^2\left[\tilde{\varphi}_2^{(n)}\right]+ab\tilde{\varphi}_0^{(n)}\right)\left(x+(-1)^k at\right)}{2a\left(a+(-1)^k b\right)}+$$

$$+\frac{\left(a^2\varphi_0^{(n)}-\Im^2\left[\tilde{\varphi}_2^{(n)}\right]\right)\left(x-bt\right)}{a^2-b^2},\ \left(t,\,x\right)\in\overline{Q^{(1)}},$$

$$v^{(n)}\left(t,\,x\right)=\sum_{k=1}^{2}\frac{(-1)^k\left(\left(a+(-1)^k b\right)\Im\left[\tilde{\varphi}_1^{(n)}\right]+\Im^2\left[\tilde{\varphi}_2^{(n)}\right]+ab\tilde{\varphi}_0^{(n)}\right)\left(x+(-1)^k at\right)}{2a\left(a+(-1)^k b\right)}+$$

$$+\frac{1}{a-b}\left(\frac{b}{a}\Im\left[\tilde{\varphi}_1^{(n)}\right]\left(at-\frac{ax}{b}\right)-ab\Im\left[\tilde{\mu}_1^{(n)}\right]\left(t-\frac{x}{b}\right)+\right.$$

$$\left.+\frac{b\Im^2\left[\tilde{\varphi}_2^{(n)}\right]\left(at-\dfrac{ax}{b}\right)}{a\left(a+b\right)}+\frac{b^2\tilde{\varphi}_0^{(n)}\left(at-\dfrac{ax}{b}\right)}{a+b}+a\tilde{\varphi}_0^{(n)}\left(0\right)-b\tilde{\mu}_0^{(n)}\left(t-\frac{x}{b}\right)\right),\ a<b,\ \left(t,\,x\right)\in\overline{Q^{(2)}},$$

$$v^{(n)}\left(t,\,x\right)=\frac{\left(\left(a+b\right)\Im\left[\tilde{\varphi}_1^{(n)}\right]+\Im^2\left[\tilde{\varphi}_2^{(n)}\right]+ab\tilde{\varphi}_0^{(n)}\right)\left(x+at\right)}{2a\left(a+b\right)}+\frac{\left(a^2\tilde{\varphi}_0^{(n)}-\Im^2\left[\tilde{\varphi}_2^{(n)}\right]\right)\left(x-bt\right)}{a^2-b^2}-\frac{1}{2a\left(a-b\right)}\times$$

$$\times\left(\left(\left(a+b\right)\Im\left[\varphi_1^{(n)}\right]+\Im^2\left[\tilde{\varphi}_2^{(n)}\right]+ab\tilde{\varphi}_0^{(n)}\right)\left(at-x\right)-2a^2b\Im\left[\tilde{\mu}_1^{(n)}\right]\left(t-\frac{x}{a}\right)+\right.$$

$$+\,2a^2\tilde{\varphi}_0^{(n)}(0)-2a^2\tilde{\mu}_0^{(n)}\left(t-\frac{x}{a}\right)\Bigg),\ b<a,\ (t,x)\in\overline{Q^{(2)}},$$

$$v^{(n)}(t,x)=\frac{1}{a-b}\Bigg(\frac{b}{a}\Im\left[\tilde{\varphi}_1^{(n)}\right]\left(at-\frac{ax}{b}\right)-ab\Im\left[\tilde{\mu}_1^{(n)}\right]\left(t-\frac{x}{b}\right)+\frac{b\Im^2\left[\tilde{\varphi}_2^{(n)}\right]\left(at-\frac{ax}{b}\right)}{a(a+b)}+\frac{b^2\tilde{\varphi}_0^{(n)}\left(at-\frac{ax}{b}\right)}{a+b}-$$

$$-\,b\tilde{\mu}_0^{(n)}\left(t-\frac{x}{b}\right)\Bigg)+\frac{\left((a+b)\Im\left[\tilde{\varphi}_1^{(n)}\right]+\Im^2\left[\tilde{\varphi}_2^{(n)}\right]+ab\tilde{\varphi}_0^{(n)}\right)(x+at)}{2a(a+b)}-$$

$$-\,\frac{1}{2a(a-b)}\Bigg(\left((a+b)\Im\left[\tilde{\varphi}_1^{(n)}\right]+\Im^2\left[\tilde{\varphi}_2^{(n)}\right]+ab\tilde{\varphi}_0^{(n)}\right)(at-x)-$$

$$-\,2a^2b\Im\left[\tilde{\mu}_1^{(n)}\right]\left(t-\frac{x}{a}\right)-2a^2\tilde{\mu}_0^{(n)}\left(t-\frac{x}{a}\right)\Bigg),\ (t,x)\in\overline{Q^{(3)}},$$

$$K_n[u](t,x)=\int\limits_0^t\Bigg\{\int\limits_0^{a(t-\tau)+x}\frac{f^{(n)}\left(\tau,z,u^{(1)}(\tau,z)\right)\left(a(t-\tau)+x-z\right)}{2a(a+b)}dz+$$

$$+\int\limits_0^{x-b(t-\tau)}\frac{f^{(n)}\left(\tau,z,u^{(1)}(\tau,z)\right)\left(b(t-\tau)-x+z\right)}{(a-b)(a+b)}dz+$$

$$+\int\limits_0^{x-a(t-\tau)}\frac{f^{(n)}\left(\tau,z,u^{(1)}(\tau,z)\right)\left(a(\tau-t)+x-z\right)}{2a(a-b)}dz\Bigg\}d\tau,\ (t,x)\in\overline{Q^{(1)}},$$

$$K_n[u](t,x)=\int\limits_{t-\frac{x}{b}}^{t}\Bigg\{\int\limits_0^{x+a(t-\tau)}\frac{\left(x-z+a(t-\tau)\right)f^{(n)}\left(\tau,z,u^{(1)}(\tau,z)\right)}{2a(a+b)}dz+$$

$$+\int\limits_0^{x-b(t-\tau)}\frac{\left(z-x+b(t-\tau)\right)f^{(n)}\left(\tau,z,u^{(1)}(\tau,z)\right)}{(a-b)(a+b)}dz+\int\limits_0^{x-a(t-\tau)}\frac{\left(x-z+a(\tau-t)\right)f^{(n)}\left(\tau,z,u^{(1)}(\tau,z)\right)}{2a(a-b)}dz\Bigg\}d\tau+$$

$$+\int\limits_0^{t-\frac{x}{b}}\Bigg\{\int\limits_0^{x+a(t-\tau)}\frac{\left(x-z+a(t-\tau)\right)f^{(n)}\left(\tau,z,u^{(2)}(\tau,z)\right)}{2a(a+b)}dz+\int\limits_0^{x-a(t-\tau)}\frac{\left(x-z+a(\tau-t)\right)f^{(n)}\left(\tau,z,u^{(2)}(\tau,z)\right)}{2a(a-b)}dz-$$

$$-\int\limits_0^{a\left(t-\tau-\frac{x}{b}\right)}\frac{\left(bz+a\left(x+b(\tau-t)\right)\right)f^{(n)}\left(\tau,z,u^{(2)}(\tau,z)\right)}{a(a-b)(a+b)}dz\Bigg\}d\tau,\ a<b,\ (t,x)\in\overline{Q^{(2)}},$$

$$K_n[u](t,x)=\int\limits_{t-\frac{x}{b}}^{t}\Bigg\{\int\limits_0^{x+a(t-\tau)}\frac{\left(x-z+a(t-\tau)\right)f^{(n)}\left(\tau,z,u^{(1)}(\tau,z)\right)}{2a(a+b)}dz+$$

$$+\int\limits_0^{x-b(t-\tau)}\frac{\left(z-x+b(t-\tau)\right)f^{(n)}\left(\tau,z,u^{(1)}(\tau,z)\right)}{(a-b)(a+b)}dz+\int\limits_0^{x-a(t-\tau)}\frac{\left(x-z+a(\tau-t)\right)f^{(n)}\left(\tau,z,u^{(1)}(\tau,z)\right)}{2a(a-b)}dz\Bigg\}d\tau+$$

$$+\int\limits_{t-\frac{x}{a}}^{t-\frac{x}{b}}\left\{\int\limits_0^{x+a(t-\tau)}\frac{\left(x-z+a(t-\tau)\right)f^{(n)}\left(\tau,z,u^{(2)}(\tau,z)\right)}{2a(a+b)}dz+\int\limits_0^{x-a(t-\tau)}\frac{\left(x-z+a(\tau-t)\right)f^{(n)}\left(\tau,z,u^{(2)}(\tau,z)\right)}{2a(a-b)}dz-\right.$$

$$\left.-\int\limits_0^{a\left(t-\frac{x}{b}-\tau\right)}\frac{\left(bz+a\left(x+b(\tau-t)\right)\right)f^{(n)}\left(\tau,z,u^{(2)}(\tau,z)\right)}{a(a-b)(a+b)}dz\right\}d\tau+$$

$$+\int\limits_0^{t-\frac{x}{a}}\left\{\int\limits_0^{x+a(t-\tau)}\frac{\left(x-z+a(t-\tau)\right)f^{(n)}\left(\tau,z,u^{(3)}(\tau,z)\right)}{2a(a+b)}dz+\int\limits_0^{a(t-\tau)-x}\frac{\left(x+z+a(\tau-t)\right)f^{(n)}\left(\tau,z,u^{(3)}(\tau,z)\right)}{2a(a-b)}dz-\right.$$

$$\left.-\int\limits_0^{a\left(t-\frac{x}{b}-\tau\right)}\frac{\left(bz+a\left(x+b(\tau-t)\right)\right)f^{(n)}\left(\tau,z,u^{(3)}(\tau,z)\right)}{a(a-b)(a+b)}dz\right\}d\tau,\ a<b,\ (t,x)\in\overline{Q^{(3)}},$$

$$K_n[u](t,x)=\int\limits_0^{t-\frac{x}{a}}\left\{\int\limits_0^{x+a(t-\tau)}\frac{\left(x-z+a(t-\tau)\right)f^{(n)}\left(\tau,z,u^{(2)}(\tau,z)\right)}{2a(a+b)}dz+\right.$$

$$\left.+\int\limits_0^{x-b(t-\tau)}\frac{\left(-x+z+b(t-\tau)\right)f^{(n)}\left(\tau,z,u^{(2)}(\tau,z)\right)}{a^2-b^2}dz+\int\limits_0^{a(t-\tau)-x}\frac{\left(x+z+a(\tau-t)\right)f^{(n)}\left(\tau,z,u^{(2)}(\tau,z)\right)}{2a(a-b)}dz\right\}d\tau+$$

$$+\int\limits_{t-\frac{x}{a}}^{t}\left\{\int\limits_0^{x+a(t-\tau)}\frac{\left(x-z+a(t-\tau)\right)f^{(n)}\left(\tau,z,u^{(1)}(\tau,z)\right)}{2a(a+b)}dz+\int\limits_0^{x-b(t-\tau)}\frac{\left(-x+z+b(t-\tau)\right)f^{(n)}\left(\tau,z,u^{(1)}(\tau,z)\right)}{(a-b)(a+b)}dz+\right.$$

$$\left.+\int\limits_0^{x-a(t-\tau)}\frac{\left(x-z+a(\tau-t)\right)f^{(n)}\left(\tau,z,u^{(1)}(\tau,z)\right)}{2a(a-b)}dz\right\}d\tau,\ b<a,\ (t,x)\in\overline{Q^{(2)}},$$

$$K_n[u](t,x)=\int\limits_{t-\frac{x}{b}}^{t-\frac{x}{a}}\left\{\int\limits_0^{x+a(t-\tau)}\frac{\left(x-z+a(t-\tau)\right)f^{(n)}\left(\tau,z,u^{(2)}(\tau,z)\right)}{2a(a+b)}dz+\right.$$

$$\left.+\int\limits_0^{x-b(t-\tau)}\frac{\left(-x+z+b(t-\tau)\right)f^{(n)}\left(\tau,z,u^{(2)}(\tau,z)\right)}{a^2-b^2}dz+\int\limits_0^{a(t-\tau)-x}\frac{\left(x+z+a(\tau-t)\right)f^{(n)}\left(\tau,z,u^{(2)}(\tau,z)\right)}{2a(a-b)}dz\right\}d\tau+$$

$$+\int\limits_{t-\frac{x}{a}}^{t}\left\{\int\limits_0^{x+a(t-\tau)}\frac{\left(x-z+a(t-\tau)\right)f^{(n)}\left(\tau,z,u^{(1)}(\tau,z)\right)}{2a(a+b)}dz+\int\limits_0^{x-b(t-\tau)}\frac{\left(-x+z+b(t-\tau)\right)f^{(n)}\left(\tau,z,u^{(1)}(\tau,z)\right)}{(a-b)(a+b)}dz+\right.$$

$$\left.+\int\limits_0^{x-a(t-\tau)}\frac{\left(x-z+a(\tau-t)\right)f^{(n)}\left(\tau,z,u^{(1)}(\tau,z)\right)}{2a(a-b)}dz\right\}d\tau+$$

$$+\int\limits_0^{t-\frac{x}{b}}\left\{\int\limits_0^{x+a(t-\tau)}\frac{\left(x-z+a(t-\tau)\right)f^{(n)}\left(\tau,z,u^{(3)}(\tau,z)\right)}{2a(a+b)}dz+\int\limits_0^{a(t-\tau)-x}\frac{\left(x+z+a(\tau-t)\right)f^{(n)}\left(\tau,z,u^{(3)}(\tau,z)\right)}{2a(a-b)}dz-\right.$$

$$\left.-\int\limits_0^{a\left(t-\frac{x}{b}-\tau\right)}\frac{\left(bz+a\left(x+b(\tau-t)\right)\right)f^{(n)}\left(\tau,z,u^{(3)}(\tau,z)\right)}{a(a-b)(a+b)}dz\right\}d\tau,\ b<a,\ (t,x)\in\overline{Q^{(3)}}.$$

For the sake of notation we also set $f^{(\infty)}=f$. We note that the operator $K_n : C\left(\bar{Q}\right)\mapsto C\left(\bar{Q}\right)$ is well-defined and continuous, since the smoothness conditions

$$z^{(n)}\in C^3\left(\bar{Q}\right),\ f^{(n)}\in C^3\left(\bar{Q}\right),\ \tilde{\varphi}_0^{(n)}\in C^3\left([0,\infty)\right),\ \tilde{\varphi}_1^{(n)}\in C^2\left([0,\infty)\right),$$
$$\tilde{\varphi}_2^{(n)}\in C^1\left([0,\infty)\right),\ \tilde{\mu}_0^{(n)}\in C^3\left([0,\infty)\right),\ \tilde{\mu}_1^{(n)}\in C^2\left([0,\infty)\right) \tag{22}$$

and the matching conditions (19) are fulfilled.

Now we construct a sequence u_n by the formula

$$u_n=K_n\left[z^{(n)}\right]+v^{(n)}. \tag{23}$$

By virtue of conditions (19) and (22), the function u_n belongs to the class $C^3\left(\bar{Q}\right)$. Taking into account the relations

$$\tilde{\varphi}_i=\frac{\partial^i u}{\partial t^i}(0,\cdot)=\lim_{n\to\infty}\frac{\partial^i z^{(n)}}{\partial t^i}(0,\cdot),\ i=0,1,2,$$

and

$$\tilde{\mu}_i=\frac{\partial^i u}{\partial x^i}(\cdot,0)=\lim_{n\to\infty}\frac{\partial^i z^{(n)}}{\partial x^i}(\cdot,0),\ i=0,1,$$

we pass to the limit in formula (23) and get

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\left(K_n\left[z^{(n)}\right]+v^{(n)}\right)=K_\infty[u]+v=u.$$

Direct verification proves that the function u_n is a solution to the problem

$$\left(\frac{\partial}{\partial t}+b\frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)u_n(t,x)=f^{(n)}\left(t,x,z^{(n)}(t,x)\right),\ (t,x)\in Q,$$

$$\frac{\partial^i u_n}{\partial t^i}(0,x)=\tilde{\varphi}_i^{(n)}(x),\ i=0,1,2,\ x\in[0,\infty),$$

$$\frac{\partial^i u_n}{\partial x^i}(t,0)=\tilde{\mu}_i^{(n)}(t),\ i=0,1,\ t\in[0,\infty).$$

Due to the continuity of the Nemytskii operator,

$$\lim_{n\to\infty}\left(\left(\frac{\partial}{\partial t}+b\frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)u_n(\cdot)-f\left(\cdot,u_n(\cdot)\right)\right)=\lim_{n\to\infty}\left(f^{(n)}\left(\cdot,z^{(n)}(\cdot)\right)-f\left(\cdot,u_n(\cdot)\right)\right)=0$$

in the space $C\left(\bar{Q}\right)$. In addition, the relations

$$\varphi_i=\frac{\partial^i u}{\partial t^i}(0,\cdot)=\lim_{n\to\infty}\frac{\partial^i u_n}{\partial t^i}(0,\cdot),\ i=0,1,2,$$

$$\mu_i=\frac{\partial^i u}{\partial x^i}(\cdot,0)=\lim_{n\to\infty}\frac{\partial^i u_n}{\partial x^i}(\cdot,0),\ i=0,1,$$

where the limits are understood in the spaces $C^{2-i}\big([0, \infty)\big)$ hold.

Thus, we have proven the following theorem.

Theorem 4. *Let the conditions* $\varphi_0 \in C^2\big([0, \infty)\big)$, $\varphi_1 \in C^1\big([0, \infty)\big)$, $\varphi_2 \in C\big([0, \infty)\big)$, $\mu_0 \in C^2\big([0, \infty)\big)$, $\mu_1 \in C\big([0, \infty)\big)$ *and* $f \in C\big(\bar{Q} \times \mathbb{R}\big)$ *be satisfied. Then a mild solution* $u : \bar{Q} \mapsto \mathbb{R}$ *of problem (1)–(3) from the class* $C^2\big(\bar{Q}\big)$ *is the limit of classical solutions, i. e. there exists a sequence of functions* $u_n \in C^3\big(\bar{Q}\big)$ *such that*

$$\lim_{n \to \infty} \rho\left(\left(\frac{\partial}{\partial t} + b\frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial t^2} - a^2\frac{\partial^2}{\partial x^2}\right)u_n(\cdot),\, f\big(\cdot,\, u_n(\cdot)\big)\right)_{C(\bar{Q} \times \mathbb{R})} = 0,$$

$$\lim_{n \to \infty} \rho\big(u,\, u_n\big)_{C(\bar{Q})} = 0,\ \lim_{n \to \infty} \rho\big(\varphi_0,\, u_n(0,\, \cdot)\big)_{C^2([0,\, \infty))} = 0,\ \lim_{n \to \infty} \rho\big(\mu_0,\, u_n(\cdot,\, 0)\big)_{C^2([0,\, \infty))} = 0,$$

$$\lim_{n \to \infty} \rho\left(\varphi_1,\, \frac{\partial u_n}{\partial t}(0,\, \cdot)\right)_{C^1([0,\, \infty))} = 0,\ \lim_{n \to \infty} \rho\left(\mu_1,\, \frac{\partial u_n}{\partial x}(\cdot,\, 0)\right)_{C^1([0,\, \infty))} = 0, \tag{24}$$

$$\lim_{n \to \infty} \rho\left(\varphi_2,\, \frac{\partial^2 u_n}{\partial t^2}(0,\, \cdot)\right)_{C([0,\, \infty))} = 0.$$

It turns out that the statement opposite to theorem 4 is also true.

Theorem 5. *Let the conditions* $\varphi_0 \in C^2\big([0, \infty)\big)$, $\varphi_1 \in C^1\big([0, \infty)\big)$, $\varphi_2 \in C\big([0, \infty)\big)$, $\mu_0 \in C^2\big([0, \infty)\big)$, $\mu_1 \in C\big([0, \infty)\big)$ *and* $f \in C\big(\bar{Q} \times \mathbb{R}\big)$ *be satisfied, and let there exists a sequence of functions* $u_n \in C^3\big(\bar{Q}\big)$ *such that equalities (24) hold, where* $u \in C^2\big(\bar{Q}\big)$. *Then the function* $u : \bar{Q} \mapsto \mathbb{R}$ *is a mild solution of problem (1)–(3).*

Proof. It is easy to see that u_n is a classical solution of the mixed problem

$$\left(\frac{\partial}{\partial t} + b\frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial t^2} - a^2\frac{\partial^2}{\partial x^2}\right)u_n(t,\, x) = f^{(n)}\big(t,\, x,\, u_n(t,\, x)\big) := f\big(t,\, x,\, u_n(t,\, x)\big) + \varepsilon_n(t,\, x),\ (t,\, x) \in Q,$$

$$\frac{\partial^i u_n}{\partial t^i}(0,\, x) := \tilde{\varphi}_i^{(n)}(x),\ i = 0,\, 1,\, 2,\ x \in [0,\, \infty),$$

$$\frac{\partial^i u_n}{\partial x^i}(t,\, 0) := \tilde{\mu}_i^{(n)}(t),\ i = 0,\, 1,\ t \in [0,\, \infty),$$

where $\lim\limits_{n \to \infty} \rho\big(\varepsilon_n,\, 0\big)_{C(\bar{Q} \times \mathbb{R})} = 0$. In this case the matching condition (19), which can be deduced according to the scheme we outlined in the work [9], is satisfied. Then by theorem 1 from the article [9], the function u_n is a solution to the integral equation

$$u_n = K_n\big[u_n\big] + v^{(n)}. \tag{25}$$

Taking into account $\lim\limits_{n \to \infty} \rho\big(\varepsilon_n,\, 0\big)_{C(\bar{Q} \times \mathbb{R})} = 0$ and the continuity of the operator K_n and passing to the limit in the equation (25), we obtain

$$u = \lim_{n \to \infty} u_n = \lim_{n \to \infty}\Big(K_n\big[u_n\big] + v^{(n)}\Big) = K_\infty\big[u\big] + v.$$

The theorem is proved.

Conclusions

The sufficient conditions under which there exists a unique mild solution of a mixed problem for a third-order non-linear strictly hyperbolic equation with the wave operator are presented.

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