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APPLICATIONS OF MATHEMATICAL AND COMPUTER MODELING IN THE SENSOR LOCATION PROBLEM

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Abstract. We consider a real-world problem of constructing such a suboptimal solution of the problem of estimating inhomogeneous flow in a bidirectional multinetwork and guaranteeing that the multinetwork is fully monitored. The sensors are placed in the multigraph nodes in order to estimate the inhomogeneous flow in the unmonitored part of the multigraph. Unlike the problem of finding the optimal solution [1], which is *NP-complete* [2] and has huge computational costs, the search for a suboptimal solution does not require to minimize the set of monitored multinetwork nodes. Our approach can be used for examining intelligent transportation systems and creating algorithms for solving Sensor Location Problem for a bidirectional multigraph.

Keywords: Sensor Location Problem; mathematical and computer modeling; rank; sparse linear system; suboptimal solution; decomposition; multigraph.

We present finite oriented connected multigraph $G=(I,U)$ with set of nodes I and set of multiarcs U, $|U| \gg |I|$, G through the set of $|K|$ connected networks $G^k=(I^k,U^k)$ corresponding to a certain type k, where I^k is the set of nodes and U^k is the set of arcs which is available on the flow of type k, $k \in K$, $K=\{1,\cdots,|K|\}$. We define for each node $i \in I$ the set of types of flows (products) $K(i)=\{k \in K : i \in I^k\}$ and for each arc $(i,j) \in U$ the set $K(i,j)=\{k \in K : (i,j)^k \in U^k\}$. In other words, $K(i)$ is the set of types of flows (products) transported through the node $i \in I$ and $K(i,j)$ is the set of types of flows (products) transported through the multiarc $(i,j) \in U$ respectively. The traffic multiflow by a network satisfies the following system:

$$\sum_{j \in I_i^+(U^k)} x_{i,j}^k - \sum_{j \in I_i^-(U^k)} x_{j,i}^k = \begin{cases} x_i^k, & i \in I_k^*, \\ 0, & i \in I^k \backslash I_k^*, \quad k \in K, \end{cases} \tag{1}$$

where $I_k^* \subseteq I^k$ is the set of nodes with unknown external flow x_i^k of node $i \in I_k^*$, $k \in K$.

In order to obtain information about the unknown $x_{i,j}^k$ for the arcs $(i,j)^k \in U^k$ and unknown external flow x_i^k of nodes $i \in I_k^*$, $k \in K$ sensors are locate at the nodes. The nodes in the multigraph $G=(I,U)$ with sensors we call monitored ones and denote the set of monitored nodes M. The set of monitored nodes in the $G^k=(I^k,U^k)$ we denote M_k, where $M_k = I^k \bigcap M$, $k \in K$.

If a node $i \in M$ is monitored, then we know the values of flows on all outgoing and all incoming arcs for each node $i \in M$, $k \in K(i)$:

$$x_{i,j}^k = f_{i,j}^k, \quad j \in I_i^+(U^k), \quad x_{j,i}^k = f_{j,i}^k, \quad j \in I_i^-(U^k), \quad k \in K(i) \text{ for each } i \in M. \tag{2}$$

If the set M includes the nodes from the set I_k^*, $k \in K(i)$ then we know the values of flows for all incoming and outgoing arcs for the nodes of the set M and besides, we know also the values:

$$x_i^k = f_i^k, \quad i \in M \bigcap I_k^*, \quad k \in K(i). \tag{3}$$

Consider any node $i \in I$. For every outgoing arc $(i,j)^k \in U^k$ for this node i we determinate a real number $p^k_{i,j} \in (0,1]$, which denotes the corresponding part of the total outgoing flow $\sum_{j \in I^+_i(U^k)} x^k_{i,j}$ from i which leaves along this arc $(i,j)^k$, for each $k \in K(i)$. That is,

$$x^k_{i,j} = p^k_{i,j} \sum_{j \in I^+_i(U^k)} x^k_{i,j}, \quad 0 < p^k_{i,j} \leqslant 1, \quad \sum_{j \in I^+_i(U^k)} p^k_{i,j} = 1.$$

If $|I^+_i(U^k)| \geqslant 2$ for the node $i \in I$ then we can write the flow along all outgoing arcs from node i in terms of a single outgoing arc, for example, $(i, v_i)^k, v_i \in I^+_i(U^k)$:

$$x^k_{i,j} = \frac{p^k_{i,j}}{p^k_{i,v_i}} x^k_{i,v_i}, \quad j \in I^+_i(U^k) \setminus v_i. \tag{4}$$

We continue this process for each node $i \in I$ if it is the case that: $|I^+_i(U^k)| \geqslant 2$ for every $k \in K(i)$.

Let's substitute (2) and (3) to the equations of system (1). If $|I^+_i(U^k)| \geqslant 2$ for the node $i \in I^k$ then we can write the flow along all outgoing arcs from node i in terms of a single known outgoing arc flow f^k_{i,v_i} for the arc $(i,v_i)^k, v_i \in I^+_i(U^k)$, where x^k_{i,v_i} is known and equal to $x^k_{i,v_i} = f^k_{i,v_i}$:

$$x^k_{i,j} = \frac{p^k_{i,j}}{p^k_{i,v_i}} f^k_{i,v_i}, \quad j \in I^+_i(U^k) \setminus v_i, \quad |I^+_i(U^k)| \geqslant 2, \quad i \in I, \quad k \in K(i). \tag{5}$$

Also, we substitute known arcs flow (5) to the equations of system (1). Let's remove from graphs $G^k = (I^k, U^k)$, $k \in K$ the set of the arcs and nodes on which the arc flows and values x^k_i are known. Then we have a new multigraph $\overline{G} = (\overline{I}, \overline{U})$, which consists from the set of graphs $\overline{G}^k = (\overline{I}^k, \overline{U}^k)$, $k \in \overline{K}$, where each $\overline{G}^k = (\overline{I}^k, \overline{U}^k)$ is, in general, a disconnected graph, corresponding to a certain type of flow $k \in \overline{K}$. We denote for each multiarc $(i,j) \in \overline{U}$ of multigraph $\overline{G}$ the set

$$\overline{K}(i,j) = \{k \in \overline{K} : (i,j)^k \in \overline{U}^k\}$$

of types of flow transported through an multiarc (i,j). We denote, also, for each node $i \in \overline{I}$ the set of types of flows $\overline{K}(i) = \{k \in \overline{K} : i \in \overline{I}^k\}$ transported through the node $i \in \overline{I}$. We note, that $|\overline{K}| \leqslant |K|$ in general case, because after we have defined the set M, for some types $k \in K$, we could have obtained the complete information about arc flows and variable intensities for the network flow of type k. So, for this type k there is no subnetwork in the new multigraph $\overline{G}$.

The new multigraph $\overline{G}$ consists from connected components. Some connected components may contain no nodes of the set $\overline{I}^*_k$, where $\overline{I}^*_k$ is the set of nodes with variable external flow of graph $\overline{G}^k$, $k \in \overline{K}$.

Relations (4) are true for all arcs in new multigraph $\overline{G}$. So, if $|I^+_i(\overline{U}^k)| \geqslant 2$ for the node $i \in \overline{I}^k$ then we can write the flow along all outgoing arcs from node i in terms of a single unknown outgoing arc flow x^k_{i,v_i}, for example, for the arc $(i,v_i)^k$, where x^k_{i,v_i} is an unknown flow:

$$x^k_{i,j} = \frac{p^k_{i,j}}{p^k_{i,v_i}} x^k_{i,v_i}, \quad j \in I^+_i(\overline{U}^k) \setminus v_i, \quad i \in \overline{I}^k, \quad k \in \overline{K}. \tag{6}$$

The system (1) and (6) for multigraph $\overline{G} = (\overline{I}, \overline{U})$ will be transformed into the following one:

$$\sum_{j \in I_i^+(\overline{U}^k)} x_{i,j}^k - \sum_{j \in I_i^-(\overline{U}^k)} x_{j,i}^k = \begin{cases} x_i^k + b_i^k, & i \in \overline{I}_k^*, \\ a_i^k, & i \in \overline{I}^k \setminus \overline{I}_k^*, \quad k \in \overline{K}. \end{cases} \tag{7}$$

$$\sum_{(i,j) \in \overline{U}} \sum_{k \in \overline{K}(i,j)} \overline{\lambda}_{i,j}^{k,p} x_{i,j}^k = 0, \quad p = \overline{1, q}, \tag{8}$$

where a_i^k, b_i^k, $\overline{\lambda}_{i,j}^{k,p}$ are constants. We call the multigraph (multinetwork) $\overline{G} = (\overline{I}, \overline{U})$ unobserved part of finite connected directional multigraph(multinetwork) $G = (I, U)$ relative to the set M of observed (monitored) nodes, $M \subseteq I$ in the the Sensor Location Problem for the multigraph of estimating nonhomogeneous flow in a directional multigraph. In [3] give the algorithm to form the multigraph (multinetwork) $\overline{G} = (\overline{I}, \overline{U})$ for Sensor Location Problem. As a result of localization $|M|$ special programmable devices (sensors) and collection the information about multiflow in G we modeling the unobserved part $\overline{G} = (\overline{I}, \overline{U})$ of bidirectional multigraph (multinetwork) $G = (I, U)$ relative to the set M of observed (monitored) nodes.

We consider the applications of the results of mathematical and computer modeling for constructing the solutions sparse underdetermined linear systems of type (7), (8) with variable external flows for the nodes to the Sensor Location Problem for the multigraph. For modeling the unobserved part of the multigraph in order to determine the values of the arc flows and the values of the external flows of the nodes so that the sparse linear system of the special type (7), (8) has an unique solution are use constructive methods, algorithms and technologies decomposition of the basis multigraphs.

The part of the unknowns of the system (7), (8) are the flows for outgoing arcs from the nodes of the set $\overline{I}^k \setminus M_k^*, k \in \overline{K}$. Also the unknowns in the system (7), (8) are the variable external flow x_i^k, where $i \in \overline{I}_k^*$, $k \in \overline{K}$ for the new multigraph $\overline{G} = (\overline{I}, \overline{U})$.

The new multigraph $\overline{G}$ consists from a connected components. If the fixed connected component of the new multigraph $\overline{G} = (\overline{I}, \overline{U})$ contain the nodes of the set $\overline{I}_k^*$, then the matrix of the sparse system (7) for this connected component equal to the full rank. If the fixed connected component of the new multigraph $\overline{G} = (\overline{I}, \overline{U})$ don't include the nodes of the set $\overline{I}_k^*$, then the rank of matrix of the sparse system (7) for this connected component equal to the not full rank. We use the constructive theory of decomposition of the basis multigraphs [3], algorithms and technologies [4] for solution the system (7), (8). The system (7), (8) has an unique solution for the given set M if and only if the rank of the matrix of system is equal to the number of unknowns.

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