$U(1)$ gauged nontopological solitons in the $3+1$ dimensional $O(3)$ sigma-model

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We present and study new nontopological soliton solutions in the $U(1)$ gauged nonlinear $O(3)$ sigma-model with a symmetry breaking potential in $3+1$ dimensional flat space-time. The configurations are endowed with an electric and magnetic field and also carry a nonvanishing angular momentum density. We discuss properties of these solitons and investigate the domains of their existence.

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I. INTRODUCTION

The nonlinear $O(3)$ sigma-model was proposed a long time ago by Gell-Mann and Levy [1] in the context of a theoretical description of the low-energy dynamics of pions. Nowadays, it is considered as a well-known prototype for a large class of field theories supporting topological solitons; see, e.g., Refs. [2,3]. The field of the $O(3)$ sigma-model is restricted to a sphere S^2; in $2+1$ dimensions, it is a topological map $S^2 \mapsto S^2$ which belongs to an equivalence class characterized by the homotopy group $\pi_2(S^2) = \mathbb{Z}$. Hence, there are soliton solutions of the model [4]. Further, in $3+1$ dimensions, the field configurations of the nonlinear $O(3)$ sigma-model define a map $\mathbb{R}^3 \mapsto S^2$, which is characterized by the third homotopy group $\pi_3(S^2)$.

However, in $3+1$ dimensions, Derrick's theorem [5] does not allow for the existence of such static finite energy solitons in the usual $O(3)$ sigma-model, which includes only the quadratic in derivative terms. Such topological solitons, referred as Hopfions, appear in the scale-invariant Nicole model [6] and, more importantly, in the Faddeev-Skyrme model [7], which contains terms with both two and four derivatives.[1] Apart from the addition of higher-derivative terms, no other mechanism to secure stability of the regular topological solitons of the $O(3)$ sigma-model in three spacial dimensions is known. Inclusion of the potential term [8] alone cannot stabilize the configuration, as well as gauging of the $U(1)$ subgroup of the $O(3)$ symmetry and adding the Maxwell term to the Lagrangian [9,10].

The global symmetry of the $O(3)$ sigma-model admits internal rotations of the components of the real scalar field; it may allow one to circumvent the restrictions of Derrick's theorem [11]. So-called Q-lumps in $2+1$ dimensions are isorotating solitons which carry both Noether and topological charges [11,12].

However, isorotations of the Hopfions are allowed only in the presence of both potential and higher-derivative terms [13,14]. On the other hand, stable isospinning nontopological solitons, Q-balls, can exist in a general class of scalar field theories with a global $U(1)$ symmetry [15–17], in particular, in the nonlinear $O(3)$ sigma-model with a non-negative potential [18]. Nonetheless, it was pointed out that the usual weakly attractive "pion mass" potential (2) cannot stabilize isorotating $O(3)$ configurations in $3+1$ dimensions [19], unless gravitational attraction is included. However, the virial theorem does not exclude the existence of regular isospinning $O(3)$ solitons stabilized by internal

[1] There is a vast amount of literature on solitons in the $2+1$ dimensional nonlinear $O(3)$ sigma-model and its generalizations, which we will not discuss here.

rotation in the flat space [19]; a certain modification of the potential term may allow for construction of such nontopological solitons in $3+1$ dimensional space [18,20].

Further, the global $U(1)$ symmetry of a model supporting Q-balls can be promoted to a local gauge symmetry, with the corresponding gauged Q-balls possessing electric charge [16,21–24]. The presence of the long-range gauge field affects the properties of the solitons, as the gauge coupling increases; the electromagnetic repulsion may destroy configurations.

The objective of this paper is to analyze properties of the regular solutions of the $U(1)$ gauged nonlinear $O(3)$ sigma-model with modified potential in $3+1$ dimensional flat space, focusing our study on nontopological localized configurations of different types, and determine their domains of existence.

This paper is organized as follows. In Sec. II, we introduce the model and the field equations with the stress-energy tensor of the system of interacting fields. Here, we describe the axially-symmetric parametrization of the matter fields and the boundary conditions imposed on the configuration. We also discuss the physical quantities of interest. In Sec. III, we present the results of our study with particular emphasis on the role of the electromagnetic interaction. We conclude with a discussion of the results and final remarks.

II. MODEL

A. Action and the field equations

We consider the $U(1)$ gauged nonlinear sigma-model in a $(3+1)$ dimensional Minkowski space-time. The corresponding matter field Lagrangian of the system is

$$L_m = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\phi^a)^2 - U(\phi), \tag{1}$$

where the real triplet of the scalar fields ϕ^a, $a = 1, 2, 3$, is restricted to the surface of the unit sphere, $(\phi^a)^2 = 1$, and $U(\phi)$ is a symmetry breaking potential. In particular, we can consider simple "pion mass" potential

$$U(\phi) = \mu^2(1-\phi^3), \tag{2}$$

where a mass parameter μ is a positive constant.

The $U(1)$ field strength tensor is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and the covariant derivative of the field ϕ^a, that minimally couples the scalar field to the gauge potential, is

$$D_\mu(\phi^1 + i\phi^2) = \partial_\mu(\phi^1 + i\phi^2) - ieA_\mu(\phi^1 + i\phi^2),$$
$$D_\mu\phi^3 = \partial_\mu\phi^3, \tag{3}$$

with e being the gauge coupling. The vacuum boundary conditions are $\phi^a_\infty = (0,0,1), D_\mu\phi^a = 0, F_{\mu\nu} = 0$. In the static gauge, where the fields have no explicit dependence on time, the asymptotic boundary conditions on the gauge potential are

$$A_0(\infty) = V, \qquad A_i(\infty) = 0, \tag{4}$$

where V is a real constant. The electromagnetic and scalar components of the energy-momentum tensor are, respectively,

$$T^{Em}_{\mu\nu} = F^\rho_\mu F_{\nu\rho} - \frac{1}{4}\eta_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma},$$
$$T^\phi_{\mu\nu} = D_\mu\phi^a D_\nu\phi^a - \eta_{\mu\nu}\left[\frac{\eta^{\rho\sigma}}{2}D_\rho\phi^a D_\sigma\phi^a + U(\phi)\right], \tag{5}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ is the Minkowski metric.

The potential $U(|\phi|)$ breaks $O(3)$ symmetry to the subgroup $SO(2) \sim U(1)$; the model (1) is invariant with respect to the local Abelian gauge transformations

$$(\phi^1 + i\phi^2) \to e^{ie\zeta}(\phi^1 + i\phi^2), \qquad A_\mu \to A_\mu + \partial_\mu\zeta, \tag{6}$$

where ζ is a real function of the coordinates. The corresponding Noether current can be written as follows:

$$j^\mu = \phi^1 D^\mu\phi^2 - \phi^2 D^\mu\phi^1. \tag{7}$$

Variation of the Lagrangian (1) with respect to the fields A_μ and ϕ_a leads to the equations

$$\partial_\mu F^{\mu\nu} = e j^\nu,$$
$$D^\mu D_\mu\phi^a + \phi^a(D^\mu\phi^b D_\mu\phi^b) + \mu^2[\phi^a(\phi^b\phi^b_\infty) - \phi^a_\infty] = 0, \tag{8}$$

where we take into account dynamical constraint imposed on the $O(3)$ field and make use of the explicit form of the potential (2).

We remark that the asymptotic value of the electric potential $A_0(\infty)$ can be adjusted via the residual $U(1)$ transformations, choosing $\zeta = -Vt$. In the stationary gauge, $A_0(\infty) = 0$, and two components (6) of the scalar triplet acquire an explicit harmonic time dependence with frequency $\omega = eV$.

B. Virial identity

The Derrick's theorem [5] indicates that the usual nonlinear $O(3)$ sigma-model does not possess stable solitonic solutions in $3+1$ dimensional flat space. By contrast, in the $U(1)$ gauged model (1), the usual scaling arguments suggest that localized regular solutions may be allowed. The total energy functional of the system can be written as

$$E = E_2 + E_0 + E_4, \tag{9}$$

where

$$E_2 = \frac{1}{2}\int d^3x D_i\phi^a D_i\phi^a, \qquad E_0 = \int d^3x U(\phi),$$
$$E_4 = \frac{1}{2}\int d^3x (E_i^2 + B_i^2), \tag{10}$$

and $E_i = F_{0i}$ and $B_i = \frac{1}{2}\varepsilon_{ijk}F_{jk}$ are the electric and magnetic fields, respectively.

The critical points of the total energy functional (9) should satisfy the arguments of Derrick's theorem. The scale transformation $x^i \to x^{i\prime} = \lambda^{-1}x^i$ does not affect the scalar field $\phi^a \to \phi^a$ because of the sigma-model constraint, and $A_\mu(x_i) \to A'_\mu(x'_i) = \lambda A_\mu(x_i)$. Then,

$$E(\lambda) = \lambda^{-1}E_2 + \lambda E_4 + \lambda^{-3}E_0, \tag{11}$$

and $\partial^2_\lambda E(\lambda = 1) = 2(E_2 + 6E_0) \geq 0$. The corresponding virial identity follows from the condition $\partial_\lambda E|_{(\lambda=1)} = 0$; it gives

$$E_2 + 3E_0 = E_4, \tag{12}$$

which suggests a possibility of the existence of soliton solutions stabilized by electromagnetic field [25] (see also Refs. [26–29]).

We also note that nontopological solitons of the model (1) in Minkowski space may exist also in the limiting case of vanishing electromagnetic interactions since the internal rotations with a nonzero angular frequency ω stabilize the configuration in the same way as happens for the usual scalar Q-balls [11]. Indeed, the total energy of a stationary spinning configuration also contains the term $E_\omega = \omega^2\Lambda/2$, where $\Lambda = \int d^2x(\phi^1)^2 + (\phi^2)^2$ is the moment of inertia. This term yields an effective (tachionic) mass to the field f. Then, under the scaling transformations in $d = 3$,

$$E(\lambda) = \lambda^{-1}E_2 + \lambda E_4 + \lambda^{-3}E_0 + \lambda^3 E_\omega, \tag{13}$$

and the corresponding virial relation becomes

$$E_2 + 3E_0 = \frac{3}{2}\omega Q,$$

where Q is the Noether charge corresponding to the current (7).

However, neither isorotations [19] nor electromagnetic interactions can support stable nontopological solitons in the $O(3)$ sigma-model. Eventually, there exists yet another loophole in the no-go arguments based on Derrick's theorem. The scalar potential may be taken as the difference of positive definite terms, $U(\phi) = U(\phi)^{(1)} - U(\phi)^{(2)}$; then, in the absence of the Maxwell term and isorotations, the corresponding virial identity becomes

$$E_2 + 3E_0^{(1)} = 3E_0^{(2)}, \tag{14}$$

where both $E_0^{(1)}$ and $E_0^{(2)}$ are positive definite. To secure existence of solitonic solutions of the model (1), the underlying potential also must possess two minima of different depths [18]. Both electromagnetic interactions and isorotations may further affect such a solution.

C. $\mathbb{C}P^1$ formulation

To gain some insight into the construction of soliton solutions of the model (1) in the flat space-time, we make use of inhomogeneous coordinates on $\mathbb{C}P^1$,

$$u = \frac{\phi^1 + i\phi^2}{1 - \phi^3}; \qquad \vec{\phi} = \frac{1}{1 + |u|^2}(u + u^*, -i(u - u^*), |u|^2 - 1). \tag{15}$$

Then, we can recast[2] the Lagrangian (1) in terms of the $\mathbb{C}P^1$ fields

$$L = -\frac{1}{8}F_{\mu\nu}F^{\mu\nu} + \frac{(D_\mu u)(D^\mu u)^*}{(1 + |u|^2)^2} - U(|u|). \tag{16}$$

Here, $D_\mu u = \partial_\mu u - ieA_\mu u$, under the $U(1)$ gauge transformation (6) $u \to e^{ie\zeta(x)}u$, $A_\mu \to A_\mu + \partial_\mu\zeta$, the derivative $D_\mu u$ transforms covariantly.

Variation of the Lagrangian (16) with respect to the field u^* yields the equation

$$(1 + |u|^2)D_\mu D^\mu u - 2u^*(D_\mu u)(D^\mu u) + u(1 + |u|^2)^3\frac{\delta U}{\delta |u|^2} = 0. \tag{17}$$

The corresponding equation for the electromagnetic field is

$$\partial_\mu F^{\mu\nu} = ej^\nu, \tag{18}$$

where the conserved current (7) is rewritten as

$$j^\nu = 2i\frac{[u(D^\nu u)^* - u^*(D^\nu u)]}{(1 + |u|^2)^2}. \tag{19}$$

To construct solutions of Eqs. (17) and (18), we may employ the polar decomposition of the complex field

$$u = Fe^{i\Theta}, \tag{20}$$

where both the phase $\Theta \in [0, 2\pi]$ and the amplitude $F \in [0, \infty]$ are real functions of the coordinates in $3 + 1$ dimensional space-time. Note that the original triplet of real scalar fields then can be written as

[2]Note that $L = L_m/2$.

$$\vec{\phi} = \left(\frac{2F}{1+F^2}\cos\Theta, \frac{2F}{1+F^2}\sin\Theta, \frac{F^2-1}{F^2+1}\right).$$

Now, denoting $F = \cot\frac{f}{2}$, $f \in [0, \pi]$, we arrive at the following "trigonometric" parametrization of the $O(3)$ field:

$$\vec{\phi} = (\sin f \cos\Theta, \sin f \sin\Theta, \cos f). \tag{21}$$

In terms of the trigonometric parametrization (21), the real and imaginary parts of the equation of motion (17) lead to the two equations

$$\partial_\mu^2 f - e^2 \sin f \cos f C_\mu^2 + \frac{\delta U}{\delta f} \tag{22}$$

and

$$\partial^\mu(\sin^2 f C_\mu) = 0, \tag{23}$$

with C_μ being the gauge-invariant vector field

$$C_\mu \equiv A_\mu - \frac{1}{e}\partial_\mu\Theta. \tag{24}$$

The equation for the electromagnetic field (18) becomes

$$\partial_\mu^2 C^\nu - \partial^\nu\partial^\mu C_\mu + e^2\sin^2 f C^\nu = 0. \tag{25}$$

Note that Eqs. (22), (23), and (25) are the same equations that one would get from the Lagrangian (1), by taking into account the constraint $\vec{\phi}^2 = 1$.

We point out that a term proportional to $(\partial_\mu f)^2$ in (22), that would come from the nonlinear term in (8) or equivalently from the second term in (17), vanishes due to the structure of the trigonometric parametrization (21).

III. RESULTS

A. Choice of the potential

In the unitary gauge, the phase function Θ can be gauged away by applying the transformation (6). However, we find it more convenient in the numerical analysis to keep the phase function. Further, let us consider, as a special case, a simple spherically symmetric ansatz parametrized in terms of the radial function $f(r)$ and the harmonic phase function which depends on time as $\Theta = -\omega t$. The only nonzero component of the gauge potential is a radially depending function $A_t(r)$, i.e.,

$$f \equiv f(r); \qquad eC_0 \equiv eA_t(r) + \omega;$$
$$C_i = 0; \qquad i = 1, 2, 3. \tag{26}$$

Note that the time-dependency of the $O(3)$ field disappears at the level of the dynamical equations and the energy-momentum tensor (5) does not depend upon time. Indeed, after inserting this ansatz into the field equations (22), (23), and (25), or equivalently (8), we obtain

$$f'' + \frac{2f'}{r} + (\omega + eA_t)^2 \sin f \cos f - \frac{\delta U}{\delta f} = 0,$$
$$A_t'' + \frac{2A_t'}{r} - e(\omega + eA_t)\sin^2 f = 0, \tag{27}$$

where a prime denotes the radial derivative and the symmetry breaking potential $U(f)$ depends on the amplitude of the field. Note that our ansatz implies that $\partial_\mu C^\mu = 0$ and $C^\mu\partial_\mu f = 0$. Therefore, Eq. (23) is automatically satisfied by the ansatz, and the middle term of (25) vanishes.

The simple spherically symmetric ansatz (26) is self-consistent, as the same equations can be derived via variation of the reduced Lagrangian

$$L_{\mathrm{eff}} = -\frac{1}{2}(A_t')^2 - \frac{1}{2}(\omega + eA_t)^2\sin^2 f + \frac{1}{2}(f')^2 - U(f)$$

with respect to the functions f and A_t.

The choice of the explicit form of the symmetry-breaking potential U has a dramatic effect on the solutions of the model (1). Considering, for example, the usual weakly attractive "pion mass" potential (2), we obtain two linearized asymptotic equations for the perturbative excitations of the fields around the vacuum $f(r) \to 0 + \rho(r) + \ldots, A_t(r) \to 0 + a(r) + \cdots$,

$$\rho'' - (\mu^2 - \omega^2)\rho = 0, \qquad a'' = 0. \tag{28}$$

Hence, a localized configuration of the $O(3)$ field with an exponentially decaying tail may exist if $\omega \leq \mu$. On the other hand, in the $3+1$ dimensional flat space-time, the electric potential remains massless.

However, numerical calculations indicate that for the pion mass potential (2) the virial identity (12) cannot be satisfied since the electrostatic energy always remains smaller than the energy of the $O(3)$ scalar field; cf. the related discussion in Ref. [19].

The situation changes dramatically when we consider a modified symmetry breaking potential (see Fig. 1),

$$\tilde{U}(\phi) = \mu^2(1 - \phi^3) - \beta(1 - \phi^3)^4$$
$$\equiv \mu^2(1 - \cos f) - \beta(1 - \cos f)^4, \tag{29}$$

where β is a real positive parameter. The potential (29), apart from the usual pion mass term, also contains a term, which yields an additional short-range interaction energy [30,31]. The corresponding additional term in the field equation decays faster than the contribution of the pion mass term; thus, the asymptotic equation (28) remains the same. Note that a similar choice of potential (with $\beta < 0$)

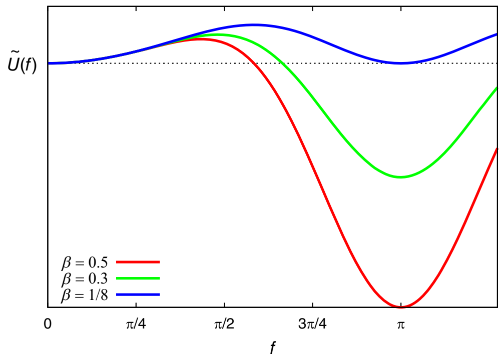

FIG. 1. The potential (29) is shown for some set of values of the parameter β and $\mu^2 = 1$.

has also been used in Refs. [29,31,32] to construct multi-soliton solutions of the Skyrme model with low binding energy.

Clearly, the modified potential $\tilde{U}(f)$ (29) displayed in Fig. 1 may support nontopological flat-space soliton solutions of the $U(1)$ gauged $O(3)$ sigma-model. It has a false minimum at $f = 0$ and a global minimum at $f = \pi$. Here, we stress an important difference between the non-topological solitons in the theory of a complex scalar field [17,33] and solutions of the $O(3)$ model with the symmetry breaking potential (29): in the former case, the true vacuum corresponds to the zero value of the field amplitude.

The depths of the global minimum at $f = \pi$ depends on the ratio of the values of the parameters β/μ^2, and for $\beta/\mu^2 = 1/8$, the vacuum becomes double degenerated. The soliton solution then represents an infinitely thin spherically symmetric wall separating two neighboring vacua $\tilde{U} = 0$ at $f_0^{(1)} = 0$ and $f_0^{(2)} = \pi$.

B. Ansatz and the boundary conditions

Nontopological solitons of the gauged $O(3)$ model are akin to the usual Q-balls in $3+1$ dimensional models of complex scalar field with a sextic potential [17,33]. Indeed, using the spherically symmetric ansatz (21) with $\Theta = -\omega t$, we can find regular finite energy solutions of the corresponding system of field equations (27) displayed in Fig. 2. In the time-dependent gauge, both profile functions are asymptotically vanishing, $f(\infty) = A_t(\infty) = 0$, and the Neumann boundary conditions at the origin are $\partial_r f(0) = \partial_r A_t(0) = 0$.

Axially symmetric solutions of the model (1) can be constructed using the stationary ansatz with a harmonic time dependence, which is similar to the corresponding parametrization of the $O(4)$ Skyrme model [34–37]

$$\vec{\phi} = (\sin f(r,\theta)\cos(n\varphi - \omega t), \sin f(r,\theta)\sin(n\varphi - \omega t), \cos f(r,\theta)), \tag{30}$$

where the profile function f depends on the radial coordinate r and the polar angle θ. In the spherically symmetric case ($n = 0$), this ansatz is reduced to (21). The corresponding ansatz for the $U(1)$ potential contains two real functions of an electric and a magnetic potential:

$$A_\mu dx^\mu = A_t(r,\theta)dt + A_\varphi(r,\theta)\sin\theta d\varphi. \tag{31}$$

Note that the ansatz (30) and (31) satisfies automatically Eq. (23). After substituting this ansatz, we get the following field equations:

$$\begin{aligned}
&\left(\partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \frac{\cos\theta}{r^2\sin\theta}\partial_\theta\right)f + \frac{1}{2}\left((eA_t + \omega)^2 - \frac{1}{r^2}\left(eA_\varphi + \frac{n}{\sin\theta}\right)^2\right)\sin 2f = \frac{\delta U}{\delta f},\\
&\left(\partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \frac{\cos\theta}{r^2\sin\theta}\partial_\theta - \frac{1}{r^2\sin^2\theta} - e^2\sin^2 f\right)A_\varphi = \frac{ne}{\sin\theta}\sin^2 f,\\
&\left(\partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \frac{\cos\theta}{r^2\sin\theta}\partial_\theta - e^2\sin^2 f\right)A_t = e\omega\sin^2 f.
\end{aligned} \tag{32}$$

The boundary conditions are $A_t = A_\varphi = 0$ and $f = 0$ at spacial infinity and $\partial_r A_t = A_\varphi = \partial_r f = 0$ at the origin (in the stationary gauge). The regularity of the solutions at the symmetry axis requires that ($n \neq 0$)

$$\partial_\theta f|_{\theta=0,\pi} = \partial_\theta A_t|_{\theta=0,\pi} = A_\varphi|_{\theta=0,\pi} = 0. \tag{33}$$

The stationary solutions are characterized by a set of physical observables, the mass M, the angular momentum J, the electric charge Q_e, and the dipole moment μ_m. Some of these quantities can be evaluated as volume integrals of the corresponding densities of the total stress-energy tensor $T_{\mu\nu} = T^{\phi}_{\mu\nu} + T^{Em}_{\mu\nu}$,

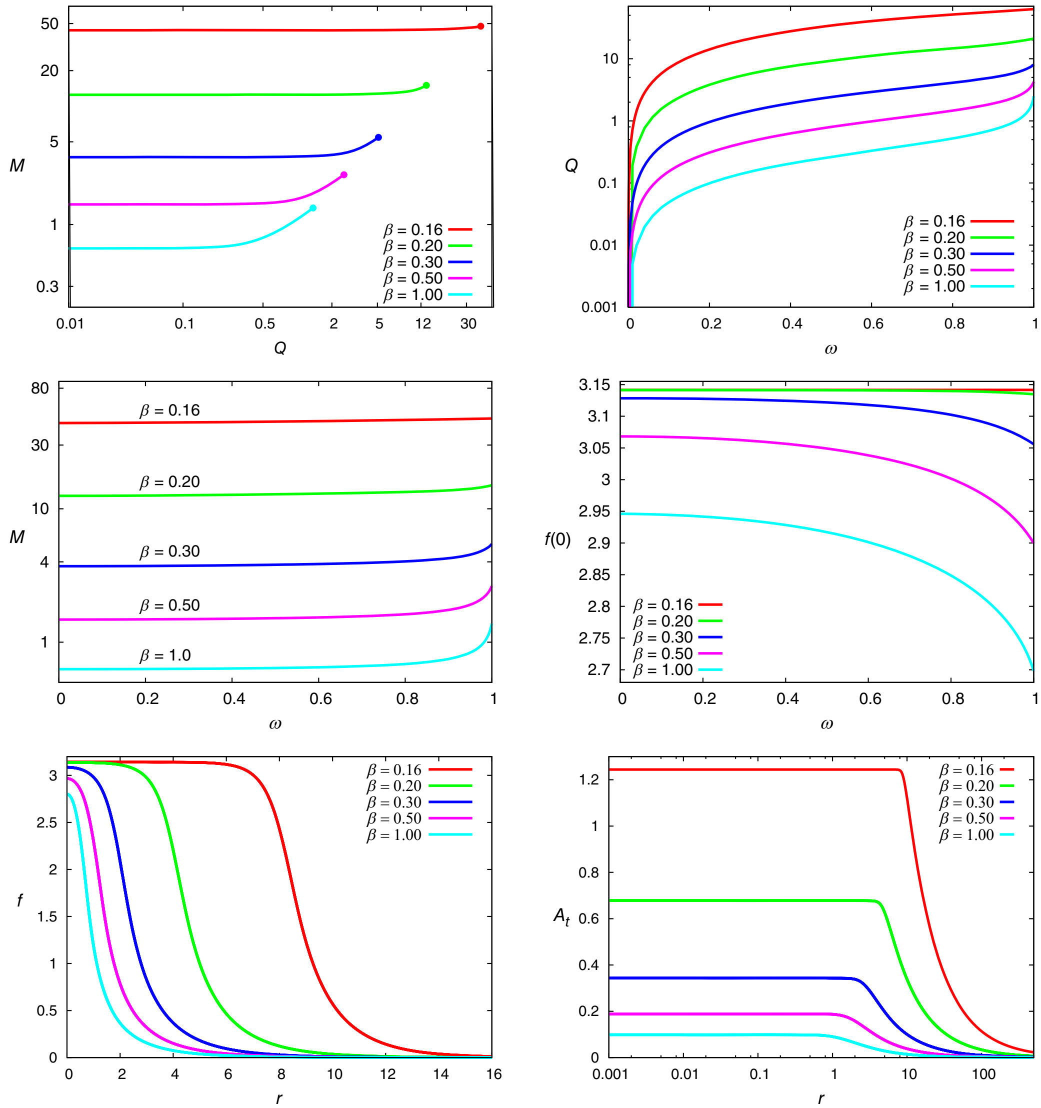

FIG. 2. Spherically symmetric $U(1)$ gauged $O(3)$ solitons: mass vs charge Q (upper left) and the charge Q vs frequency ω are shown for some set of values of the parameter β and $\mu^2 = 1$. The dots on the upper left plot indicate the limiting solutions with maximal mass and charge at the threshold $\omega = \mu$. Middle plots: mass M (left) and the central value of the profile function $f(0)$ (right) vs the angular frequency ω. Bottom plots: the radial profiles of the fields $f(r)$ and $A_t(r)$ are displayed for some set of values of the parameter β of the potential (29) at frequency $\omega = 0.90$ and gauge coupling $e = 0.1$.

$$M = \int d^3x(T^\mu_\mu - 2T^t_t), \qquad J = \int d^3xT^t_\varphi, \tag{34}$$

and taking into account (5), (30), and (31), we can write

$$\begin{aligned} M &= 2\pi \int_0^\pi \int_0^\infty r^2 \sin\theta dr d\theta \left(\partial_r f^2 + \frac{\partial_\theta f^2}{r^2} + \frac{1}{r^2}\left(eA_\varphi + \frac{n}{\sin\theta}\right)^2 \sin^2 f + (gA_t + \omega)^2 \sin^2 f + U + E_{em} \right), \\ E_{em} &= \frac{1}{2}\left((\partial_r A_0)^2 + \frac{1}{r^2}(\partial_\theta A_t)^2 + \frac{1}{r^2}(\partial_r A_\varphi)^2 + \frac{1}{r^4 \sin^2\theta}(\partial_\theta(A_\varphi \sin\theta))^2 \right). \end{aligned} \tag{35}$$

$$J = 2\pi \int_0^\pi \int_0^\infty r^2 \sin\theta dr d\theta ((eA_t + \omega)(n + eA_\varphi \sin\theta)\sin^2 f + J_{em}),$$
$$J_{em} = \sin\theta \partial_r A_t \partial_r A_\varphi + \frac{\partial_\theta A_t (A_\varphi \cos\theta + \sin\theta \partial_\theta A_\varphi)}{r^2}. \quad (36)$$

Also, they can be extracted from the asymptotic decay of the gauge field functions, as

$$A_t \to \frac{Q_e}{r} + O\left(\frac{1}{r^2}\right), \qquad A_\varphi \to \frac{\mu_m \sin^2\theta}{r^2} + O\left(\frac{1}{r^3}\right). \quad (37)$$

The associated Noether charge of the solitons of the $O(3)$ sigma-model can be evaluated as a volume integral over the temporal component of the conserved 4-current (7),

$$Q = \int d^3x j^0 = 2\pi \int_0^\infty r^2 dr \int_0^\pi \sin\theta d\theta \sin^2 f(\omega + eA_t), \quad (38)$$

where we make use of the parametrization of the $O(3)$ field (30). It can be shown that $T^t_\varphi \sqrt{-g} = -\partial_\mu[(A_\varphi \sin\theta + \frac{n}{e}) F^{\mu 4} r^2 \sin\theta]$ is the total derivative, $F^{\mu 4} = (-\partial_r A_t, -\frac{1}{r^2}\partial_\theta A_t, 0, 0)$. Also, $(T^\phi)^t_\varphi = (n + eA_\varphi \sin\theta) j^0$. Substituting it into (34) and taking into account the field equations (37) together with the boundary conditions (33) and (37), one finds that in the $U(1)$ gauged nonlinear sigma-model, J, Q, and the electric charge Q_e are proportional in the same way, as in the usual model with a single complex scalar field (see, e.g., Refs. [38–42]):

$$\begin{aligned} J &= \int d^3x\, T^t_\varphi \\ &= -2\pi \iint dr d\theta\, \partial_k \left[\left(A_\varphi \sin\theta + \frac{n}{e}\right) F^{kt} r^2 \sin\theta\right] \\ &= -2\pi \int d\theta \frac{n}{e} \frac{Q_e}{r^2} r^2 \sin\theta = -4\pi \frac{nQ_e}{e}, \end{aligned} \quad (39)$$

$$\begin{aligned} Q &= 2\pi \iint dr d\theta r^2 \sin\theta \sin^2 f(\omega + eA_t) \\ &= \frac{2\pi}{e} \iint dr d\theta r^2 \sin\theta \left(\partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \frac{\cos\theta}{r^2 \sin\theta}\partial_\theta\right) A_t \\ &= -\frac{2\pi}{e} \int_0^\pi d\theta r^2 \sin\theta \frac{Q_e}{r^2} = -\frac{4\pi Q_e}{e}. \end{aligned} \quad (40)$$

Hence,

$$J = nQ = \frac{nQ_e}{e}, \quad (41)$$

where n is the winding number of the scalar field. We made use of this relation to check correctness of our numerical results.

C. Stationary soliton solutions

In this section, we discuss properties of the soliton configurations. Solitons with winding number $n = 1$ are constructed numerically as solutions of the set of coupled partial differential equations (32); the spherically symmetric ($n = 0$) configurations are solutions of the reduced system of the radial equations (27). In general, the numerical calculations are performed on an equidistant grid in spherical coordinates r and θ, employing the compact radial coordinate $x = r/(C + r) \in [0:1]$, where C is an arbitrary constant used to adjust the grid according to the contraction of the solutions, and the polar angle $\theta \in [0:\pi]$. In our numerical calculations, we have made use of a sixth-order finite difference scheme, where the system of equations is discretized on a grid with a typical size of about 530 points in the radial direction. The emerging system of nonlinear algebraic equations has been solved using the Newton-Raphson scheme. Calculations have been performed by employing a professional solver [43], with typical errors of order of 10^{-5}.

There are a few important differences between the Q-balls in scalar theory and nontopological solitons of the $O(3)$ sigma-model. First, usual Q-balls exist for a finite interval of the frequency $\omega \in [\omega_{\min}, \mu]$, where a minimal allowed value of the frequency depends on the form of the potential. In particular, the minimal frequency can be zero, as happens in the Fridberg-Lee-Sirlin model in Minkowski space-time [15]. In the model under consideration, there is no lower bound on the frequency, and the solutions exist for all range of values of the angular frequency. The limiting fundamental solution at $\omega = 0$ is uncharged; it corresponds to the nontopological soliton stabilized by the balance of repulsive and attractive scalar interactions [18].

For both spherically ($n = 0$) and axially ($n = 1$) symmetric configurations, there is a single branch of solutions which extends all the way up from $\omega = 0$ to the mass threshold μ, the charge Q, as a function of the angular frequency, increases monotonically from zero to some maximal value $Q_{\max}$ as the frequency approaches the upper bound; see Figs. 2 and 6, right upper plots. This value depends on the shape of the potential $\tilde{U}$ (29).

In Fig. 2, middle plots, we present the total mass M of the $n = 0$ solutions and the central value of the profile function $f(0)$ versus the angular frequency ω for a set of values of the parameter β and fixed gauge coupling $e = 0.1$. The mass of the solitons weakly increases as the frequency grows, and the increase of the mass becomes more explicit for larger values of β. The central value of the function $f(0)$ decreases

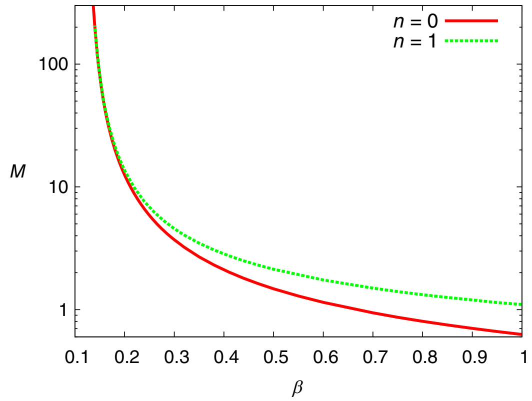

FIG. 3. The mass of the fundamental ($\omega = 0$, $e = 0$) soliton solutions of the model (1) with the potential (29) is shown as function of the parameter β for the spherical ($n = 0$) and the axial ($n = 1$) configurations.

with increase of ω; it approaches the vacuum $f(0) \sim \pi$ as the ratio β/μ^2 decreases toward the critical value $1/8$.

Note that the size of the soliton also depends of the ratio of the parameters β/μ^2; for a fixed value of the frequency ω, it increases as the potential becomes more shallow (see Figs. 2 and 6, which exhibit the profiles of the corresponding solutions). As $\beta/\mu^2 \to 1/8$, the potential becomes degenerated, $\tilde{U}(f) \underset{\beta\to 1/8}{\longrightarrow} 2\sin^2\frac{f}{2}(1-\sin^6\frac{f}{2})$, and the interior region of all solutions, for any values of the winding number, angular frequency, and the gauge coupling, turns into a rapidly expanding ball with the vacuum $f = \pi$ inside. Decreasing of the parameter β increases the mass of the configurations, as seen in the left upper plot of Figs. 2 and 6.

The mass of the solutions diverge in the limit $\beta/\mu^2 \to 1/8$; see Fig. 3.

Second, the Noether charge Q (38) cannot be interpreted as the particle number; the norm of the $O(3)$ field is fixed, and only two components of the triplet contribute to the charge (38). In Fig. 5, we displayed an isosurface of the charge density distribution of an illustrative $n = 1$ solution. Unlike the corresponding distribution of the charge density of an axially symmetric Q-ball in a scalar theory, which represent a torus, the charge distribution of the $O(3)$ soliton has a shell-like structure, as seen in Fig. 5, left plot. This is related with the form of the profile functions which are shown in Fig. 4. The central value of the profile function interpolates between $f|_0 \sim \pi$ and $f|_\infty = 0$; thus, the charge density (38) has a minimum at the center of the soliton, and it is vanishing on the spacial infinity with a maximum at the wall separating the vacua.

Indeed, the amplitude of the scalar field features a plateau around the center at which a false vacuum is formed; see the left bottom plot of Figs. 2 and 6. The plateau ends with two steps on the symmetry axis, where the field f drops to the true vacuum. The energy density distribution has two sharp peaks at these points, as seen in Fig. 5, right plot. Notice that the electric potential A_t is a constant in the interior of the soliton, as displayed in the Figs. 2 and 6, bottom right plots. Thus, the electric field of the gauged configuration is vanishing inside. Outside the core of the soliton, it possesses a Coulomb-like asymptotic tail. It corresponds to the long-range field of the charge $Q_e = eQ$. On the other hand, the magnetic field of the $n = 1$ gauged soliton is toroidal, as in the case of the usual gauged Q-balls.

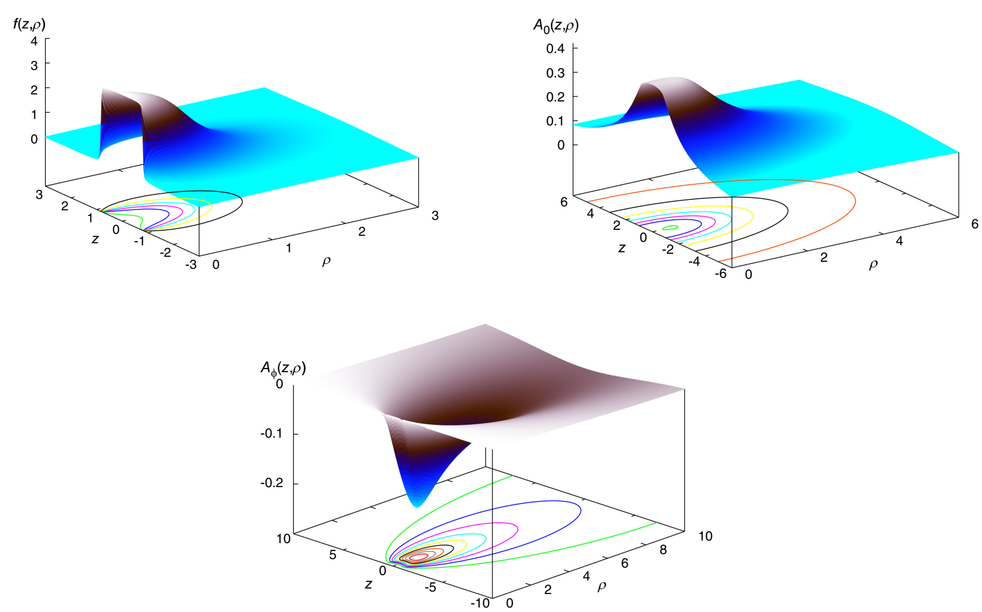

FIG. 4. Axially symmetric $U(1)$ gauged $O(3)$ solitons: profiles of the scalar field amplitude and the components of the electromagnetic potential are shown for an $n = 1$ solution of the model (1) with the potential (29) at $\omega = 0.90$ and $e = 0.1$. The axis are $\rho = r\sin\theta$ and $z = r\cos\theta$.

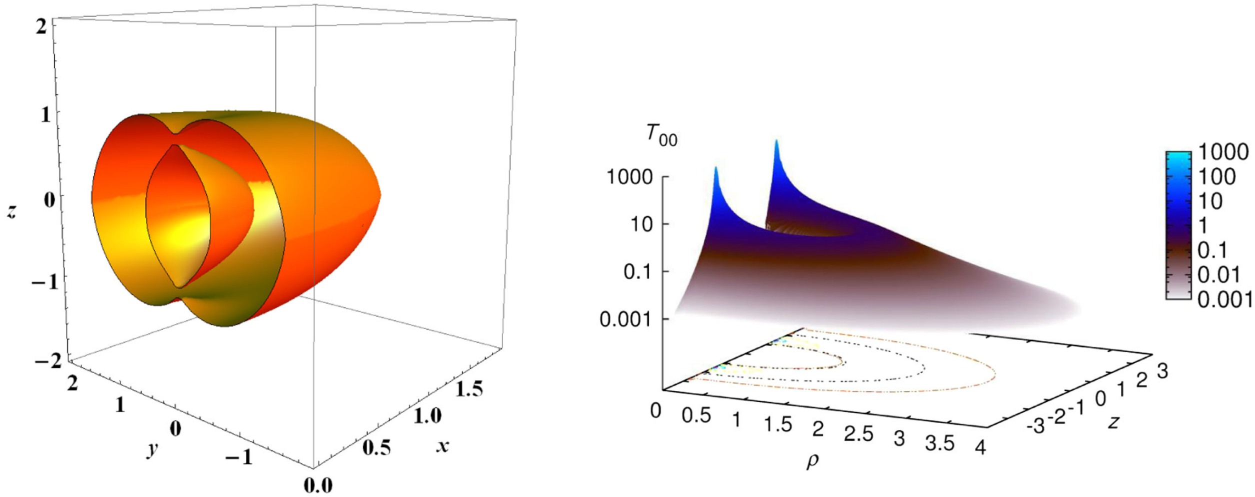

FIG. 5. Axially symmetric $U(1)$ gauged $O(3)$ solitons. Left plot: the charge Q isosurface at $1/5$ of the maximum density of an $n=1$ solution of the model (1) with the potential (29) at $\beta = 0.5$, $\omega = 0.90$, and $e = 0.1$. Right plot: total energy density of this solution is shown as function of the coordinates $\rho = r\sin\theta$ and $z = r\cos\theta$.

Third, in the case of the usual ungauged Q-balls in a complex scalar theory, the size of the soliton grows indefinitely as the angular frequency approaches the lower minimal value; the value of the field amplitude f at the origin rapidly increases, and both the mass and the charge of the configuration diverge in this limit.

In the case of solitonic solutions of the $O(3)$ model, the maximum of the field amplitude $f(0)$ is restricted by the constraint. When ω is decreased, it slowly approaches its maximal value, which remains slightly below $f_{cr} = \pi$, even in limit $\omega \to 0$, while the interior region of the soliton is slowly increasing. Thus, in this limit, the mass of the

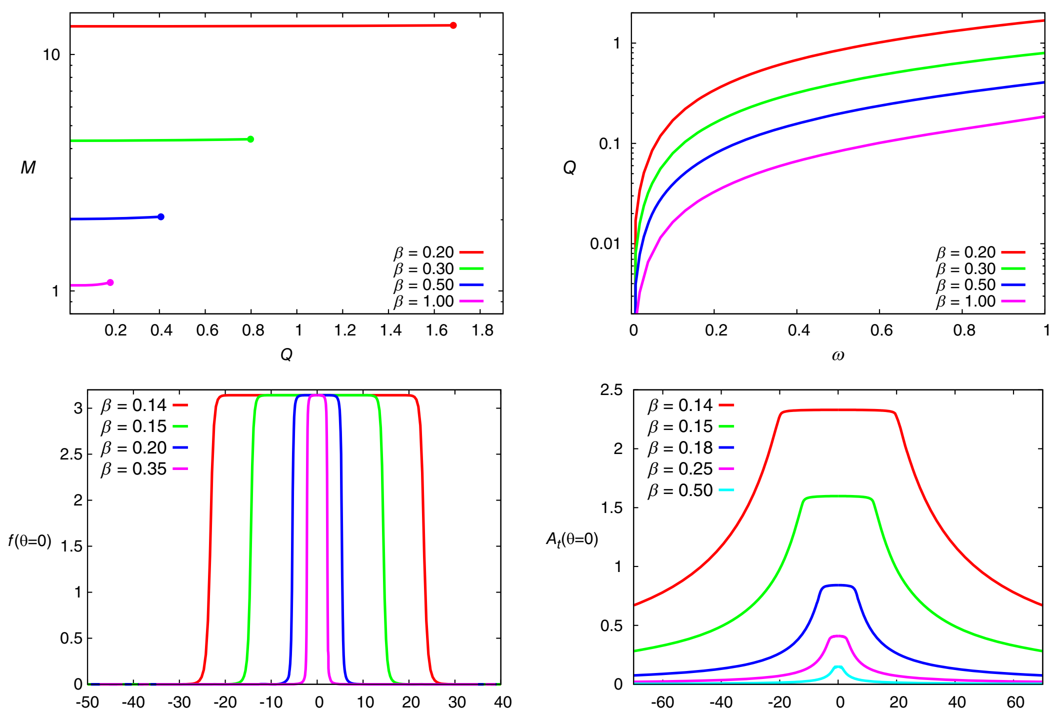

FIG. 6. Axially symmetric $n=1$ $U(1)$ gauged $O(3)$ solitons. Mass vs charge Q (upper left) and the charge Q vs frequency ω (upper right) are shown for some set of values of the parameter β and $\mu^2 = 1$. The dots on the upper left plot indicate the limiting solutions with maximal mass and charge at the threshold $\omega = \mu$. The profiles of the field $f(r)$ of the fundamental ($\omega = 0$, $e = 0$) solution (bottom left) and the profiles of the field $A_t(r)$ at frequency $\omega = 0.90$ and gauge coupling $e = 0.1$ (bottom right) on the symmetry axis are displayed for some set of values of the parameter β of the potential (29).

configuration remains finite, albeit the Noether charge $Q \sim \omega$ is vanishing.

Fourth, the usual evolution pattern of the gauged Q-balls in a complex scalar theory is that there are two branches of solutions, the first of which arises from the pertubative excitations as the angular frequency is decreasing below the mass threshold [21–24]. A bifurcation with the second, upper in energy branch occurs at some minimal nonzero value of the angular frequency. The second branch of gauged Q-balls extends forward up to the maximal value of the frequency; along this branch, the characteristic size of the gauged Q-balls rapidly increases, and it inflates as ω approaches its upper critical value. The minimal critical value of the frequency increases with the increase of the gauge coupling e; an additional repulsive interaction arises from the gauge sector, and the solutions cease to exist at some maximal critical value of the gauge coupling.

On the contrary, there is only one branch of charged Q-balls in the $O(3)$ sigma-model; the increase of the gauge coupling affects the value of the Noether charge and drives the configurations closer to the limit $f_{cr} = \pi$. However, there is no second branch of solutions for any values of the gauge coupling and the parameters of the potential μ and β.

Fifth, in the theory of a complex scalar field, Q-balls typically arise smoothly from perturbative excitations about the vacuum, as the angular frequency is decreasing below the mass threshold, and scalar quanta condense into a nontopological soliton. It is not the case for the solitons of the $O(3)$ sigma-model; they are disconnected from the vacuum excitations for most of the range of values of the potential parameter β.

Finally, we note that stable soliton solutions of the model do not exist as the ratio β/μ^2 becomes smaller than $1/8$, so there are no flat space solutions in the gauged $O(3)$ sigma-model with the pion mass potential.

IV. CONCLUSIONS

The main purpose of this paper is to show the existence of new type of nontopological soliton solutions of the $U(1)$ gauged nonlinear $O(3)$ sigma-model in $3+1$ dimensional flat space-time. Unlike usual Q-balls in a complex field theory, they are stabilized by the special choice of the symmetry breaking potential rather than isorotations. Considering dependency of the solutions on the frequency, we observe only one branch of charged Q-balls in the $O(3)$ sigma-model; it originates from static ($\omega = 0$) configuration and extends up to the mass threshold.

The solitons of the $U(1)$ gauged nonlinear $O(3)$ sigma-model exhibit examples of the configurations with a quantized angular momentum, $J = nQ$, and with both the electric charge and toroidal magnetic field, which forms a vortex encircling the soliton.

The work here should be taken further by considering a family of potentials which would support nonzero vacuum expectation value of the scalar field. It may allow one to introduce Higgs-like mechanism providing a mass to the gauge field. Another direction can be related with investigation of properties of the self-gravitating soltions in the $U(1)$ gauged nonlinear $O(3)$ sigma-model minimally coupled to Einstein gravity. We hope to address these problems in our future work.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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