BLOW-UP PROBLEM FOR POROUS MEDIUM EQUATION WITH ABSORPTION UNDER NONLINEAR NONLOCAL BOUNDARY CONDITION

ALEXANDER GLADKOV

Abstract. In this paper, we consider an initial boundary value problem for the porous medium equation with absorption under a nonlinear nonlocal boundary condition and a nonnegative initial datum. We prove the local existence of solutions, establish a comparison principle, and demonstrate both global existence and blow-up of solutions.

1. Introduction

We consider the initial boundary value problem for the nonlinear parabolic equation

$$u_t = \Delta u^\mu - a u^\nu,\ x \in \Omega,\ t > 0, \tag{1.1}$$

with a nonlinear nonlocal boundary condition

$$\frac{\partial u(x,t)}{\partial \mathbf{n}} = \int_\Omega k(x,y,t) u^l(y,t)\,dy,\ x \in \partial\Omega,\ t > 0, \tag{1.2}$$

and initial datum

$$u(x,0) = u_0(x),\ x \in \Omega, \tag{1.3}$$

where $\mu > 1,$ and a, ν, l are positive numbers, Ω is a bounded domain in $\mathbb{R}^N$ for $N \geq 1$ with smooth boundary $\partial\Omega$, $\mathbf{n}$ is the unit outward normal on $\partial\Omega$.

Throughout this paper we suppose that nonnegative functions $k(x,y,t)$ and $u_0(x)$ satisfy the following conditions

$$k(x,y,t) \in L^\infty_{loc}(\partial\Omega \times \overline{\Omega} \times [0,\infty)),\ u_0(x) \in L^\infty(\Omega).$$

Various phenomena in the natural sciences and engineering lead to the nonclassical mathematical models subject to nonlocal boundary conditions. For global existence and blow-up of solutions for parabolic equations and systems with nonlocal boundary conditions we refer to [1] – [16] and the references therein. In particular, the blow-up problem for parabolic equations with nonlocal boundary condition

$$u(x,t) = \int_\Omega k(x,y,t) u^l(y,t)\,dy,\ x \in \partial\Omega,\ t > 0,$$

was considered in [17] – [24]. Initial boundary value problems for parabolic equations with nonlocal boundary condition (1.2) were addressed in many papers also (see, for example, [25] – [30]). So, the problem (1.1)–(1.3) with $\mu = 1$ was studied in [31, 32]. Uniqueness and blow-up problems for the porous medium equation with absorption and local nonlinear boundary condition were analyzed in [33].

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The aim of this paper is to investigate the global existence and blow-up of solutions to (1.1)–(1.3).

This paper is organized as follows. In the next section we prove local existence of solutions. A comparison principle is established in Section 3. We provide a general analysis of the blow-up problem in the last two sections. The global existence of solutions for any initial data is proved in Section 4. In Section 5 we present results on finite time blow-up.

2. Local existence

In this section, we will prove the local existence of solutions to (1.1)–(1.3). We begin with the definitions of a supersolution, a subsolution and a solution of (1.1)–(1.3). Let $Q_T = \Omega \times (0,T)$, $S_T = \partial\Omega \times (0,T)$, $T > 0$.

Definition 2.1. We say that a nonnegative function $u(x,t) \in C([0,T]; L^1(\Omega)) \cap L^\infty(Q_T)$ is a supersolution of (1.1)–(1.3) in Q_T if

$$\int_\Omega u(x,t)\varphi(x,t)dx \geq \int_\Omega u_0(x)\varphi(x,0)dx + \int_0^t \int_{\partial\Omega} \varphi(x,\tau) \int_\Omega k(x,y,\tau)u^l(y,\tau)dydS_x d\tau$$

$$+ \int_0^t \int_\Omega \left[u(x,\tau)\varphi_\tau(x,\tau) + u^\mu(x,\tau)\Delta\varphi(x,\tau) - au^\nu\varphi(x,\tau)\right] dxd\tau \qquad (2.1)$$

for every $t \in (0,T]$ and every nonnegative function $\varphi(x,t) \in C^{2,1}(Q_T) \cap C^{1,0}(\overline{Q_T})$ such that $\varphi_t, \Delta\varphi \in L^2(Q_T)$ and $\frac{\partial\varphi(x,t)}{\partial\nu} = 0$ for $(x,t) \in S_T$. A nonnegative function $u(x,t) \in C([0,T]; L^1(\Omega)) \cap L^\infty(Q_T)$ is called a subsolution of (1.1)–(1.3) in Q_T if it satisfies (2.1) in the reverse order. We say that $u(x,t)$ is a solution of problem (1.1)–(1.3) in Q_T if $u(x,t)$ is both a subsolution and a supersolution of (1.1)–(1.3) in Q_T.

From [34, 35] we immediately infer the following lemma.

Lemma 2.2. *There exists a sequence of positive functions $u_{0m}(x) \in L^\infty(\Omega)$, $m \in \mathbb{N}$, possessing the following properties:*

$u_{0(m+1)}(x) \leq u_{0m}(x)$ *and* $u_{0m}(x) \to u_0(x)$ *as* $m \to \infty$ *almost everywhere (a.e.) in* Ω,

$$\frac{1}{m} \leq u_{0m}(x) \text{ a.e. in } \Omega.$$

For every $m \in \mathbb{N}$, there is a sequence of positive functions $u_{0mj}(x) \in C(\Omega)$, $j \in \mathbb{N}$, satisfying the conditions

$$u_{0(m+1)j}(x) \leq u_{0mj}(x) \leq u_{0m(j+1)}(x) \text{ in } \Omega,$$

$$u_{0mj}(x) \to u_{0m}(x) \text{ as } j \to \infty \text{ a.e. in } \Omega,$$

$$\frac{1}{m} \leq u_{0mj}(x) \text{ in } \Omega.$$

Theorem 2.3. *Problem (1.1)–(1.3) has a solution in Q_T for small values of T.*

Proof. From Lemma 2.2 it is easy to deduce that

$$u_{0mj}(x) \leq \operatorname*{ess\,sup}_\Omega u_{01}(x),\ m, j \in \mathbb{N}.$$

We set $u_{m0}(x,t) \equiv 0$ and consider the following initial boundary value problem for $m, j \in \mathbb{N}$

$$\begin{cases} L_m u_{mj} \equiv u_{mjt} - \Delta u_{mj}^{\mu} + a u_{mj}^{\nu} - a/m^{\nu} = 0 & \text{for } (x,t) \in Q_T, \\ \frac{\partial u_{mj}(x,t)}{\partial \mathbf{n}} = \int_{\Omega} k(x,y,t) u_{m(j-1)}^{l}(y,t) dy & \text{for} (x,t) \in S_T, \\ u_{mj}(x,0) = u_{0mj}(x) & \text{for } x \in \Omega. \end{cases} \tag{2.2}$$

It is well known that problem (2.2) has a classical solution.

Let us consider the following auxiliary function:

$$w(x,t) = [1 - \alpha(\mu - 1)t]^{-\frac{1}{\mu-1}} \zeta(x),$$

where

$$\zeta(x) \in C^2(\overline{\Omega}), \ \inf_{\Omega} \zeta(x) \geq \max\{1, \operatorname*{ess\,sup}_{\Omega} u_{01}(x)\}, \ \alpha > \sup_{\Omega} \frac{|\Delta \zeta^{\mu}|}{\zeta},$$

$$\inf_{\partial\Omega} \frac{\partial \zeta(x)}{\partial \mathbf{n}} \geq \max\{1, 2^{(l-1)/(\mu-1)}\} \operatorname*{ess\,sup}_{\partial\Omega \times \Omega \times [0,1/\{2\alpha(\mu-1)\}]} k(x,y,t) \int_{\Omega} \zeta^{l}(y)\, dy.$$

It is easy to verify that $\underline{u}(x,t) = 1/m$ and $w(x,t)$ are subsolution and supersolution of (2.2) for $j = 1$ in Q_T with $T = 1/[2\alpha(\mu - 1)]$, respectively. By a comparison principle for (2.2) we have

$$\frac{1}{m} \leq u_{m1}(x,t) \leq w(x,t) \text{ in } Q_T, m \in \mathbb{N}.$$

Then using the induction on j and a comparison principle for (2.2), we deduce

$$\frac{1}{m} \leq u_{mj}(x,t) \leq w(x,t) \text{ in } Q_T, j \in \mathbb{N}. \tag{2.3}$$

Obviously,

$$u_{m1}(x,t) \geq u_{m0}(x,t) \text{ in } Q_T \text{ for } m \in \mathbb{N}.$$

Let us assume

$$u_{mj}(x,t) \geq u_{m(j-1)}(x,t) \text{ in } Q_T \text{ for } m \in \mathbb{N} \text{ and for some } j \in \mathbb{N}. \tag{2.4}$$

Using (2.2), (2.4) and Lemma 2.2, we obtain

$$L_m u_{m(j+1)}(x,t) = L_m u_{mj}(x,t) = 0 \text{ in } Q_T,$$

$$\frac{\partial u_{m(j+1)}(x,t)}{\partial \mathbf{n}} \geq \frac{\partial u_{mj}(x,t)}{\partial \mathbf{n}} \text{ on } S_T,$$

$$u_{0m(j+1)}(x) \geq u_{0mj}(x) \text{ in } \Omega.$$

Applying a comparison principle, we find that

$$u_{m(j+1)}(x,t) \geq u_{mj}(x,t) \text{ in } Q_T \text{ for } m, j \in \mathbb{N}.$$

We note that

$$L_m u_{m1}(x,t) = 0, \ L_m u_{(m+1)1}(x,t) = L_{m+1} u_{(m+1)1}(x,t) - \frac{a}{m^{\nu}} + \frac{a}{(m+1)^{\nu}} \leq 0 \text{ in } Q_T,$$

$$\frac{\partial u_{(m+1)1}(x,t)}{\partial \mathbf{n}} = \frac{\partial u_{m1}(x,t)}{\partial \mathbf{n}} = 0 \text{ on } S_T,$$

$$u_{0(m+1)1}(x) \leq u_{0m1}(x) \text{ in } \Omega.$$

Then by a comparison principle we obtain

$$u_{(m+1)1}(x,t) \leq u_{m1}(x,t) \text{ in } Q_T \text{ for } m \in \mathbb{N}.$$

In a similar manner, using the induction on j and a comparison principle, we deduce

$$u_{(m+1)j}(x,t) \le u_{mj}(x,t) \text{ in } Q_T \text{ for } m,j \in \mathbb{N}. \tag{2.5}$$

Multiplying the first equation in (2.2) by $\varphi(x,t)$ from Definition 2.1 and then integrating over Q_t for $t \in (0,T]$, we get

$$\begin{aligned} \int_\Omega u_{mj}(x,t)\varphi(x,t)\,dx &= \int_0^t \int_\Omega \left[u_{mj}\varphi_\tau + u_{mj}^\mu \Delta\varphi - a u_{mj}^\nu \varphi + \frac{a}{m^\nu}\varphi \right] dx d\tau \\ &+ \int_0^t \int_{\partial\Omega} \varphi(x,\tau) \int_\Omega k(x,y,\tau) u_{m(j-1)}^l (y,\tau)\, dy dS_x d\tau \\ &+ \int_\Omega u_{0mj}(x)\varphi(x,0)\,dx. \end{aligned}$$

Since the sequence $u_{mj}(x,t)$ is monotone in j and bounded, we can define

$$u_m(x,t) = \lim_{j\to\infty} u_{mj}(x,t), \tag{2.6}$$

and it is easy to see that $u_m(x,t)$ satisfies the following equation

$$\begin{aligned} \int_\Omega u_m(x,t)\varphi(x,t)\,dx &= \int_0^t \int_\Omega \left[u_m\varphi_\tau + u_m^\mu \Delta\varphi - a u_m^\nu \varphi + \frac{a}{m^\nu}\varphi \right] dx d\tau \\ &+ \int_0^t \int_{\partial\Omega} \varphi(x,\tau) \int_\Omega k(x,y,\tau) u_m^l (y,\tau)\, dy dS_x d\tau \\ &+ \int_\Omega u_{0m}(x)\varphi(x,0)\,dx. \end{aligned} \tag{2.7}$$

Moreover, from (2.3), (2.5), (2.6) we have

$$\frac{1}{m} \le u_m(x,t) \le w(x,t), \ \ u_{m+1}(x,t) \le u_m(x,t) \text{ in } Q_T \text{ for } m \in \mathbb{N}. \tag{2.8}$$

Now we define

$$u(x,t) = \lim_{m\to\infty} u_m(x,t) \tag{2.9}$$

and prove that $u(x,t) \in C([0,T]; L^1(\Omega))$. To show this, we integrate the first equation in (2.2) over Q_t for $t \in (0,T]$ to obtain

$$\begin{aligned} \int_\Omega u_{mj}(x,t)\,dx &= \int_0^t \int_\Omega \left[\frac{a}{m^\nu} - a u_{mj}^\nu \right] dx d\tau \\ &+ \int_0^t \int_{\partial\Omega} \int_\Omega k(x,y,\tau) u_{m(j-1)}^l (y,\tau)\, dy dS_x d\tau \\ &+ \int_\Omega u_{0mj}(x)\,dx. \end{aligned} \tag{2.10}$$

Substracting from (2.10) the similar equality with $m = k\,(k > m)$ and taking (2.5) into account, we get

$$\begin{aligned} &\int_\Omega [u_{mj}(x,t) - u_{kj}(x,t)]\,dx \le aT|\Omega| \left(\frac{1}{m^\nu} - \frac{1}{k^\nu} \right) + \int_\Omega [u_{0mj}(x) - u_{0kj}(x)]\,dx \\ &+ \ |\partial\Omega| \operatorname*{ess\,sup}_{\partial\Omega\times\Omega\times[0,T]} k(x,y,t) \int_0^T \int_\Omega [u_{m(j-1)}^l(y,\tau) - u_{k(j-1)}^l(y,\tau)]\,dy d\tau, \end{aligned} \tag{2.11}$$

where $|\partial\Omega|$ and $|\Omega|$ are the Lebesgue measures of $\partial\Omega$ in $\mathbb{R}^{N-1}$ and Ω in $\mathbb{R}^N$, respectively. Passing to the limit in (2.11) as $j \to \infty$, by virtue of (2.3), (2.6) – (2.9) and Lemma 2.2, we conclude

$$\lim_{m\to\infty} \sup_{[0,T]} ||u_m(x,t) - u_k(x,t)||_{L^1(\Omega)} = 0.$$

Thus, u_m is a Cauchy sequence in $C([0,T]; L^1(\Omega))$, and the limit function u is continuous in $L^1(\Omega)$. Now passing to the limit in (2.7) as $m \to \infty$, we prove that $u(x,t)$ is a solution of (1.1)–(1.3) in Q_T. □

3. Comparison principle

In this section, a comparison principle for (1.1)–(1.3) will be proved.

Theorem 3.1. *Let $\overline{u}$ and $\underline{u}$ be a supersolution and a subsolution of problem (1.1)–(1.3) in Q_T, respectively. Suppose that $\underline{u}(x,t) > 0$ or $\overline{u}(x,t) > 0$ a.e. in Q_T if $l < 1$. Then $\overline{u}(x,t) \geq \underline{u}(x,t)$ a.e. in Q_T.*

Proof. Suppose that $l \geq 1$. Let $u_m(x,t)$ be defined in (2.6). Then it satisfies (2.8). To establish the theorem we will show that

$$\underline{u}(x,t) \leq u(x,t) \leq \overline{u}(x,t) \text{ a.e. in } Q_T, \tag{3.1}$$

where $u(x,t)$ was defined in (2.9). We will prove only the second inequality in (3.1) since the proof of the first one is similar. Let $\varphi(x,t) \in C^{2,1}(\overline{Q_T})$ be a nonnegative function such that

$$\frac{\partial\varphi(x,t)}{\partial\mathbf{n}} = 0, \ (x,t) \in S_T.$$

Set $w(x,t) = u_m(x,t) - \overline{u}(x,t)$. Then $w(x,t)$ satisfies

$$\begin{aligned}\int_\Omega w(x,t)\varphi(x,t)\,dx \quad &\leq \quad \int_\Omega w(x,0)\varphi(x,0)\,dx + \frac{a}{m^\nu}\int_0^t\int_\Omega \varphi(x,\tau)\,dxd\tau \\ &+ \quad \int_0^t\int_{\partial\Omega}\varphi(x,\tau)\int_\Omega k(x,y,\tau)d_m(y,\tau)w(y,\tau)\,dydS_xd\tau \\ &+ \quad \int_0^t\int_\Omega(\varphi_\tau + a_m\Delta\varphi - ab_m\varphi)w\,dxd\tau, \end{aligned}\tag{3.2}$$

where

$$a_m = \begin{cases} \frac{u_m^\mu - \overline{u}^\mu}{u_m - \overline{u}}, \ u_m \neq \overline{u}, \\ \mu u_m^{\mu-1}, \ u_m = \overline{u}, \end{cases} \quad b_m = \begin{cases} \frac{u_m^\nu - \overline{u}^\nu}{u_m - \overline{u}}, \ u_m \neq \overline{u}, \\ \nu u_m^{\nu-1}, \ u_m = \overline{u}, \end{cases} \quad d_m = \begin{cases} \frac{u_m^l - \overline{u}^l}{u_m - \overline{u}}, \ u_m \neq \overline{u}, \\ l u_m^{l-1}, \ u_m = \overline{u}. \end{cases}$$

Note that by the hypotheses for $k(x,y,t)$, $u_m(x,t)$ and $\overline{u}(x,t)$, we have

$$\begin{aligned}&0 \leq \overline{u}(x,t) \leq M, \ \frac{1}{m} \leq u_m(x,t) \leq M, \\ &r_m \leq a_m(x,t) \leq M, \ r_m \leq d_m(x,t) \leq M, \ r_m \leq b_m(x,t) \leq M_m \text{ a. e. in } Q_T, \\ &\text{and } \ 0 \leq k(x,y,t) \leq M \text{ a. e. in } \partial\Omega \times Q_T, \end{aligned}\tag{3.3}$$

where M, r_m, M_m are some positive constants, with r_m and M_m potentially depending on m. Let $\{a_{mk}\}, \{b_{mk}\}$ be sequences of functions having the following properties: $a_{mk} \in C^\infty(\overline{Q_T})$, $b_{mk} \in C^\infty(\overline{Q_T})$,

$$a_{mk} \to a_m \text{ as } k \to \infty \text{ in } L^2(Q_T), \ b_{mk} \to b_m \text{ as } k \to \infty \text{ in } L^1(Q_T) \tag{3.4}$$

and

$$r_m \le a_{mk}(x,t) \le M+1,\ \ r_m \le b_{mk}(x,t) \le M_m+1 \ \text{ in } \overline{Q_T}. \tag{3.5}$$

Now consider a backward problem given by

$$\begin{cases} \varphi_\tau + a_{mk}\Delta\varphi - ab_{mk}\varphi = 0 & \text{for } x\in\Omega,\ \ 0<\tau<t,\\ \frac{\partial\varphi(x,\tau)}{\partial \mathbf{n}} = 0 & \text{for } x\in\partial\Omega,\ \ 0<\tau<t,\\ \varphi(x,t)=\psi(x) & \text{for } x\in\Omega, \end{cases} \tag{3.6}$$

where $\psi(x)\in C_0^\infty(\Omega)$ and $0\le\psi(x)\le 1$. Denote the solution of (3.6) as $\varphi_{mk}(x,\tau)$. By the standard theory for linear parabolic equations (see [36], for example), we find that $\varphi_{mk}\in C^{2,1}(\overline{Q}_t)$, $0\le\varphi_{mk}(x,\tau)\le 1$ in $\overline{Q_t}$. Substituting $\varphi=\varphi_{mk}$ into (3.2), we infer

$$\int_\Omega w(x,t)\psi(x)\,dx \le \int_\Omega w(x,0)_+\,dx + \frac{a}{m^\nu}T|\Omega| + |\partial\Omega|M^2\int_0^t\int_\Omega w(y,\tau)_+\,dyd\tau$$
$$+\quad \int_0^t\int_\Omega \{(a_m-a_{mk})\Delta\varphi_{mk} - a(b_m-b_{mk})\varphi_{mk}\}\,w(x,\tau)\,dxd\tau, \tag{3.7}$$

where $w_+=\max\{w,0\}$.

To estimate last integral on the right-hand side of (3.7), we multiply the equation in (3.6) by $\Delta\varphi_{mk}$ and integrate the result over Q_t :

$$\begin{aligned} \int_0^t\int_\Omega a_{mk}(\Delta\varphi_{mk})^2\,dxd\tau &= -\int_0^t\int_\Omega \varphi_{mk\tau}\Delta\varphi_{mk}\,dxd\tau + a\int_0^t\int_\Omega b_{mk}\varphi_{mk}\Delta\varphi_{mk}\,dxd\tau\\ &\le \frac12\int_0^t\int_\Omega |\nabla\psi(x)|^2\,dx + \frac{a^2}{2}\int_0^t\int_\Omega \frac{b_{mk}^2}{a_{mk}}\varphi_{mk}^2\,dxd\tau\\ &+ \frac12\int_0^t\int_\Omega a_{mk}(\Delta\varphi_{mk})^2\,dxd\tau. \end{aligned} \tag{3.8}$$

From (3.8) we conclude that

$$\int_0^t\int_\Omega a_{mk}(\Delta\varphi_{mk})^2\,dxd\tau \le C_m, \tag{3.9}$$

where C_m is some positive constant that may depend on m. Taking into account (3.3) – (3.5), (3.9) and Hölder's inequality, we obtain

$$\Big|\int_0^t\int_\Omega \{(a_m-a_{mk})\Delta\varphi_{mk} - a(b_m-b_{mk})\varphi_{mk}\}\,w(x,\tau)\,dxd\tau\Big|$$
$$\le M\left(\int_0^t\int_\Omega \frac{(a_m-a_{mk})^2}{a_{mk}}w^2\,dxd\tau\right)^{\frac12}\left(\int_0^t\int_\Omega a_{mk}(\Delta\varphi_{mk})^2\,dxd\tau\right)^{\frac12}$$
$$+aM\int_0^t\int_\Omega |b_m-b_{mk}|\,dxd\tau \to 0 \ \text{ as } \ k\to\infty.$$

Now passing in (3.7) to the limit as $k\to\infty$, we infer

$$\int_\Omega w(x,t)\psi(x)\,dx \le \int_\Omega w(x,0)_+\,dx + \frac{a}{m^\nu}T|\Omega| + |\partial\Omega|M^2\int_0^t\int_\Omega w(y,\tau)_+\,dyd\tau. \tag{3.10}$$

Since (3.10) holds for every $\psi(x)$, we can choose a sequence $\{\psi_n\}$ converging in $L^1(\Omega)$ to $\psi(x) = 1$ if $w(x,t) > 0$ and $\psi(x) = 0$ otherwise. Hence, from (3.10) we get

$$\int_\Omega w(x,t)_+ \, dx \leq \int_\Omega w(x,0)_+ \, dx + \frac{a}{m^\nu} T|\Omega| + |\partial\Omega| M^2 \int_0^t \int_\Omega w(y,\tau)_+ \, dy d\tau.$$

Applying Gronwall's inequality and passing to the limit as $m \to \infty$, the conclusion of the theorem follows for $l \geq 1$. For the case $l < 1$ we can consider $w(x,t) = \underline{u}(x,t) - \overline{u}(x,t)$ and prove the theorem in a similar way using the positiveness of a subsolution or a supersolution. □

Remark 3.2. It is not difficult to show that a classical subsoluiton and a classical supersoluiton of (1.1)–(1.3) are also a subsoluiton and a supersoluiton, respectively.

4. Global existence

To formulate global existence result we need the following condition

$$||k(x,y,t)||_{L^\infty(\partial\Omega\times\overline{\Omega}\times[0,\infty))} = K_\infty < \infty.$$

Theorem 4.1. *Let at least one of the following conditions hold:*

1) $l + \mu \leq 2$,

2) $\nu > \mu + l - 1$,

3) $l + \mu > 2,\ \nu = \mu + l - 1,\ a/K_\infty$ *is large enough.*

Then every solution of (1.1)–(1.3) is global.

Proof. In order to prove global existence of solutions we construct a suitable explicit supersolution of (1.1)–(1.3) in Q_T for any positive T. Suppose first that $l + \mu < 2$. Let

$$K_T = ||k(x,y,t)||_{L^\infty(\partial\Omega\times\overline{\Omega}\times[0,T])}.$$

Now we construct a supersolution of (1.1)–(1.3) in Q_T as follows

$$\overline{u}(x,t) = \left\{(1-l)\left[\psi(x) + (\alpha t + \beta)^{\frac{1-l}{2-l-\mu}}\right]\right\}^{\frac{1}{1-l}}, \tag{4.1}$$

where positive constants α, β will be chosen later and $\psi(x)$ is some positive solution of the following problem

$$\begin{cases} \Delta\psi(x) = b,\ x \in \Omega, \\ \frac{\partial\psi(x)}{\partial \mathbf{n}} = \frac{b|\Omega|}{|\partial\Omega|},\ x \in \partial\Omega \end{cases} \tag{4.2}$$

with $b > 0$. Due to (4.1), (4.2) we have

$$\begin{aligned} L\overline{u} \equiv \overline{u}_t - \Delta\overline{u}^\mu + a\overline{u}^\nu \geq{} & \frac{\alpha(1-l)}{2-l-\mu}\overline{u}^{\mu+l-1}\frac{(\alpha t+\beta)^{\frac{\mu-1}{2-l-\mu}}}{\overline{u}^{\mu-1}} \\ & - \mu b\overline{u}^{\mu+l-1} - \mu(\mu+l-1)|\nabla\psi|^2\overline{u}^{\mu+2l-2} \geq 0 \end{aligned} \tag{4.3}$$

in Q_T for large values of α and β, and

$$\frac{\partial\overline{u}}{\partial\nu} = \frac{b|\Omega|}{|\partial\Omega|}\overline{u}^l \geq \int_\Omega k(x,y,t)\overline{u}^l(y,t)\, dy \tag{4.4}$$

on S_T for large values of b and β. Finally,

$$\overline{u}(x,0) \geq u_0(x) \quad \text{a.e. in } \Omega \tag{4.5}$$

for β large enough. By virtue of (4.3)–(4.5) and Theorem 3.1 every solution of (1.1)–(1.3) exists globally.

For $l+\mu=2$ it is easy to check that

$$\overline{u}(x,t)=\{(1-l)\,[\psi(x)+\beta\exp(\alpha t)]\}^{\frac{1}{1-l}}\,, \tag{4.6}$$

is a supersolution of (1.1)–(1.3) in Q_T for large values of α and β.

Suppose now that $\nu>\mu+l-1$ and $l<1$. Then the function in (4.6) with $\alpha=0$ is a supersolution of (1.1)–(1.3) in Q_T for β large enough.

Let $\nu>\mu+l-1$ and $l\geq 1$. To construct a supersolution we use the change of variables in a neighborhood of $\partial\Omega$ as in [37]. Let $\overline{x}\in\partial\Omega$ and $\widehat{n}(\overline{x})$ be the inner unit normal to $\partial\Omega$ at the point $\overline{x}$. Since $\partial\Omega$ is smooth it is well known that there exists $\delta>0$ such that the mapping $\psi:\partial\Omega\times[0,\delta]\to\mathbb{R}^n$ given by $\psi(\overline{x},s)=\overline{x}+s\widehat{n}(\overline{x})$ defines new coordinates $(\overline{x},s)$ in a neighborhood of $\partial\Omega$ in $\overline{\Omega}$. A straightforward computation shows that, in these coordinates, Δ applied to a function $g(\overline{x},s)=g(s)$, which is independent of the variable $\overline{x}$, evaluated at a point $(\overline{x},s)$ is given by

$$\Delta g(\overline{x},s)=\frac{\partial^2 g}{\partial s^2}(\overline{x},s)-\sum_{j=1}^{n-1}\frac{H_j(\overline{x})}{1-sH_j(\overline{x})}\frac{\partial g}{\partial s}(\overline{x},s), \tag{4.7}$$

where $H_j(\overline{x})$ for $j=1,...,n-1$, denote the principal curvatures of $\partial\Omega$ at $\overline{x}$. For $0\leq s\leq\delta$ and small δ we have

$$\left|\sum_{j=1}^{n-1}\frac{H_j(\overline{x})}{1-sH_j(\overline{x})}\right|\leq\overline{c}. \tag{4.8}$$

Let $\rho>0,\ 0<\varepsilon<\omega<\min(\delta\rho,1),\ \max\{\mu/l,2\mu/(\nu-\mu)\}<\beta,\ \beta<2\mu/(l-1)$ for $l>1$, $0<\gamma<\beta/2$, $A^\mu\geq\operatorname{ess\,sup}_\Omega u_0(x)$. For points in $Q_{\delta,T}=\partial\Omega\times[0,\delta]\times[0,T]$ with coordinates $(\overline{x},s,t)$ define

$$\overline{u}(x,t)=\overline{u}((\overline{x},s),t)=\left(\left[(\rho s+\varepsilon)^{-\gamma}-\omega^{-\gamma}\right]_+^{\frac{\beta}{\gamma}}+A\right)^{\frac{1}{\mu}}, \tag{4.9}$$

where $s_+=\max(s,0)$. For points in $\overline{Q_T}\setminus Q_{\delta,T}$ we set $\overline{u}(x,t)=A$. We will prove that $\overline{u}(x,t)$ is the supersolution of (1.1)–(1.3) in Q_T. It is not difficult to check that

$$\left|\frac{\partial\overline{u}^\mu}{\partial s}\right|\leq\rho\beta\min\left([D(s)]^{\frac{\gamma+1}{\gamma}}\left[(\rho s+\varepsilon)^{-\gamma}-\omega^{-\gamma}\right]_+^{\frac{\beta+1}{\gamma}},\ (\rho s+\varepsilon)^{-(\beta+1)}\right), \tag{4.10}$$

$$\left|\frac{\partial^2\overline{u}^\mu}{\partial s^2}\right|\leq\rho^2\beta(\beta+1)\min\left([D(s)]^{\frac{2(\gamma+1)}{\gamma}}\left[(\rho s+\varepsilon)^{-\gamma}-\omega^{-\gamma}\right]_+^{\frac{\beta+2}{\gamma}},\ (\rho s+\varepsilon)^{-(\beta+2)}\right), \tag{4.11}$$

where

$$D(s)=\frac{(\rho s+\varepsilon)^{-\gamma}}{(\rho s+\varepsilon)^{-\gamma}-\omega^{-\gamma}}.$$

Then $D'(s)>0$ and for any $\overline{\varepsilon}>0$

$$1\leq D(s)\leq 1+\overline{\varepsilon},\ 0<s\leq\overline{s}, \tag{4.12}$$

where $\overline{s}=([\overline{\varepsilon}/(1+\overline{\varepsilon})]^{1/\gamma}\omega-\varepsilon)/\rho,\ \varepsilon<[\overline{\varepsilon}/(1+\overline{\varepsilon})]^{1/\gamma}\omega$. By (4.7)–(4.12) we can choose $\overline{\varepsilon}$ small enough so that in $Q_{\overline{s},T}$

$$L\overline{u}\geq a\left(\left[(\rho s+\varepsilon)^{-\gamma}-\omega^{-\gamma}\right]_+^{\frac{\beta}{\gamma}}+A\right)^{\frac{\nu}{\mu}}-\rho^2\beta(\beta+1)\,[D(s)]^{\frac{2(\gamma+1)}{\gamma}}\left[(\rho s+\varepsilon)^{-\gamma}-\omega^{-\gamma}\right]_+^{\frac{\beta+2}{\gamma}}$$

$$- \rho\beta\overline{c}\,[D(s)]^{\frac{\gamma+1}{\gamma}} \left[(\rho s+\varepsilon)^{-\gamma} - \omega^{-\gamma}\right]_+^{\frac{\beta+1}{\gamma}} \geq 0.$$

Let $s \in [\overline{s}, \delta]$. From (4.7)–(4.11) we have

$$|\Delta \overline{u}^\mu| \leq \rho^2\beta(\beta+1)\omega^{-(\beta+2)} \left(\frac{1+\overline{\varepsilon}}{\overline{\varepsilon}}\right)^{\frac{\beta+2}{\gamma}} + \rho\beta\overline{c}\omega^{-(\beta+1)} \left(\frac{1+\overline{\varepsilon}}{\overline{\varepsilon}}\right)^{\frac{\beta+1}{\gamma}}$$

and $L\overline{u} \geq 0$ for A large enough. Obviously, in $\overline{Q_T} \setminus Q_{\delta,T}$

$$L\overline{u} = aA^{\frac{\nu}{\mu}} \geq 0.$$

Now we prove the following inequality

$$\frac{\partial \overline{u}}{\partial \nu}(\overline{x}, 0, t) \geq \int_\Omega K_T \overline{u}^l(\overline{x}, s, t)\, dy, \quad (x,t) \in S_T \tag{4.13}$$

To estimate the integral I on the right hand side of (4.13) we use the change of variables in a neighborhood of $\partial\Omega$ as above. Let

$$\overline{J} = \sup_{0<s<\delta} \int_{\partial\Omega} |J(\overline{y}, s)|\, d\overline{y},$$

where $J(\overline{y}, s)$ is Jacobian of the change of variables. Then we have

$$\begin{aligned} I \leq & \theta K_T \int_\Omega \left[(\rho s+\varepsilon)^{-\gamma} - \omega^{-\gamma}\right]_+^{\frac{\beta l}{\gamma\mu}} dy + \theta K_T A^{\frac{l}{\mu}} |\Omega| \\ \leq & \theta K_T \overline{J} \int_0^{(\omega-\varepsilon)/\rho} \left[(\rho s+\varepsilon)^{-\gamma} - \omega^{-\gamma}\right]^{\frac{\beta l}{\gamma\mu}} ds + \theta K_T A^{\frac{l}{\mu}} |\Omega| \\ \leq & \frac{\mu\theta K_T \overline{J}}{\rho(\beta l - \mu)} \left[\varepsilon^{-\left(\frac{\beta l}{\mu}-1\right)} - \omega^{-\left(\frac{\beta l}{\mu}-1\right)}\right] + \theta K_T A^{\frac{l}{\mu}} |\Omega|, \end{aligned}$$

where $\theta = \max(2^{l/\mu - 1}, 1)$. On the other hand, since

$$\frac{\partial \overline{u}}{\partial \nu}(\overline{x}, 0, t) = -\frac{\partial \overline{u}}{\partial s}(\overline{x}, 0, t) = \frac{\rho\beta}{\mu} \varepsilon^{-\gamma-1} \left[\varepsilon^{-\gamma} - \omega^{-\gamma}\right]^{\frac{\beta-\gamma}{\gamma}} \left(\left[\varepsilon^{-\gamma} - \omega^{-\gamma}\right]^{\frac{\beta}{\gamma}} + A\right)^{\frac{1-\mu}{\mu}},$$

(4.13) holds if ε is small enough. Finally,

$$u_0(x) \leq \overline{u}(x, 0) \quad \text{a.e. in} \quad \Omega.$$

Hence, by Theorem 3.1 we obtain

$$u(x,t) \leq \overline{u}(x,t) \quad \text{a.e. in} \quad \overline{Q}_T.$$

Suppose now that $l + \mu > 2$, $\nu = \mu + l - 1$ and $l < 1$. Then the function in (4.6) with $\alpha = 0$ is a supersolution of (1.1)–(1.3) in Q_T for suitable choices of b and β if $a/K_\infty > \mu|\partial\Omega|$.

For $l = 1$ and $\nu = \mu + l - 1 = \mu$ it is not difficult to check that

$$\overline{u}(x,t) = [\psi(x) + B]^{\frac{1}{\mu}},$$

is a supersolution of (1.1)–(1.3) in Q_T for $a/K_\infty > \mu|\partial\Omega|$ under suitable choices of b and B.

For $l > 1$ and $\nu = \mu + l - 1$ we can show in the same way as above that $\overline{u}$ in (4.9) with $\beta = 2\mu/(l-1)$ is a supersolution of (1.1)–(1.3) in Q_T under suitable choices of ρ, ε and A if

$$\frac{a}{K_\infty} > \frac{\theta\mu(2\mu + l - 1)\overline{J}}{l+1}.$$

□

5. Blow-up in finite time

To formulate finite time blow-up result we suppose that for some $\tau > 0$

$$\operatorname*{ess\,inf}_{\partial\Omega\times\overline{\Omega}\times[0,\tau]} k(x,y,t) = k_0 > 0. \tag{5.1}$$

Theorem 5.1. *Let (5.1) hold, $l+\mu>2$ and either $\nu<\mu+l-1$ or $\nu=\mu+l-1$ and a/k_0 be small enough. Then there exist solutions of (1.1)–(1.3) with finite time blow-up.*

Proof. To prove the theorem we construct a suitable subsolution of (1.1)–(1.3) with finite time blow-up and use a comparison argument. Suppose first that $l\le 1$ and $\nu<\mu+l-1$. Then there exists $\gamma>0$ such that $\gamma<l$, $\nu<\mu+\gamma-1$ and $\gamma+\mu>2$. We set

$$\underline{u}(x,t) = \left\{(1-\gamma)[\psi(x)+(T-t)^{-\alpha}+A]\right\}^{\frac{1}{1-\gamma}}, \tag{5.2}$$

where $A>0,\ T\in(0,\tau],\ \alpha>(1-\gamma)/(\gamma+\mu-2)$ and $\psi(x)$ was defined in (4.2). It is easy to check that

$$L\underline{u} \le \alpha(1-\gamma)^{-\frac{\alpha+1}{\alpha}}\underline{u}^{\frac{\alpha+1-\gamma}{\alpha}} - \mu b\underline{u}^{\mu+\gamma-1} + a\underline{u}^{\nu} \le 0 \quad \text{in} \quad Q_T \tag{5.3}$$

for large values of A. For $x\in\partial\Omega$ and $t\in(0,T)$ we have

$$\frac{\partial\underline{u}(x,t)}{\partial\mathbf{n}} = \frac{b|\Omega|}{|\partial\Omega|}\underline{u}^{\gamma}(x,t) \le k_0\int_{\Omega}\underline{u}^{l}(y,t)\,dy \tag{5.4}$$

for large enough A since $\gamma<l$. By (5.1) – (5.4) and Theorem 3.1 we conclude that any solution $u(x,t)$ of (1.1)–(1.3) blows up in finite time if

$$u_0(x) \ge \underline{u}(x,0) \quad \text{a.e. in} \quad \Omega. \tag{5.5}$$

If $l<1$ and $\nu=\mu+l-1$ then similarly we show that $\underline{u}$ in (5.2) with $\gamma=l$ is a subsolution of (1.1)–(1.3) in Q_σ, $\sigma\in(0,T)$ for $a/k_0<\mu|\partial\Omega|$ provided (5.5) holds and $A,\ b$ are appropriately chosen.

For $l=1$ and $\nu=\mu+l-1=\mu$ we introduce

$$\underline{u}(x,t) = B(T-t)^{-\alpha}\exp[\psi(x)], \tag{5.6}$$

where $B\ge 1,\ \alpha>1/(\mu-1),\ \psi(x)$ satisfies (4.2). Then

$$L\underline{u} \le \alpha(T-t)^{-1}\underline{u} - b\mu\underline{u}^{\mu} + a\underline{u}^{\mu} \le 0 \quad \text{in} \quad Q_T \tag{5.7}$$

for $b>a/\mu$ and large values of B. For $x\in\partial\Omega$ and $t\in(0,T)$ we obtain

$$\frac{\partial\underline{u}(x,t)}{\partial\mathbf{n}} = \frac{b|\Omega|}{|\partial\Omega|}\underline{u}(x,t) \le k_0\int_{\Omega}\underline{u}(y,t)\,dy \tag{5.8}$$

if

$$b \le k_0\frac{|\partial\Omega|\int_{\Omega}\exp[\psi(y)]\,dy}{|\Omega|\sup_{\partial\Omega}\exp[\psi(x)]}. \tag{5.9}$$

Thus, by (4.2), (5.1), (5.6) – (5.9) and Theorem 3.1 we conclude that for $a/k_0<\mu|\partial\Omega|$ under suitable choices of $\psi(x),\ b,\ B$ any solution $u(x,t)$ of (1.1)–(1.3) blows up in finite time if (5.5) holds.

Let $l>1,\ \nu<\mu+l-1$ and $u(x,t)$ be defined in (2.9). It is not difficult to check that the function

$$w_\nu(t) = \begin{cases} \left[A^{1-\nu}-2(1-\nu)at\right]^{1/(1-\nu)} & \text{for } 0<\nu<1, \\ A\exp(-2at) & \text{for } \nu=1 \end{cases}$$

is a subsolution of (2.2) in Q_τ for $\tau \le 1/(2a)$ and large values of m if

$$u_0(x) \ge A > 1.$$

Additionally, the function

$$w_\nu(t) = [2(\nu-1)a(t+t_0)]^{-\frac{1}{\nu-1}},\ t_0 > 0, \text{ for } \nu > 1$$

is a subsolution of (2.2) in Q_τ for m large enough if

$$u_0(x) \ge [2(\nu-1)at_0]^{-\frac{1}{\nu-1}}.$$

Applying a comparison principle to (2.2), we have

$$w_\nu(t) \le u_{mj}(x,t) \text{ in } Q_\tau,\ j \in \mathbb{N} \tag{5.10}$$

for large values of m. Then from (2.6), (2.9), (5.10) we obtain

$$w_\nu(t) \le u(x,t) \quad \text{in } Q_\tau. \tag{5.11}$$

Now we use the change of variables in a neighborhood of $\partial\Omega$ as in Theorem 4.1. Set $\Omega_\gamma = \{(\overline{x}, s) : \overline{x} \in \partial\Omega, 0 < s < \gamma\}$.

Let us consider the following initial boundary value problem:

$$\begin{cases} v_t = \Delta v^\mu - av^\nu & \text{for } x \in \Omega_\gamma,\ 0 < t < t_0, \\ \frac{\partial v(x,t)}{\partial \mathbf{n}} = \int_{\Omega_\gamma} k(x,y,t) v^l(y,t)\,dy & \text{for } x \in \partial\Omega,\ 0 < t < t_0, \\ v(x,t) = u(x,t) & \text{for } x \in \partial\Omega_\gamma \setminus \partial\Omega,\ 0 < t < t_0, \\ v(x,0) = u_0(x) & \text{for } x \in \Omega_\gamma, \end{cases} \tag{5.12}$$

where γ and $t_0 \le \tau$ will be chosen later. We can define the notions of a supersolution and a subsolution of (5.12)in a manner similar to that in Definition 2.1.

We define

$$\xi(s,t) = C(t_0 + s - t)^{-\sigma}, \tag{5.13}$$

where $C > 0,\ \sigma > 2/(l-1)$ for $\nu \le \mu$ and $2/(l-1) < \sigma < 2/(\nu-\mu)$ for $\nu > \mu$.

We will show that $\xi(s,t)$ is a subsolution of (5.12) in $Q(\gamma, t_0)$ under suitable choices of t_0 and γ. Using (4.7), (4.8), (5.13), we find

$$\begin{aligned} -\xi_t + \Delta\xi^\mu - a\xi^\nu \ge (t_0 + s - t)^{-\sigma\mu-2} \Big\{ &\sigma\mu(\sigma\mu+1)C^\mu - \sigma(t_0+\gamma)^{\sigma(\mu-1)+1}C \\ &- \sigma\mu\overline{c}(t_0+\gamma)C^\mu - a(t_0+\gamma)^{\sigma(\mu-\nu)+2}C^\nu \Big\} \ge 0 \end{aligned} \tag{5.14}$$

in $Q(\gamma, t_0)$ if we take γ and t_0 small enough. Next, we check the inequality on the boundary $\partial\Omega \times (0, t_0)$. Let

$$\underline{J} = \inf_{0<s<\gamma} \int_{\partial\Omega} |J(\overline{y}, s)|\, d\overline{y}.$$

In view of (5.13) we have

$$\begin{aligned} &\frac{\partial\xi}{\partial\nu}(0,t) - k_0 \int_{\Omega_\gamma} \xi^l(s,t)\,dy \le \sigma C(t_0-t)^{-\sigma-1} - k_0\underline{J}C^l \int_0^\gamma (t_0+s-t)^{-\sigma l} ds \\ \le &\sigma C(t_0-t)^{-\sigma-1} - k_0\underline{J}C^l \frac{(t_0-t)^{-\sigma l+1}}{\sigma l - 1}\left[1 - \left(\frac{t_0}{t_0+\gamma}\right)^{\sigma l - 1}\right] \le 0 \end{aligned} \tag{5.15}$$

for $x \in \partial\Omega$, $0 < t < t_0$ and small enough t_0. Let

$$u_0(x) \ge \gamma^{-\sigma} \tag{5.16}$$

and

$$C\gamma^{-\sigma} \le \left[A^{1-\nu} - 2(1-\nu)at_0\right]^{1/(1-\nu)} \quad \text{for } 0<\nu<1, \tag{5.17}$$

$$C\gamma^{-\sigma} \le A\exp\left(-2at_0\right) \quad \text{for } \nu=1, \tag{5.18}$$

$$C\gamma^{-\sigma} \le \left[4(\nu-1)at_0\right]^{-\frac{1}{\nu-1}} \quad \text{for } \nu>1. \tag{5.19}$$

Using (5.11), (5.16) – (5.19), we obtain

$$\xi(s,t) \le u(x,t) \ \text{ for } x\in\Omega_\gamma,\ t=0 \ \text{ and } \ x\in\partial\Omega_\gamma\setminus\partial\Omega,\ \ 0<t<t_0. \tag{5.20}$$

From (5.14), (5.15), (5.20) we conclude that $\xi(s,t)$ is a subsolution of (5.12) in $Q(\gamma,t_0)$. It is easy to check that $u(x,t)$ is a supersolution of (5.12) in $Q(\gamma,t_0)$. Arguing as in Theorem 3.1, we prove $u(x,t) \ge \xi(s,t)$ in $Q(\gamma,t_0)$. Thus, $u(x,t)$ blows up in finite time since $\xi(0,t)\to\infty$ as $t\to t_0$.

In the case $l>1$ and $\nu=\mu+l-1$ we can similarly show that $\underline{u}$ in (5.13) with $\sigma=2/(l-1)$ is a subsolution of (1.1)–(1.3) in $Q(\gamma,t_0)$ under suitable choices of t_0, γ, C and $u_0(x)$ if

$$\frac{a}{k_0} < \frac{\mu(2\mu+l-1)\underline{J}}{l+1}.$$

□

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Alexander Gladkov, Department of Mechanics and Mathematics, Belarusian State University, 4 Nezavisimosti Avenue, 220030 Minsk, Belarus

Email address: `gladkoval@bsu.by`