УДК 517.988

ОБОБЩЕННЫЙ МЕТОД НЬЮТОНА – КАНТОРОВИЧА ПРИ МОДИФИЦИРОВАННОМ УСЛОВИИ РЕГУЛЯРНОЙ ГЛАДКОСТИ

А. Н. ТАНЫГИНА[1)]

[1)]*Белорусский государственный университет, пр. Независимости, 4, 220030, г. Минск, Беларусь*

Аннотация. Рассматривается обобщенный метод Ньютона – Канторовича для решения в банаховых пространствах нелинейных операторных уравнений вида $f(x) + g(x) = 0$, где f – регулярно гладкий оператор; g – недифференцируемый оператор, удовлетворяющий условию Липшица. Приводится доказательство основной теоремы о сходимости метода при модифицированном условии регулярной гладкости, в записи которого приращения производной оператора f мажорируются приращениями скалярной функции.

Ключевые слова: обобщенный метод Ньютона – Канторовича; условие регулярной гладкости; нелинейное операторное уравнение.

Образец цитирования:
Таныгина АН. Обобщенный метод Ньютона – Канторовича при модифицированном условии регулярной гладкости. *Журнал Белорусского государственного университета. Математика. Информатика.* 2025;3:6–14 (на англ.).
EDN: FYYKIA

For citation:
Tanyhina AN. Generalised Newton – Kantorovich method under the modified regular smoothness condition. *Journal of the Belarusian State University. Mathematics and Informatics.* 2025;3:6–14.
EDN: FYYKIA

Автор:
Анастасия Николаевна Таныгина – кандидат физико-математических наук, доцент; доцент кафедры фундаментальной математики и интеллектуальных систем факультета прикладной математики и информатики.

Author:
Anastasija N. Tanyhina, PhD (physics and mathematics), docent; associate professor at the department of fundamental mathematics and intelligent systems, faculty of applied mathematics and computer science.
tanyhina@bsu.by
https://orcid.org/0009-0004-6451-9669

GENERALISED NEWTON – KANTOROVICH METHOD UNDER THE MODIFIED REGULAR SMOOTHNESS CONDITION

A. N. TANYHINA[a]

[a]*Belarusian State University, 4 Niezaliezhnasci Avenue, Minsk 220030, Belarus*

Abstract. The article deals with the generalised Newton – Kantorovich method for solving non-linear operator equations of the form $f(x) + g(x) = 0$ in Banach spaces, where f is the operator satisfying the regular smoothness condition; g is the non-differentiable operator satisfying Lipschitz condition. The main convergence theorem is proved under the modified regular smoothness condition in which increments of the operator f derivative are majorised by the increments of a scalar function.

Keywords: generalised Newton – Kantorovich method; regular smoothness condition; non-linear operator equation.

Introduction

Let X and Y be Banach spaces, f and g be non-linear operators defined on the closed ball $\overline{B(x_0, R)} \subset X$ and taking values in Y, where the operator f is differentiable at every interior point of $\overline{B(x_0, R)}$ and the operator g is non-differentiable. One of the most effective iterative methods for solving operator equations of the form

$$f(x) + g(x) = 0 \tag{1}$$

is the generalised Newton – Kantorovich method with successive approximations

$$x_{n+1} = x_n - \left[f'(x_n)\right]^{-1}\left(f(x_n) + g(x_n)\right), n = 0, 1, \ldots, \tag{2}$$

where x_0 is given.

A thorough convergence analysis of the sequence (2) was carried out in the work [1] by means of the approach based on the application of majorant scalar equations and originating from L. V. Kantorovich's investigations [2]. However, the hypotheses given there are difficult to verify and for this reason a more flexible approach for solving the equation (1) was proposed in the research [3].

In the case when $g = 0$, the most precise error estimates for the process (2) were obtained by A. Galperin and Z. Waksman in [4; 5]. These results were generalised in the article [6] under the assumption that the operator f satisfies the regular smoothness condition introduced in the works [4; 5], and the operator g satisfies Lipschitz condition from the paper [3]

$$\left\|g(x'') - g(x')\right\| \le \psi(t)\left\|x'' - x'\right\| \ \forall x', x'' \in \overline{B(x_0, t)}, \tag{3}$$

where ψ is non-decreasing function on $[0, R]$. However, the meaning of the regular smoothness concept from the works [4; 5] is quite complex and it was shown in the research [7] that it may be replaced by a simpler one in which increments of the derivative f' are majorised by increments of a scalar function. The aim of this article is to prove the main convergence theorem for the process (2) under the modification of Galperin – Waksman condition from the paper [7].

Main concepts and preliminary results

Let $\omega : [0, \infty) \to [0, \infty)$ is a continuous strictly increasing concave function that vanishes at zero: $\omega(0) = 0$. Assume without loss of generality that $f'(x_0) = I$. Let

$$h(f) = \inf\left\{\left\|f'(x)\right\| : x \in \overline{B(x_0, R)}\right\}.$$

In accordance with the article [5] the operator f is ω-regularly smooth on $\overline{B(x_0, R)}$ (or, equivalently, ω is a regular smoothness modulus of f on $\overline{B(x_0, R)}$), if there exists $h \in \left[0, h(f)\right]$ such that the inequality

$$\omega^{-1}\left(h_f(x', x'') + \left\|f'(x'') - f'(x')\right\|\right) - \omega^{-1}\left(h_f(x', x'')\right) \le \left\|x'' - x'\right\|, \tag{4}$$

where

$$h_f(x', x'') = \min\left\{\left\|f'(x')\right\|, \left\|f'(x'')\right\|\right\} - h,$$

holds for all $x', x'' \in \overline{B(x_0, R)}$. The operator f is called regularly smooth on $\overline{B(x_0, R)}$, if it is ω-regularly smooth on $\overline{B(x_0, R)}$ for some ω with such properties.

It was shown in the paper [7] that the condition (4) may be replaced by a simpler condition

$$\left\|f'(x'') - f'(x')\right\| \le \omega\left(\left(\chi - r - \left\|x'' - x'\right\|\right)^+ + \left\|x'' - x'\right\|\right) - \omega\left(\left(\chi - r - \left\|x'' - x'\right\|\right)^+\right), \tag{5}$$

where $\chi = \omega^{-1}(1 - h)$; $r = \min\left\{\left\|x' - x_0\right\|, \left\|x'' - x_0\right\|\right\}$; $\lambda^+ = \max\{\lambda, 0\}$. This condition is clearer than the condition (4). Moreover, in the work [5] this condition is used in the proof of some auxiliary statements and the main theorem about the convergence of the classical Newton – Kantorovich method. If χ increases, then the value of $\left(\chi - r - \left\|x'' - x'\right\|\right)^+$ also increases and the right part of the condition (5) decreases. Therefore, the higher is the value of χ, the better is the estimate for $\left\|f'(x'') - f'(x')\right\|$, which leads to more accurate estimates for successive approximations. In the paper [8] the comparative analysis of the regular smoothness and the Hölder continuity conditions for the equation (1) in the case, when $g = 0$, was carried out.

Let $\Omega(t) = \int_0^t \omega(\tau)d\tau$, $\Psi(t) = \int_0^t \psi(\tau)d\tau$, a is a positive number such that $a \ge \left\|f(x_0) + g(x_0)\right\|$, $\chi \in \left[0, \omega^{-1}(1)\right]$ is a constant. Let us define a function with numeric argument

$$W(t) = a - \Omega(\chi) + \Omega(\chi - t) - t\left(1 - \omega(\chi)\right) + \Psi(t), \tag{6}$$

and the numerical sequence $\{t_n\}$ as follows:

$$t_{n+1} = t_n + \frac{W(t_n)}{1 - \left[\omega(\chi) - \omega(\chi - t_n)\right]}, \; n = 0, 1, \ldots, \tag{7}$$

where $t_0 = 0$.

Lemma 1. *Let us suppose that the function (6) has a unique zero* $t_* \in [0, \chi]$ *and*

$$a < \Omega(\chi) - \chi\omega(\chi) + \chi - \Psi(\chi). \tag{8}$$

Then the sequence (7) is defined for all n, monotonically increases and converges to t_*.

P r o o f. The function W is positive on $[0, t_*)$, since t_* is a unique zero of the equation $W(t) = 0$, $W(0) = a > 0$ and W is continuous on $[0, \chi]$. Hence the function

$$u(t) = \frac{W(t)}{1 - \left[\omega(\chi) - \omega(\chi - t)\right]}$$

is positive on $[0, t_*)$.

Let us show that the function $t + u(t)$ is non-decreasing on $[0, t_*)$. In fact,

$$\left(t + u(t)\right)' = 1 + u'(t) = 1 + \left(\frac{W(t)}{1 - \left[\omega(\chi) - \omega(\chi - t)\right]}\right)' =$$

$$= 1 + \frac{W'(t)\left(1 + \omega(\chi - t) - \omega(\chi)\right) + W(t)\omega'(\chi - t)}{\left(1 + \omega(\chi - t) - \omega(\chi)\right)^2} =$$

$$= \frac{\Psi'(t)\left(1 + \omega(\chi - t) - \omega(\chi)\right) + W(t)\omega'(\chi - t)}{\left(1 + \omega(\chi - t) - \omega(\chi)\right)^2} \ge 0$$

on $[0, t_*)$. This implies that the sequence $\{t_n\}$ monotonically increases and

$$t_{n+1} = t_n + u(t_n) \le t_* + u(t_*) = t_*$$

for $t_n \le t_*$. Consequently, the sequence $\{t_n\}$ converges to $t_{**} \in [0, t_*]$ and $t_{**} = t_{**} + u(t_{**})$, hence $W(t_{**}) = 0$. Since t_* is a unique zero of W in $[0, \chi]$, it follows that $t_{**} = t_*$.

The sequence $\{t_n\}$ is defined for all n. In fact, it is clear from the condition (8) that $W(\chi) < 0 < a = W(0)$ and hence there exists $\theta \in (0, \chi)$ such that $W(\theta) = 0$. Consequently, $\theta = t_* = \lim\limits_{n \to \infty} t_n$ and $t_n \le \theta < \chi$ for all $n = 0, 1, \ldots$. Because of the monotonicity of ω, the inequality $\omega(\chi - t_n) > 0$ is true for all $n = 0, 1, \ldots$. Lemma 1 is proved.

Lemma 2. *Let us suppose that there exists a constant* $\chi \in \left[0,\, \omega^{-1}(1)\right]$ *satisfying the condition (8), the operator f satisfies the condition (5) on* $\overline{B(x_0,\, R)}$ *with such* χ, *the operator g satisfies the condition (3), and the function (6) has a unique zero* $t_* \le R$ *in* $[0,\, \chi]$. *Then the equation (1) has a unique solution in* $\overline{B(x_0,\, t_*)}$.

P r o o f. Let us prove the existence of a solution in $\overline{B(x_0,\, t_*)}$. We consider the sequence

$$u_{n+1} = Du_n,\ n = 0, 1, \ldots;\ u_0 = x_0,$$

where $D = I - \left[f'(x_0)\right]^{-1}(f + g) = I - (f + g)$, and the numerical sequence

$$\rho_{n+1} = d(\rho_n),\ n = 0, 1, \ldots;\ \rho_0 = 0,$$

where $d(t) = t + W(t)$. Since

$$d'(t) = 1 + W'(t) = \omega(\chi) - \omega(\chi - t) + \psi(t) \ge 0$$

for all $t \in [0,\, \chi]$, the function d is monotonically increasing on $[0,\, \chi]$.

For all n = 0, 1, … the inequality

$$\rho_n \le t_* \tag{9}$$

holds. In fact, for $n = 0$ the inequality (9) is obvious: $\rho_0 = 0 \le t_*$. Let us suppose that the inequality (9) holds for all $n \le k$. Then from $\rho_k \le t_*$, because of the monotonicity of d, we obtain $d(\rho_k) \le d(t_*)$, that is $\rho_{k+1} \le t_*$. Consequently, by the induction hypothesis, the inequality (9) is true for all n.

Let us prove by induction that the sequence $\{\rho_n\}$ is monotone. Clearly, $0 = \rho_0 \le \rho_1 = a$. We suppose that $\rho_k \le \rho_{k+1}$. Then $\rho_{k+1} = d(\rho_k) \le d(\rho_{k+1}) = \rho_{k+2}$. Thus, the sequence $\{\rho_n\}$ is monotonically increasing and bounded from above. Consequently, it converges to some $\tilde{\rho} \in [0,\, t_*]$. If $n \to \infty$ in $\rho_{n+1} = \rho_n + W(\rho_n)$, we obtain $W(\tilde{\rho}) = 0$ and $\tilde{\rho} = t_*$.

Let us show that for all n = 0, 1, … the inequality

$$\|u_{n+1} - u_n\| \le \rho_{n+1} - \rho_n \tag{10}$$

holds. For $n = 0$ the inequality (10) is obvious:

$$\|u_1 - u_0\| = \|x_0 - (f(x_0) + g(x_0)) - x_0\| = \|f(x_0) + g(x_0)\| \le a = W(0) = \rho_1 - \rho_0.$$

We suppose that the inequality (10) holds for all $n < k$. Then

$$\|u_{k+1} - u_k\| = \|Du_k - Du_{k-1}\| = \left\|u_k - u_{k-1} - \left(f(u_k) - f(u_{k-1})\right) - \left(g(u_k) - g(u_{k-1})\right)\right\| \le$$

$$\le \left\|u_k - u_{k-1} - \left(f(u_k) - f(u_{k-1})\right)\right\| + \left\|g(u_k) - g(u_{k-1})\right\| \le$$

$$\le \int_0^1 \|f'(u_t) - f'(x_0)\| \|u_k - u_{k-1}\| dt + \left\|g(u_k) - g(u_{k-1})\right\| \le$$

$$\le \int_0^1 \left(\omega\left((\chi - \|u_t - x_0\|)^+ + \|u_t - x_0\|\right) - \omega\left((\chi - \|u_t - x_0\|)^+\right)\right) \|u_k - u_{k-1}\| dt + \left\|g(u_k) - g(u_{k-1})\right\|,$$

where $u_t = u_{k-1} + t(u_k - u_{k-1}),\ 0 \le t \le 1$. By the induction hypothesis,

$$\|u_k - x_0\| = \|u_k - u_0\| \le \sum_{j=1}^{k} \|u_j - u_{j-1}\| \le \sum_{j=1}^{k} (\rho_j - \rho_{j-1}) = \rho_k.$$

Consequently,

$$\|u_t - x_0\| = \left\|(1 - t)(u_{k-1} - u_0) + t(u_k - u_0)\right\| \le (1 - t)\|u_{k-1} - u_0\| + t\|u_k - u_0\| \le (1 - t)\rho_{k-1} + t\rho_k.$$

From the condition (3) and the proposition 1 in article [3] it follows that

$$\|g(x'') - g(x')\| \le \Psi(t + \|x'' - x'\|) - \Psi(t)\ \forall x' \in \overline{B(x_0,\, t)},\ \|x'' - x'\| \le R - t. \tag{11}$$

Because of the concavity of ω and the inequality (11), we have

$$\left\|u_{k+1}-u_k\right\| \le \int_0^1 \left(\omega\left(\left(\chi-\left\|u_t-x_0\right\|\right)^+ + \left\|u_t-x_0\right\|\right)-\omega\left(\left(\chi-\left\|u_t-x_0\right\|\right)^+\right)\right)\left\|u_k-u_{k-1}\right\| dt +$$
$$+\Psi\left(\rho_{k-1}+\left\|u_k-u_{k-1}\right\|\right)-\Psi\left(\rho_{k-1}\right) \le$$
$$\le \int_0^1 \left(\omega(\chi)-\omega\left(\chi-\left\|u_t-x_0\right\|\right)\right)\left(\rho_k-\rho_{k-1}\right)dt+\Psi\left(\rho_k\right)-\Psi\left(\rho_{k-1}\right) \le$$
$$\le \int_0^1 \left(\omega(\chi)-\omega\left(\chi-\left((1-t)\rho_{k-1}+t\rho_k\right)\right)\right)\left(\rho_k-\rho_{k-1}\right)dt+\Psi\left(\rho_k\right)-\Psi\left(\rho_{k-1}\right)=$$
$$=\int_{\rho_{k-1}}^{\rho_k}\left(\omega(\chi)-\omega(\chi-\theta)\right)d\theta+\Psi\left(\rho_k\right)-\Psi\left(\rho_{k-1}\right)=d\left(\rho_k\right)-d\left(\rho_{k-1}\right)=\rho_{k+1}-\rho_k.$$

Thus, the inequality (10) holds for $n = k$.

It follows from the inequality (10) that for $m > n$

$$\left\|u_m-u_n\right\| \le \left\|u_m-u_{m-1}\right\|+\ldots+\left\|u_{n+1}-u_n\right\| \le \rho_m-\rho_{m-1}+\ldots+\rho_{n+1}-\rho_n=\rho_m-\rho_n.$$

Hence for all m and n

$$\left\|u_m-u_n\right\| \le \left|\rho_m-\rho_n\right|. \tag{12}$$

Since the sequence $\{\rho_n\}$ converges to t_*, it follows from the inequality (12) that the sequence $\{u_n\}$ also converges to some x_*. Further

$$\left\|u_n-u_0\right\| \le \rho_n \le t_*,\ n=0, 1, \ldots,$$

and, consequently, all u_n with x_* belong to $\overline{B\left(x_0, t_*\right)}$. If $n\to\infty$ in $u_{n+1}=Du_n$, we obtain that $x_*=D\left(x_*\right)$, or $f\left(x_*\right)+g\left(x_*\right)=0$. Thus, x_* is a solution of the equation (1) in $\overline{B\left(x_0, t_*\right)}$.

To prove the uniqueness of the solution x_* in $\overline{B\left(x_0, t_*\right)}$ let us consider the second solution $x_{**}\in\overline{B\left(x_0, t_*\right)}$ of the equation (1) and show that for all $n = 0, 1, \ldots$ the inequality

$$\left\|x_{**}-u_n\right\| \le t_*-\rho_n \tag{13}$$

holds. For $n = 0$ the inequality (13) is obvious:

$$\left\|x_{**}-x_0\right\| \le t_*-\rho_0=t_*.$$

We suppose that inequality (13) holds for all $n \le k$. Then

$$\left\|x_{**}-u_{k+1}\right\|=\left\|x_{**}-Du_k\right\|=\left\|x_{**}-u_k+f\left(u_k\right)+g\left(u_k\right)\right\|=$$
$$=\left\|f\left(u_k\right)-f\left(x_{**}\right)-\left(u_k-x_{**}\right)+g\left(u_k\right)-g\left(x_{**}\right)\right\| \le$$
$$\le \left\|f\left(u_k\right)-f\left(x_{**}\right)-f'\left(x_0\right)\left(u_k-x_{**}\right)\right\|+\left\|g\left(x_{**}\right)-g\left(u_k\right)\right\| \le$$
$$\le \int_0^1 \left\|f'\left(\tilde{u}_t\right)-f'\left(x_0\right)\right\|\left\|u_k-x_{**}\right\|dt+\left\|g\left(x_{**}\right)-g\left(u_k\right)\right\| \le$$
$$\le \int_0^1 \left(\omega\left(\left(\chi-\left\|\tilde{u}_t-x_0\right\|\right)^+ + \left\|\tilde{u}_t-x_0\right\|\right)-\omega\left(\left(\chi-\left\|\tilde{u}_t-x_0\right\|\right)^+\right)\right)\left\|u_k-x_{**}\right\|dt+\left\|g\left(x_{**}\right)-g\left(u_k\right)\right\|,$$

where $\tilde{u}_t=x_{**}+t\left(u_k-x_{**}\right)$, $0\le t\le 1$. Further

$$\left\|\tilde{u}_t-x_0\right\|=\left\|(1-t)\left(x_{**}-x_0\right)+t\left(u_k-x_0\right)\right\| \le (1-t)\left\|x_{**}-x_0\right\|+t\left\|u_k-x_0\right\| \le (1-t)t_*+t\rho_k.$$

Because of the concavity of ω, the inequality (11) and the induction hypothesis, we have

$$\left\|x_{**}-u_{k+1}\right\| \le \int_0^1 \left(\omega(\chi)-\omega\left(\chi-\left\|\tilde{u}_t-x_0\right\|\right)\right)\left(t_*-\rho_k\right)dt+\Psi\left(\rho_k+\left\|x_{**}-u_k\right\|\right)-\Psi\left(\rho_k\right) \le$$

$$\le \int_0^1 \left(\omega(\chi)-\omega\left(\chi-\left((1-t)t_*+t\rho_k\right)\right)\right)\left(t_*-\rho_k\right)dt+\Psi\left(t_*\right)-\Psi\left(\rho_k\right)=$$

$$=\int_{\rho_k}^{t_*}\left(\omega(\chi)-\omega(\chi-\theta)\right)d\theta+\Psi\left(t_*\right)-\Psi\left(\rho_k\right)=d\left(t_*\right)-d\left(\rho_k\right)=t_*-\rho_{k+1}.$$

Hence the inequality (13) holds for $n=k+1$. If $n\to\infty$ in the inequality (13), we obtain that

$$\left\|x_{**}-x_*\right\| \le t_*-t_*=0$$

and hence $x_{**}=x_*$. Lemma 2 is proved.

Let us denote for all $n=1, 2, \ldots$

$$r_n=\left\|f\left(x_n\right)-f\left(x_{n-1}\right)-f'\left(x_{n-1}\right)\left(x_n-x_{n-1}\right)\right\|.$$

Lemma 3. *Let us suppose that there exists a constant* $\chi\in\left[0,\ \omega^{-1}(1)\right]$ *satisfying the condition (8), the operator f satisfies the condition (5) on* $\overline{B\left(x_0,\ R\right)}$ *with such* χ*, the operator g satisfies the condition (3), the function (6) has a unique zero* $t_*\in[0,\ \chi]$*, and the sequence* $\{t_n\}$ *is defined by the recurrence formula (7). If for all* $1\le k\le n$ *successive approximations* x_k *are defined and satisfy the inequality*

$$\left\|x_k-x_{k-1}\right\| \le t_k-t_{k-1}, \tag{14}$$

then

$$r_n \le a-\Omega(\chi)+\Omega\left(\chi-t_n\right)-t_n\left(1-\omega(\chi)\right)+\Psi\left(t_{n-1}\right). \tag{15}$$

P r o o f. Let $x_t=x_{n-1}+t\left(x_n-x_{n-1}\right)$, $0\le t\le 1$. Then

$$r_n \le \int_0^1 \left\|f'\left(x_t\right)-f'\left(x_{n-1}\right)\right\|\left\|x_n-x_{n-1}\right\|dt \le$$

$$\le \int_0^1 \left(\omega\left(\left(\chi-r-\left\|x_t-x_{n-1}\right\|\right)^+ +\left\|x_t-x_{n-1}\right\|\right)-\omega\left(\left(\chi-r-\left\|x_t-x_{n-1}\right\|\right)^+\right)\right)\left\|x_n-x_{n-1}\right\|dt,$$

where $r=\left\|x_{n-1}-x_0\right\|$.

Since for all $1\le k\le n$ the inequality (14) holds, it follows that

$$\left\|x_{n-1}-x_0\right\| \le \sum_{k=1}^{n-1}\left\|x_k-x_{k-1}\right\| \le \sum_{k=1}^{n-1}\left(t_k-t_{k-1}\right)=t_{n-1}$$

and

$$\left\|x_t-x_{n-1}\right\|=\left\|x_{n-1}+t\left(x_n-x_{n-1}\right)-x_{n-1}\right\|=t\left\|x_n-x_{n-1}\right\|\le t\left(t_n-t_{n-1}\right).$$

According to lemma 1, $t_n<\chi$ for all $n=0, 1, \ldots$. Hence

$$\chi-r-\left\|x_t-x_{n-1}\right\| \ge \chi-t_{n-1}-t\left(t_n-t_{n-1}\right)=t\left(\chi-t_n\right)+(1-t)\left(\chi-t_{n-1}\right)>0$$

and, because of the concavity and the monotonicity of ω,

$$r_n \le \int_0^1\left(\omega\left(\chi-t_{n-1}\right)-\omega\left(\chi-t_{n-1}-t\left(t_n-t_{n-1}\right)\right)\right)\left(t_n-t_{n-1}\right)dt=$$

$$=\omega\left(\chi-t_{n-1}\right)\left(t_n-t_{n-1}\right)-\int_{\chi-t_{n-1}}^{\chi-t_n}\omega(\theta)d\theta=$$

$$= \omega(\chi - t_{n-1})(t_n - t_{n-1}) + \int_0^{\chi - t_n} \omega(\theta) d\theta - \int_0^{\chi - t_{n-1}} \omega(\theta) d\theta =$$

$$= \omega(\chi - t_{n-1})(t_n - t_{n-1}) + \Omega(\chi - t_n) - \Omega(\chi - t_{n-1}).$$

Let us show that for all $n = 0, 1, \ldots$ the equality

$$\omega(\chi - t_n)(t_{n+1} - t_n) - \Omega(\chi - t_n) + t_{n+1}(1 - \omega(\chi)) - \Psi(t_n) = a - \Omega(\chi) \tag{16}$$

holds. In fact, by the definition of the sequence $\{t_n\}$

$$(t_{n+1} - t_n)(1 - \omega(\chi) + \omega(\chi - t_n)) = a - \Omega(\chi) + \Omega(\chi - t_n) - t_n(1 - \omega(\chi)) + \Psi(t_n)$$

and

$$(t_n - t_{n-1})(1 - \omega(\chi) + \omega(\chi - t_{n-1})) = a - \Omega(\chi) + \Omega(\chi - t_{n-1}) - t_{n-1}(1 - \omega(\chi)) + \Psi(t_{n-1}).$$

It follows from the first of these equalities that

$$a - \Omega(\chi) = t_{n+1}(1 - \omega(\chi)) + \omega(\chi - t_n)(t_{n+1} - t_n) - \Omega(\chi - t_n) - \Psi(t_n)$$

and from the second that

$$a - \Omega(\chi) = t_n(1 - \omega(\chi)) + \omega(\chi - t_{n-1})(t_n - t_{n-1}) - \Omega(\chi - t_{n-1}) - \Psi(t_{n-1}).$$

Consequently,

$$\omega(\chi - t_n)(t_{n+1} - t_n) - \Omega(\chi - t_n) + t_{n+1}(1 - \omega(\chi)) - \Psi(t_n) =$$

$$= \omega(\chi - t_{n-1})(t_n - t_{n-1}) - \Omega(\chi - t_{n-1}) + t_n(1 - \omega(\chi)) - \Psi(t_{n-1})$$

for all $n = 1, 2, \ldots$ and

$$\omega(\chi - t_n)(t_{n+1} - t_n) - \Omega(\chi - t_n) + t_{n+1}(1 - \omega(\chi)) - \Psi(t_n) =$$

$$= \omega(\chi - t_0)(t_1 - t_0) - \Omega(\chi - t_0) + t_1(1 - \omega(\chi)) - \Psi(t_0) =$$

$$= \omega(\chi)a - \Omega(\chi) + a(1 - \omega(\chi)) = a - \Omega(\chi).$$

Thus, the equality (16) holds for all $n = 0, 1, \ldots$ and the estimate for r_n may be rewritten in the form of the inequality (15). Lemma 3 is proved.

Convergence theorem

Theorem. *Let us suppose that there exists a constant* $\chi \in [0, \omega^{-1}(1)]$ *satisfying the condition (8), the operator f satisfies the condition (5) on* $\overline{B(x_0, R)}$ *with such* χ*, the operator g satisfies the condition (3), and the function (6) has a unique zero* $t_* \le R$ *in* $[0, \chi]$*. Then the following conditions are met:*

1) the equation (1) has a unique solution x_* *in* $\overline{B(x_0, t_*)}$*;*

2) the successive approximations (2) are defined for all $n = 0, 1, \ldots$ *and belong to* $\overline{B(x_0, t_*)}$ *as well as converge to* x_**;*

3) for all $n = 0, 1, \ldots$ *the inequalities*

$$\|x_{n+1} - x_n\| \le t_{n+1} - t_n, \tag{17}$$

$$\|x_* - x_n\| \le t_* - t_n \tag{18}$$

hold, where the sequence $\{t_n\}$ *is defined by the recurrence formula (7), monotonically increases and converges to* t_*.

P r o o f. In order to prove the theorem it is sufficient to show that successive approximations (2) are defined for all $n = 0, 1, \ldots$ and belong to $\overline{B(x_0, t_*)}$ as well as satisfy the inequalities (17) and (18). The other assertions of the theorem follow from lemmas 1 and 2.

Since the inequality (18) is a direct consequence of the inequality (17), it is sufficient to prove the inequality (17). For $n = 0$ the inequality (17) is obvious:

$$\|x_1 - x_0\| = \left\|\left[f'(x_0)\right]^{-1}\left(f(x_0) + g(x_0)\right)\right\| \leq a = t_1 - t_0.$$

We suppose that the inequality (17) holds for all $n < k$. Let us show that the operator $f'(x_k)$ is invertible. In fact,

$$\left\|\left[f'(x_0)\right]^{-1}\left(f'(x_k) - f'(x_0)\right)\right\| = \left\|f'(x_k) - f'(x_0)\right\| \leq$$

$$\leq \omega\left(\left(\chi - \|x_k - x_0\|\right)^+ + \|x_k - x_0\|\right) - \omega\left(\left(\chi - \|x_k - x_0\|\right)^+\right).$$

By the induction hypothesis,

$$\|x_k - x_0\| \leq \sum_{j=1}^{k}\left\|x_j - x_{j-1}\right\| \leq \sum_{j=1}^{k}\left(t_j - t_{j-1}\right) = t_k$$

and hence $\chi - \|x_k - x_0\| \geq \chi - t_k > 0$ ($t_k < \chi$ for all $k = 0, 1, \ldots$ as it was shown in lemma 1). Because of the concavity and the monotonicity of ω, we have

$$\omega\left(\left(\chi - \|x_k - x_0\|\right)^+ + \|x_k - x_0\|\right) - \omega\left(\left(\chi - \|x_k - x_0\|\right)^+\right) \leq$$

$$\leq \omega(\chi - t_k + t_k) - \omega(\chi - t_k) < \omega(\chi) - \omega(0) = \omega(\chi) \leq 1.$$

Thus, $\left\|\left[f'(x_0)\right]^{-1}\left(f'(x_k) - f'(x_0)\right)\right\| < 1$ and, consequently, the operator

$$T = I + \left[f'(x_0)\right]^{-1}\left(f'(x_k) - f'(x_0)\right)$$

is invertible. Since $f'(x_k) = f'(x_0)T = T$, the operator $f'(x_k)$ is also invertible and

$$\left\|\left[f'(x_k)\right]^{-1}\right\| = \left\|T^{-1}\right\| \leq \frac{1}{1 - \|T - I\|} \leq \frac{1}{1 - \left[\omega(\chi) - \omega(\chi - t_k)\right]}.$$

Using the estimate for r_k from lemma 3 and the inequality (11), we get

$$\|x_{k+1} - x_k\| = \left\|\left[f'(x_k)\right]^{-1}\left(f(x_k) + g(x_k)\right)\right\| =$$

$$= \left\|\left[f'(x_k)\right]^{-1}\left(f(x_k) - f(x_{k-1}) - f'(x_{k-1})(x_k - x_{k-1}) + g(x_k) - g(x_{k-1})\right)\right\| \leq$$

$$\leq \left\|\left[f'(x_k)\right]^{-1}\right\|\left\|f(x_k) - f(x_{k-1}) - f'(x_{k-1})(x_k - x_{k-1})\right\| + \left\|\left[f'(x_k)\right]^{-1}\right\|\left\|g(x_k) - g(x_{k-1})\right\| \leq$$

$$\leq \frac{r_k + \Psi(t_k) - \Psi(t_{k-1})}{1 - \left[\omega(\chi) - \omega(\chi - t_k)\right]} \leq \frac{a - \Omega(\chi) + \Omega(\chi - t_k) - t_k(1 - \omega(\chi)) + \Psi(t_k)}{1 - \left[\omega(\chi) - \omega(\chi - t_k)\right]} = t_{k+1} - t_k.$$

Consequently, the inequality (17) holds for $n = k$.

Since for all $n = 0, 1, \ldots$ the operator $f'(x_n)$ is invertible and $\|x_n - x_0\| \leq t_n \leq t_*$, the successive approximations (2) are defined for all $n = 0, 1, \ldots$ and belong to $\overline{B(x_0, t_*)}$. The convergence of the successive approximations to x_* follows from the inequality (18). The theorem is proved.

Conclusions

In this paper the generalised Newton – Kantorovich method for solving non-linear operator equations with non-differentiable operators in Banach spaces was considered. The regular smoothness condition of the operator involved, which was proposed by A. Galperin and Z. Waksman, was replaced by a simpler one in which increments of the operator derivative are majorised by increments of a scalar function. The convergence theorem was proved by means of majorant scalar equations.

It should be noted that each Lipschitz smooth operator is also regularly smooth but the opposite is not true. So the theorem is applicable to more wide class of non-linear operator equations of the form (1) than the corresponding convergence theorems from articles [1; 3].

References

1. Забрейко ПП, Злепко ПП. Об обобщении метода Ньютона – Канторовича на уравнения с недифференцируемыми операторами. *Украинский математический журнал.* 1982;34(3):365–369.

2. Kantorovich LV, Akilov GP. *Functional analysis in normed spaces*. Moscow: Fizmatgiz; 1959. 684 p. Russian.

3. Zabrejko PP, Nguen DF. The majorant method in the theory of Newton – Kantorovich approximations and the Pták error estimates. *Numerical Functional Analysis and Optimization*. 1987;9(5–6):671–684. DOI: 10.1080/01630568708816254.

4. Galperin A, Waksman Z. Newton's method under a weak smoothness assumption. *Journal of Computational and Applied Mathematics*. 1991;35(1–3):207–215. DOI: 10.1016/0377-0427(91)90208-2.

5. Galperin A, Waksman Z. Regular smoothness and Newton's method. *Numerical Functional Analysis and Optimization*. 1994; 15(7–8):813–858. DOI: 10.1080/01630569408816595.

6. Таныгина АН. Обобщенный метод Ньютона – Канторовича для уравнений с недифференцируемыми операторами. *Доклады Национальной академии наук Беларуси.* 2011;55(6):17–22. EDN: YPSBZL.

7. Забрейко ПП, Таныгина АН. Модификация условия Гальперина – Ваксмана для решения нелинейных операторных уравнений методом Ньютона – Канторовича. *Доклады Национальной академии наук Беларуси.* 2013;57(6):8–12. EDN: WIQWOR.

8. Таныгина АН. Сравнительный анализ условий сходимости метода Ньютона – Канторовича для приближенного решения нелинейных операторных уравнений. *Вестник Белорусского государственного университета. Серия 1, Физика. Математика. Информатика.* 2014;2:97–103.

Received 11.05.2025 / revised 15.11.2025 / accepted 15.11.2025.