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ОБОБЩЕНИЕ ТЕОРЕМЫ ШТЕЙНЕРА – ЛЕМУСА И ТРАНСЦЕНДЕНТНОСТЬ КРИТИЧЕСКИХ ЗНАЧЕНИЙ ЕЕ ПАРАМЕТРОВ

М. М. ВАСЬКОВСКИЙ [1]***, М. А. ФИРСОВ*** [1]***, П. Д. БАБАЕВА*** [1]

[1] *Белорусский государственный университет, пр. Независимости, 4, 220030, г. Минск, Беларусь*

Аннотация. Внутренняя n-линия треугольника является отрезком, проходящим через вершину треугольника и делящим его противоположную сторону в отношении n-х степеней прилежащих сторон. Рассматривается аналог теоремы Штейнера – Лемуса для внутренних n-линий треугольника. Находятся все значения $n \in \mathbb{R}$, для которых аналог данной теоремы выполняется. Кроме того, определяются все значения $n \in \mathbb{R}$, для которых существует неравносторонний треугольник с тремя равными внутренними n-линиями. Доказывается трансцендентность положительных критических значений n обобщенной теоремы Штейнера – Лемуса.

Ключевые слова: внутренняя n-линия треугольника; трансцендентное число; алгебраическое числовое поле; поверхность.

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Авторы:
Максим Михайлович Васьковский – доктор физико-математических наук, профессор; заведующий кафедрой фундаментальной математики и интеллектуальных систем факультета прикладной математики и информатики.
Максим Алексеевич Фирсов – ассистент кафедры фундаментальной математики и интеллектуальных систем факультета прикладной математики и информатики.
Полина Дмитриевна Бабаева – студентка факультета прикладной математики и информатики. Научный руководитель – М. М. Васьковский.

Authors:
Maksim M. Vaskouski, doctor of science (physics and mathematics), full professor; head of the department of fundamental mathematics and intelligence systems, faculty of applied mathematics and computer science.
vaskovskii@bsu.by
https://orcid.org/0000-0001-5769-3678
Maksim A. Firsau, assistant at the department of fundamental mathematics and intelligence systems, faculty of applied mathematics and computer science.
firsovm23@gmail.com
Palina D. Babayeva, student at the faculty of applied mathematics and computer science.
palinababayeva@gmail.com

A GENERALISATION OF THE STEINER – LEHMUS THEOREM AND CRITICAL VALUES TRANSCENDENCE OF ITS PARAMETERS

***M. M. VASKOUSKI*[a], *M. A. FIRSAU*[a], *P. D. BABAYEVA*[a]**

[a]*Belarusian State University, 4 Niezaliezhnasci Avenue, Minsk 220030, Belarus*
Corresponding author: M. M. Vaskouski (vaskovskii@bsu.by)

Abstract. The internal n-line of a triangle is a segment from the vertex to the opposite side dividing this side into segments proportionally to the n^{th} powers of the adjacent sides. An analogue of the Steiner – Lehmus theorem for the internal n-lines of a triangle is considered. All values $n \in \mathbb{R}$ for which the mentioned analogue of the Steiner – Lehmus theorem holds are found. Also all values $n \in \mathbb{R}$ for which there exists a non-equilateral triangle with three equal internal n-lines are determined. The transcendence of positive critical values of n of the generalised Steiner – Lehmus theorem is proved.

Keywords: internal n-line of a triangle; transcendental number; algebraic number field; surface.

Introduction

The Steiner – Lehmus theorem mentioned for the first time in 1840 states: «Any triangle that has two angle bisectors… is isosceles» [1, p. 14]. This theorem allows a trivial algebraic negative proof. Existence of a direct proof in the frame of classical and intuitionistic logic was shown in 2018 [2], and the first such proof was found in 2022 [3]. Also there are known analogues of the Steiner – Lehmus theorem in other geometries different from the Euclidean geometry [4]. In the present paper, we consider the generalisation of the Steiner – Lehmus theorem, which is studied by methods of differential geometry. Let n be a real number. We examine a triangle ABC on the Euclidean plane $\mathbb{R}^2$ with sides $a = BC$, $b = AC$, $c = AB$. Let AA_1, BB_1, CC_1 be the internal n-lines (shortly n-lines) through sides a, b, c respectively [5], i. e. each n-line divides the corresponding side of the triangle into segments with lengths proportional to the n^{th} powers of the adjacent sides. For example, $BA_1 : A_1C = (AB : AC)^n$. Particularly, AA_1 is the median for $n = 0$, AA_1 is the bisector for $n = 1$, AA_1 is the symmedian for $n = 2$. Denote the lengths of the corresponding n-lines by $l_{a,n}$, $l_{b,n}$, $l_{c,n}$.

If we assume that points A_1, B_1, C_1 divide externally the corresponding sides of a triangle into segments proportionally to the n^{th} powers of adjacent sides, we have definition of the external n-lines. Their lengths will be denoted by $l_{a,n}^{ext}$, $l_{b,n}^{ext}$, $l_{c,n}^{ext}$.

It is well-known that any triangle having two equal medians, two equal bisectors or two equal symmedians is isosceles. In the article [5], an analogous statement was proved for n-lines, where $n \in [-0.5, 2]$, and there also was shown that for all sufficiently large real n there exists a non-isosceles triangle with two equal n-lines. In the present paper, we give a complete answer to the question: «For which real n can one find a non-isosceles triangle with two equal n-lines?» We shall say that for a given $n \in \mathbb{R}$ an analogue of the Steiner – Lehmus theorem for two n-lines holds if any triangle with two equal n-lines is isosceles. Similarly, we say that for a given $n \in \mathbb{R}$ an analogue of the Steiner – Lehmus theorem for three n-lines holds if any triangle with three equal n-lines is equilateral. The main results of the paper are the following two theorems.

Theorem 1. *An analogue of the Steiner – Lehmus theorem for two n-lines holds if and only if* $n \in [-1, N_1]$. *An analogue of the Steiner – Lehmus theorem for three n-lines holds if and only if* $n \in [-2, N_0)$. *Here*

$$N_1 = \frac{4(\beta_1 + 1)(2\beta_1 + 1)}{3\beta_1 + 1 - \sqrt{\beta_1^2 + 6\beta_1 + 1 - 16\beta_1^3}} = 24.506\,13\ldots,$$

$$N_0 = 2 \min_{x \in (0,1)} \frac{\log x}{\log(8x + 4) - \log(x^2 + 6x + 5)} = 29.143\,359\ldots$$

and β_1 *is a unique positive solution to equation*

$$\frac{3\beta + 1 - \sqrt{\beta^2 + 6\beta + 1 - 16\beta^3}}{2(\beta + 1)(2\beta + 1)} + \frac{\log\left(\beta - 1 + \sqrt{\beta^2 + 6\beta + 1 - 16\beta^3}\right) - \log\left(2\beta^2 + 2\beta\right)}{\log\beta} = 0.$$

Theorem 2. *The numbers N_0 and N_1 defined in theorem 1 are transcendental.*

We will consider the following three properties directly related to the mentioned above generalisations of the Steiner – Lehmus theorem. Let $n \in \mathbb{R}$.

Property 1. There exists a non-isosceles triangle ABC ($a \neq c$) such that $l_{a,n} = l_{c,n}$.

Property 2. There exists an isosceles triangle ABC ($a \neq c$) such that $l_{a,n} = l_{c,n}$.

Property 3. There exists a non-equilateral triangle ABC such that $l_{a,n} = l_{b,n} = l_{c,n}$.

Using elementary computations, we deduce the formula for length of a n-line:

$$l_{a,n} = \sqrt{\frac{b^2c^n + c^2b^n}{b^n + c^n} - \frac{a^2b^nc^n}{\left(b^n + c^n\right)^2}}.$$

Without loss of generality we may assume that $b = 1$ and $a \geq c$. Since the case of $n = 0$ is trivial, we will assume further that $n \neq 0$.

Let us rewrite the equality of two n-lines in terms of barycentric coordinates. We denote $\delta = a^n$, $\beta = c^n$, $k = \frac{2}{n}$. Then the equality $l_{a,n} = l_{c,n}$ is equivalent to $g(\delta, \beta, k) = 0$, where

$$g(\delta, \beta, k) = \delta^k\left(\beta(\delta+1)^2 + (\beta+1)^2(\delta+1)\right) -$$
$$- \beta^k\left(\delta(\beta+1)^2 + (\beta+1)(\delta+1)^2\right) + (\beta+1)(\delta+1)(\delta-\beta).$$

Proof of theorem 1: case $n > 0$

We assume that $n > 0$.

Proposition 1. *If there holds either property 1 or 2, then* $\delta\beta \leq 1$ *and* $k \in (0, 1)$.

P r o o f. We suppose that either property 1 or 2 holds. Then there exists a triangle ABC such that $\delta > \beta$ and $g(\delta, \beta, k) = 0$.

Let us show that $k \in (0, 1)$. Assume the contrary, $k \geq 1$. As it was shown in theorem 2.2 presented in the paper [5], the assumption of $g'_\delta(\delta, \beta, k) \leq 0$ for $\delta \geq \beta > 0$, $k \geq 1$ implies the inequalities $\beta \geq 1$ and

$$k\beta(\delta+1) + k(\beta+1)^2 \leq 2\delta.$$

From this relation we obtain $2 > k\beta$ and $\delta \geq \frac{k\left(\beta^2 + 3\beta + 1\right)}{2 - k\beta}$, so $(2-k)\delta \geq 5k$. The last inequality implies that $k \in [1, 2)$. Since $k \in [1, 2)$ and $\beta \geq 1$, we get $2 - k\beta \leq 1$. Hence $\delta \geq \beta^2 + 3\beta + 1$. Moreover, $\delta < \left(\beta^{\frac{k}{2}} + 1\right)^{\frac{2}{k}} \leq$ $\leq \left(\beta^{\frac{k}{2}} + 1\right)^2 = \beta^k + 2\beta^{\frac{k}{2}} + 1$. Therefore, $\beta^k + 2\beta^{\frac{k}{2}} + 1 > \beta^2 + 3\beta + 1$, which is impossible for $\beta \geq 1$ and $k \in [1, 2)$. Consequently, we obtain that $g'_\delta(\delta, \beta, k) > 0$ for $\delta^{\frac{k}{2}} < \beta^{\frac{k}{2}} + 1$, $\delta \geq \beta \geq 1$, $k \in [1, 2)$, and $g'_\delta(\delta, \beta, k) > 0$ for $\delta \geq \beta > 0$, $k \geq 2$ and for $0 < \beta < 1$, $\delta \geq \beta$, $k \in [1, 2)$. Because of $g(\beta, \beta, k) = 0$, we get a contradiction to $g(\delta, \beta, k) = 0$ in the domain coloured in red (fig. 1). Hence $k \in (0, 1)$.

Now we need to show that $\delta\beta \leq 1$. Let us assume that $\delta\beta > 1$. If we assume that $\delta > 1 \geq \beta$, then we obtain $g(\delta, \beta, k) \geq g(\delta, \beta, k)0 = (\delta - \beta)(\delta\beta - 1) > 0$. Hence $\delta > \beta > 1$. Taking into account $\delta > \beta > 1$, $k \in (0, 1)$, we have

$$g(\delta, \beta, k) = \left(\delta^k - \beta^k\right)\left(\beta(\delta+1)^2 + \delta(\beta+1)^2\right) + \delta^k(\beta+1)^2 - \beta^k(\delta+1)^2 + (\delta+1)(\beta+1)(\delta-\beta) >$$
$$> \delta^k(\beta+1)^2 - \beta^k(\delta+1)^2 + (\beta\delta + \beta + \delta + 1)(\delta - \beta) > \delta^k(\beta+1)^2 - \beta^k(\delta+1)^2 + \left(\beta\delta + \beta^k + \delta^k + 1\right)(\delta-\beta) =$$
$$= \delta^k\left(\beta^2 + 1 + \beta + \delta\right) - \beta^k\left(\delta^2 + 1 + \delta + \beta\right) + (\beta\delta + 1)(\delta - \beta) > \delta^k\beta^2 - \beta^k\delta^2 + (\beta\delta + 1)(\delta - \beta).$$

Let us show that

$$\delta^k\beta^2 - \beta^k\delta^2 + (\beta\delta + 1)(\delta - \beta) > 0,$$

which is equivalent to the inequality

$$\left(\delta^{\frac{k}{2}}\beta-\beta^{\frac{k}{2}}\delta\right)\left(\delta^{\frac{k}{2}}\beta+\beta^{\frac{k}{2}}\delta\right)>\beta\delta(\beta-\delta)\frac{\beta\delta+1}{\beta\delta}.$$

Since the function $x^{\frac{k}{2}-1}+x$ is strictly increasing at $x\ge 1$ for $k\in(0,1)$, we get $\delta^{\frac{k}{2}-1}+\delta>\beta^{\frac{k}{2}-1}+\beta$. Hence $\delta^{\frac{k}{2}}\beta-\beta^{\frac{k}{2}}\delta>\beta\delta(\beta-\delta)$. It remains to prove that $\delta^{\frac{k}{2}}\beta+\beta^{\frac{k}{2}}\delta>\frac{\beta\delta+1}{\beta\delta}$. Indeed $\delta^{\frac{k}{2}+1}\beta^2>\beta\delta$, $\beta^{\frac{k}{2}+1}\delta^2>1$. Therefore, $\delta\beta\le 1$.

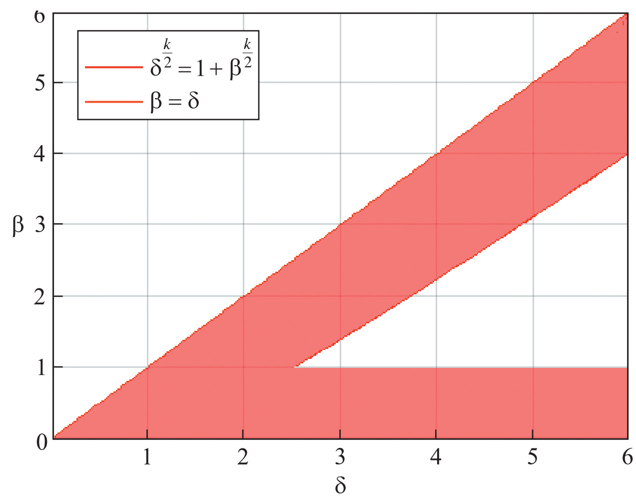

Fig. 1. Illustration to the proof of proposition 1 ($k=1.5$)

Proposition 2. *If there holds property 2, then $a=b=1>c$.*

P r o o f. It follows from proposition 1 that property 2 is not compatible with the case $ac>1$, hence $ac\le 1$. If we assume that $b=c=1$, then $a>c$, and we get $ac>1$. So $a=b=1$. But then $c<1$, as it is required.

Proposition 3. *Property 2 holds if and only if $n\ge N_0$, where*

$$N_0=2\min_{x\in(0,1)}\frac{\log x}{\log(8x+4)-\log\left(x^2+6x+5\right)}=29.143\,359\ldots.$$

P r o o f. It follows from proposition 2 that $a=b=1$, $c<1$. Thus, we need to determine all possible values $k\in(0,1)$ such that there exists $\beta\in(0,1)$ such that $g(1,\beta,k)=0$.

It is easy to see that the equation $g(1,\beta,k)=0$ is equivalent to the equivalent

$$\beta^k=\frac{8\beta+4}{\beta^2+6\beta+5}.$$

Let us consider the function

$$k(\beta)=\frac{\log(8\beta+4)-\log\left(\beta^2+6\beta+5\right)}{\log\beta},\ \beta\in(0,1).$$

Critical points of this functions are the solutions of the equation

$$\frac{2\beta(\beta+2)(1-\beta)}{(2\beta+1)(\beta+5)(\beta+1)}=\frac{\log(8\beta+4)-\log(\beta+1)-\log(\beta+5)}{\log\beta},$$

which has a unique solution $\beta_0\in(0,1)$, $\beta_0=0.139\,697\,7\ldots$. This can be verified with the help of the results of the paper [6].

Since $k(+0)=k(1-0)=0$, we deduce that the value set of the function $k(\beta)$ is the segment $(0,k_0]$, where $k_0=k(\beta_0)=0.068\,626\,2\ldots$. Therefore, property 2 holds if and only if $k\le k_0$, which is equivalent to $n\ge N_0$.

Proposition 4. *For any $n>N_0=29.143\,359\ldots$ there exists a triangle with sides a, b, c such that $a>1=b>c$ and $l_{a,n}=l_{c,n}$.*

P r o o f. By proposition 3, there exists a triangle with sides $a_0 = b_0 = 1 > c_0$ such that $g(\delta_0, \beta_0, k_0) = 0, k_0 = \frac{2}{N_0}$. It is clear that $g(\delta_0, \beta_0, k) < 0$ for any $k \in (0, k_0)$. Also for any $\varepsilon > 0$ we have $g(\delta_1, \beta_1, k) > g(\delta_1, \beta_1, 0) = 0$, where $\delta_1 = \delta_0 + \varepsilon$; $\beta_1 = \frac{1}{\delta_1}$. By the intermediate values theorem, for any $k \in (0, k_0)$ one can find the point (δ_k, β_k), which belongs to the segment with endpoints at (δ_0, β_0), (δ_1, β_1) such that $g(\delta_k, \beta_k, k) = 0$. It remains to check that there exists a triangle with sides $a_k = \delta_k^{\frac{k}{2}}$, $b_k = 1$, $c_k = \beta_k^{\frac{k}{2}}$. Taking $\varepsilon \in (0, 1)$, we obtain the needed triangle inequality

$$a_k = \delta_k^{\frac{k}{2}} < \delta_1^{\frac{k}{2}} = (1 + \varepsilon)^{\frac{k}{2}} < (1 + \varepsilon)^{\frac{k_0}{2}} < 2^{\frac{k_0}{2}} < 1 + \beta_0^{\frac{k_0}{2}} < 1 + \beta_0^{\frac{k}{2}} < 1 + \beta_k^{\frac{k}{2}} = b_k + c_k.$$

Remark 1. It follows from proposition 4 that we may assume that $n \le N_0$ while investigating property 1.

Let us consider the function

$$L(\delta, \beta, k) = \frac{g(\delta, \beta, k)}{\delta - \beta}$$

in the domain $1 > \delta > \beta > 0$, $k \in (0, 1)$. It is easy to see that

$$L(\delta, \beta, k) = \frac{\delta^k - \beta^k}{\delta - \beta}\left(\beta(\delta + 1)^2 + (\beta + 1)^2(\delta + 1)\right) - \beta^k(\delta + \beta + 2) + (\beta + 1)(\delta + 1).$$

We shall examine the function $L(\delta, \beta, k)$ in the domain $(0, 1) \times (0, 1) \times (0, 1)$, assuming that the function is defined by continuity at the line $\delta = \beta$.

We are going to find all values of k such that the curve $L(\delta, \beta, k) = 0$ has common points with the line $\delta = \beta$. Let us consider the function $H(\beta, k) = L(\beta, \beta, k)$. We get

$$H(\beta, k) = k\beta^{k-1}(\beta + 1)^2(2\beta + 1) - (2\beta + 2)\beta^k + (\beta + 1)^2 = (\beta + 1)H_1(\beta, k).$$

It is sufficient to find all k such that the following equation has a solution $\beta \in (0, 1)$:

$$H_1(\beta, k) = k\beta^{k-1}\left(2\beta^2 + 3\beta + 1\right) - 2\beta^k + \beta + 1 = 0.$$

Multiplying by $\beta > 0$, we obtain

$$\hat{H}(\beta, k) = \beta^k\left(2k\beta^2 + (3k - 2)\beta + k\right) + \beta(\beta + 1) = 0.$$

We consider the implicit function $k = k(\beta)$ defined by the last equation. It is sufficient to check the critical points of this function, the boundary points and the points in which the implicit function theorem fails. Critical points can be found by solving the equation

$$\hat{H}'_\beta = 2k(k + 2)\beta^{k+1} + (3k - 2)(k + 1)\beta^k + k^2\beta^{k-1} + 2(\beta + 1) = 0.$$

Multiplying by β and subtracting the equation $\hat{H}(\beta, k) = 0$, we obtain

$$\beta^k\left(\left(2k^2 + 2k\right)\beta^2 + \left(3k^2 - 2k\right)\beta + k^2 - k\right) + \beta^2 = 0.$$

So we get

$$\frac{\beta + 1}{2k\beta^2 + (3k - 2)\beta + k} = \frac{\beta}{\left(2k^2 + 2k\right)\beta^2 + \left(3k^2 - 2k\right)\beta + k^2 - k}.$$

Solving this equation with respect to k, we obtain

$$k = \frac{3\beta + 1 \pm \sqrt{\beta^2 + 6\beta + 1 - 16\beta^3}}{2(\beta + 1)(2\beta + 1)}.$$

It is clear that the largest root does not work, since

$$k \ge \frac{3\beta + 1}{2(\beta + 1)(2\beta + 1)} > \frac{1}{3},$$

and the equality $H_1(\beta, k) = 0$ fails in this case because of

$$0 = H_1(\beta, k) > \frac{1}{3}\left(2\beta^2 + 3\beta + 1\right) - 2\beta^k + \beta + 1$$

and $\beta^{\frac{1}{3}} > \beta^k > \frac{1}{3}\beta^2 + \beta + \frac{2}{3}$. Then $\beta > \frac{8}{27} \Rightarrow \beta > 0.97$, and we get $\frac{1}{3}\beta^2 + \beta + \frac{2}{3} > 1$. Hence

$$k = \frac{3\beta + 1 - \sqrt{\beta^2 + 6\beta + 1 - 16\beta^3}}{2(\beta + 1)(2\beta + 1)}.$$

Let us obtain an equation on β for finding the critical point. We have

$$\beta^k = -\frac{\beta(\beta + 1)}{k(\beta + 1)(2\beta + 1) - 2\beta},$$

then

$$\beta^{-k} = \frac{2}{\beta + 1} - \frac{k(2\beta + 1)}{\beta}.$$

Substituting the found value k, we obtain

$$\beta^{-k} = \frac{\beta - 1 + \sqrt{\beta^2 + 6\beta + 1 - 16\beta^3}}{2\beta(\beta + 1)}.$$

Finally, we get

$$\frac{3\beta + 1 - \sqrt{\beta^2 + 6\beta + 1 - 16\beta^3}}{2(\beta + 1)(2\beta + 1)} + \frac{\log\left(\beta - 1 + \sqrt{\beta^2 + 6\beta + 1 - 16\beta^3}\right) - \log\left(2\beta^2 + 2\beta\right)}{\log\beta} = 0.$$

Applying an algorithm from the work [7], one can show that this equation has a unique root $\beta_1 = 0.304\,55\ldots$. The corresponding value of k is $k_1 = 0.081\,612\ldots$.

Let us investigate the existence of points in which the explicit function theorem fails:

$$H_k'(\beta, k) = 0.$$

By routine computations, we find

$$k = \frac{2\beta}{(\beta + 1)(2\beta + 1)} - \frac{1}{\log\beta}.$$

Taking into account

$$\beta^{-k} = \frac{2}{\beta + 1} - \frac{k(2\beta + 1)}{\beta},$$

we obtain the equation

$$\beta^{-k} = \frac{2\beta + 1}{\beta\log\beta},$$

which does not have solutions $\beta \in (0, 1)$, since $\log\beta < 0$.

So the equation $L(\beta, \beta, k) = 0$ has a solution $\beta \in (0, 1)$ if and only if $k \in (0, k_1]$ with $k_1 = 0.081\,612\ldots$. We set $N_1 = \frac{2}{k_1} = 24.506\,13\ldots$.

Proposition 5. *For any $n \in [0, N_1]$ property 1 does not hold.*

Proof. By proposition 1, we may assume that $n > 2$, which means $k \in [k_1, 1)$, where $k_1 = 0.081\,612\ldots$. For any fixed $k \in [k_1, 1)$ let us consider the function $g(\delta, \beta, k)$ on the compact set $1 \geq \delta \geq \beta \geq 0$. We are going to prove that $g(\delta, \beta, k) > 0$ at all points of this compact set except the line $\beta = \delta$. Let us show that $g_\delta'(\delta, \beta, k) > 0$ for $1 > \delta > \beta > 0$:

$$g_\delta'(\delta, \beta, k) = k\delta^{k-1}\left(\beta(\delta + 1)^2 + (\delta + 1)(\beta + 1)^2\right) + \delta^k\left(2\beta(\delta + 1) + (\beta + 1)^2\right) -$$

$$- \beta^k\left((\beta + 1)^2 + 2(\delta + 1)(\beta + 1)\right) + (\beta + 1)(2\delta - \beta + 1) >$$

$$> k\delta^{k-1}\left(\beta(\delta+1)^2 + (\delta+1)(\beta+1)^2\right) - 2\beta^k(\delta+1) + (\beta+1)(2\delta-\beta+1) >$$

$$> k\beta^{k-1}(\beta+1)^2(2\beta+1) - 2\beta^k(\delta+1) + (\beta+1)(2\delta-\beta+1) =$$

$$= k\beta^{k-1}(\beta+1)^2(2\beta+1) - (\beta+1)^2 + 2(\delta+1)\left(\beta+1-\beta^k\right) >$$

$$> (\beta+1)\left(k\beta^{k-1}(\beta+1)(2\beta+1) - (\beta+1) + 2\left(\beta+1-\beta^k\right)\right) =$$

$$= (\beta+1)\beta^{k-1}\left(k(\beta+1)(2\beta+1) - 2\beta + \beta(\beta+1)\beta^{-k}\right) \ge (\beta+1)\beta^{k-1}h(\beta),$$

where $h(\beta) = k_1(\beta+1)(2\beta+1) - 2\beta + \beta(\beta+1)\beta^{-k_1}$. Since $h(\beta) = \beta^{-k_1}\hat{H}(\beta, k_1)$, it follows from the arguments of above proposition 5 that the equality $h(\beta) = 0$, $\beta \in (0, 1)$, holds only if $\beta = \beta_1$. Because of $h(0) = k_1 > 0$ and $h(1) = 6k_1 > 0$, we get the inequality $h(\beta) \ge 0$ for all $\beta \in [0, 1]$.

Proposition 6. *Let $N_1 < n \le N_0$. For any $\varepsilon > 0$ one can find δ, β such that $1 > \delta > \beta > \beta_1$, $0 < \delta - \beta \le \varepsilon$ and $L(\delta, \beta, k) = 0$, where $\beta_1 = 0.304\,55\ldots$; $k = \frac{2}{n}$.*

P r o o f. Let us show that for any fixed $n \in (N_1, N_0]$ one can find a solution $\beta_2 \in (\beta_1, 1)$ of the equation $L(\beta, \beta, k) = 0$. It is sufficient to prove that functions $L(\beta_1, \beta_1, k)$ and $L(1, 1, k)$ have different signs. Clearly, $L(1, 1, k) = 12k > 0$. To prove that $L(\beta_1, \beta_1, k) < 0$ it is enough to check that the function $H_1(\beta_1, k)$ defined before proposition 5 is increasing as a function of k, since $H_1(\beta_1, k_1) = 0$. We consider the derivative

$$\frac{\partial H_1(\beta_1, k)}{\partial k} = \beta_1^k \log\beta_1\left(2k\beta_1^2 + (3k-2)\beta_1 + k\right) + \beta_1^k\left(2\beta_1^2 + 3\beta_1 + 1\right) = \beta_1^k(ka + b),$$

where $a = -2.49\ldots$; $b = 2.82\ldots$. It is easy to see that $\frac{\partial H_1(\beta_1, k)}{\partial k} > 0$ for all $k \in (0, 1)$. Hence $L(\beta_1, \beta_1, k) < 0$.

We are going to prove that $L'_\beta(\beta, \beta, k) > 0$ for any $\beta \in (\beta_1, 1)$, $k \in \left[\frac{2}{N_0}, \frac{2}{N_1}\right]$. So we have

$$L'_\beta(\beta, \beta, k) = \left(k^2 + 2k\right)\beta^{k+1} + \left(\frac{5}{2}k^2 + \frac{3}{2}k - 1\right)\beta^k + \left(2k^2 - k\right)\beta^{k-1} + \frac{k^2 - k}{2}\beta^{k-2} + \beta + 1.$$

Let us find the minimum of the function $R(\beta, k) = L'_\beta(\beta, \beta, k)$ within the closed rectangle $[\beta_1, 1] \times \left[\frac{2}{N_0}, \frac{2}{N_1}\right]$.

We will prove that the function $R(\beta, k)$ has no critical points within the open rectangle $(\beta_1, 1) \times \left(\frac{2}{N_0}, \frac{2}{N_1}\right)$. Let us consider the derivative

$$R'_\beta(\beta, k) = \left(k^2 + 2k\right)(k+1)\beta^k + k\left(\frac{5}{2}k^2 + \frac{3}{2}k - 1\right)\beta^{k-1} +$$

$$+ (k-1)\left(2k^2 - 2\right)\beta^{k-2} + (k-2)\frac{k^2 - k}{2}\beta^{k-3} + 1.$$

It is clear that $R'_\beta(\beta, k) > -k\beta^{k-1} + 1 > 0$ in the mentioned domain. Hence, the minimum of the function $R(\beta, k)$, $(\beta, k) \in [\beta_1, 1] \times \left[\frac{2}{N_0}, \frac{2}{N_1}\right]$, is achieved on the boundary of the rectangle. Let us prove that $R\left(\beta_1, \frac{2}{N_1}\right) = 0$ and $R\left(\beta_1, \frac{2}{N_1}\right) > 0$ on the boundary of the rectangle except the point $\beta = \beta_1$, $k = \frac{2}{N_1}$. To prove the equality $R\left(\beta_1, \frac{2}{N_1}\right) = 0$ let us recall properties of functions $H(\beta, k)$, $\hat{H}(\beta, k)$ defined before proposition 5. Since $H'_\beta(\beta, k) = L'_\delta(\beta, \beta, k) + L'_\beta(\beta, \beta, k) = 2R(\beta, k)$, it is sufficient to show that $H'_\beta\left(\beta_1, \frac{2}{N_1}\right) = 0$. Taking into account

$H(\beta, k) = \beta(\beta+1)\hat{H}(\beta, k)$, we get $H'_\beta = (2\beta+1)\hat{H} + (\beta^2+\beta)\hat{H}'_\beta$. Since $\hat{H}\left(\beta_1, \frac{2}{N_1}\right) = 0$, $\hat{H}'_\beta\left(\beta_1, \frac{2}{N_1}\right) = 0$ by definition of β_1 and N_1, we obtain the required $H'_\beta\left(\beta_1, \frac{2}{N_1}\right) = 0$.

Let us examine the boundary $\beta = 1$, $k \in \left[\frac{2}{N_0}, \frac{2}{N_1}\right]$. We have $R(1, k) = 6k^2 + 2k + 2 > 0$. Let us start with the boundary $k = \frac{2}{N_0}$, $\beta \in [\beta_1, 1]$. Since $R'_\beta(\beta, k) > 0$, it is enough to show that $R\left(\beta_1, \frac{2}{N_0}\right) > 0$. The latter can be verified straightforwardly using decimal approximations of β_1 and N_1. Now we consider the boundary $k = \frac{2}{N_1}$, $\beta \in [\beta_1, 1]$. Since $R'_\beta(\beta, k) > 0$, we obtain $R\left(\beta, \frac{2}{N_1}\right) > R\left(\beta_1, \frac{2}{N_1}\right) = 0$. So it remains to examine the boundary $\beta = \beta_1$, $k \in \left[\frac{2}{N_0}, \frac{2}{N_1}\right]$. We consider the derivative

$$R'_k(\beta_1, k) = (2k+2)\beta_1^{k+1} + \left(5k + \frac{3}{2}\right)\beta_1^k + (4k-1)\beta_1^{k-1} + \left(k - \frac{1}{2}\right)\beta_1^{k-2} +$$

$$+\left((k^2+2k)\beta_1^{k+1} + \left(\frac{5}{2}k^2 + \frac{3}{2}k - 1\right)\beta_1^k + (2k^2-k)\beta_1^{k-1} + \frac{k^2-k}{2}\beta_1^{k-2}\right)\log\beta_1.$$

We will show that $R'_k(\beta_1, k) < 0$. Indeed,

$$R'_k(\beta_1, k) < 2.2 \cdot 0.3 + 2 - 0.6 \cdot 3.1 - 0.4 \cdot 10.3 + 1.2(1 + 0.1 \cdot 3.2 + 0.05 \cdot 10.4) < -1.1.$$

Here we get the needed $R(\beta_1, k) > R\left(\beta_1, \frac{2}{N_1}\right) = 0$.

Since $L'_\delta(\beta_2, \beta_2, k) > 0$, $L'_\beta(\beta_2, \beta_2, k) > 0$, there exists a neighbourhood of the point (β_2, β_2) such that these derivatives have a constant sign. By the Lagrange theorem, for any $m \ge m_0(\beta_2, k)$ there hold $L\left(\beta_2 + \frac{1}{m}, \beta_2, k\right) > 0$, $L\left(\beta_2, \beta_2 - \frac{1}{m}, k\right) < 0$. Hence, the intermediate values theorem implies the existence of $\theta_m \in (0, 1)$ such that $L\left(\beta_2 + \frac{\theta_m}{m}, \beta_2 - \frac{\theta_m}{m}, k\right) = 0$. The relative position of the functions $L(\delta, \beta, k) = 0$ for particular values of the parameter k is illustrated in fig. 2.

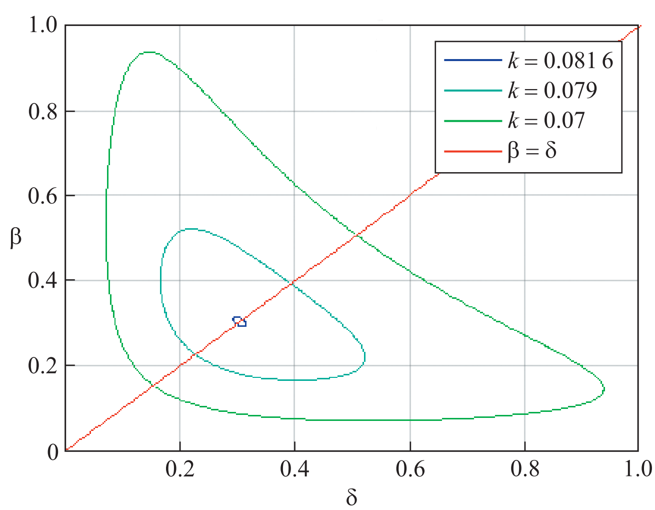

Fig. 2. Illustration to the proof of proposition 6

Corollary 1. *For any $n > N_1$ property 1 holds.*

P r o o f. If $n > N_0$, the required statement follows from proposition 4. If $n \in (N_1, N_0]$, by proposition 6 one can find δ, β such that $\delta > \beta > \beta_1$ and $L(\delta, \beta, k) = 0$. Additionally,

$$a + c = \delta^{\frac{k}{2}} + \beta^{\frac{k}{2}} > 2\beta_1^{\frac{k_1}{2}} > 1,$$

so a triangle with sides a, b, c exists.

Proposition 7. *Property 2 is equivalent to property 3.*

P r o o f. We suppose that property 2 holds for some fixed n, i. e. there exists a triangle such that $a = 1 = b > c$ and $l_{a,n} = l_{c,n}$. It follows from the equality $a = b$ that $l_{a,n} = l_{b,n}$. Hence, property 3 is valid as well. Now we suppose that property 3 holds, but property 2 fails. Since property 2 is not valid, by proposition 3 we get $n < N_0 = 29.143\,359\ldots$. Thus, in a triangle, which satisfies property 3, all sides must have different length. Without loss of generality we may assume that the shortest side c has length with value of 1. Then $a > 1$, $b > 1$. It follows from the equality $l_{a,n} = l_{b,n}$ and proposition 1 that $ab \le 1$, which is impossible. So property 2 must hold.

Remark 2. Validity of theorem 1 for $n > 0$ follows from propositions 3–5 and corollary 1.

Proof of theorem 1: case $n < 0$

We assume that $n < 0$, $b = 1$, $a \le c$. Then $\delta = a^n \ge c^n = \beta$.

Proposition 8. *If there holds either property 1 or 2, then* $\delta\beta \ge 1$.

P r o o f. We assume that there holds property 1 or 2, but $\delta\beta < 1$. Without loss of generality $\delta > \beta$. Since $\delta^k - \beta^k < 0$, we get

$$g(\delta, \beta, k) = \left(\delta^k - \beta^k\right)\left(\beta(\delta + 1)^2 + \delta(\beta + 1)^2\right) + \delta^k(\beta + 1)^2 - \beta^k(\delta + 1)^2 + (\delta + 1)(\beta + 1)(\delta - \beta) <$$
$$< \delta^k(\beta + 1)^2 - \beta^k(\delta + 1)^2 + (\delta + 1)(\beta + 1)(\delta - \beta) := z(\delta, \beta, k).$$

If $\beta \le 1 \le \delta$, then

$$z(\delta, \beta, k) \le (\beta + 1)^2 - (\delta + 1)^2 + (\delta + 1)(\beta + 1)(\delta - \beta) = (\delta - \beta)(\beta\delta - 1) < 0.$$

Thus, $0 < \beta < \delta < 1$. We have

$$z(\delta, \beta, k) = \delta^k(\beta + 1)^2 - \beta^k(\delta + 1)^2 + (\delta + 1)(\beta + 1)(\delta - \beta) <$$
$$< \delta^k\left(\beta^2 + 2\beta\right) - \beta^k\left(\delta^2 + 2\delta\right) + (\beta\delta + \beta + \delta + 1)(\delta - \beta) =$$
$$= \left(2\delta^k\beta - 2\beta^k\delta + (\beta + 1)(\delta - \beta)\right) + \left(\delta^k\beta^2 - \beta^k\delta^2 + (\delta\beta + \delta)(\delta - \beta)\right) := z_1(\delta, \beta, k) + z_2(\delta, \beta, k).$$

Let us prove that $z_1(\delta, \beta, k) < 0$ and $z_2(\delta, \beta, k) < 0$. We obtain

$$z_1(\delta, \beta, k) = \beta\left(2\delta^k - \beta - 1\right) - \delta\left(2\beta^k - \beta - 1\right).$$

Since $2\delta^k - \beta - 1 < 2\beta^k - \beta - 1$ and $0 < \beta < \delta < 1$, we have $z_1(\delta, \beta, k) < 0$. The inequality $z_2(\delta, \beta, k) < 0$ is equivalent to the inequality

$$\delta^k\beta^2 - \beta^k\delta^2 < (\beta\delta + \delta)(\beta - \delta),$$

which is equivalent to the inequality

$$\delta^{k-2} - \beta^{k-2} < \left(1 + \frac{1}{\beta}\right)\left(\frac{1}{\delta} - \frac{1}{\beta}\right).$$

We rewrite it in the from

$$\frac{1}{\delta}\left(\delta^{k-1} - 1 - \frac{1}{\beta}\right) < \frac{1}{\beta}\left(\beta^{k-1} - 1 - \frac{1}{\beta}\right).$$

The last inequality holds, since $\frac{1}{\delta} < \frac{1}{\beta}$ and $\delta^{k-1} < \beta^{k-1}$. Therefore, $z_2(\delta, \beta, k) < 0$ and $g(\delta, \beta, k) < 0$. The proposition is proved.

We set $k = -\frac{2}{n}$, $a = \delta^{-\frac{k}{2}}$, $c = \beta^{-\frac{k}{2}}$, $b = 1$, $\delta \ge \beta > 0$. Then

$$g(\delta, \beta, -k) = \delta^{-k}\Big(\beta(\delta+1)^2 + (\beta+1)^2(\delta+1)\Big) -$$
$$-\beta^{-k}\Big(\delta(\beta+1)^2 + (\beta+1)(\delta+1)^2\Big) + (\beta+1)(\delta+1)(\delta-\beta).$$

Proposition 9. *If there holds either property 1 or 2, then* $k \in (0, 2)$.

P r o o f. We suppose that there holds property 1 or 2, but $k \geq 2$. We examine the derivative

$$g'_\beta(\delta, \beta, -k) = \delta^{-k}\Big((\delta+1)^2 + 2(\beta+1)(\delta+1)\Big) + k\beta^{-k-1}\Big(\delta(\beta+1)^2 + (\beta+1)(\delta+1)^2\Big) -$$
$$-\beta^{-k}\Big(2\delta(\beta+1) + (\delta+1)^2\Big) + (\delta+1)^2 - 2(\beta+1)(\delta+1).$$

Let us prove that $g'_\beta(\delta, \beta, -k) > 0$ for all $\delta \geq \beta > 0$ such that corresponding a, b, c satisfy the triangle inequality. As it was shown in the proof of theorem 2.2 presented in the article [5], the needed inequality is obtained for $k \geq 4$. So it remains to prove this inequality for $k \in [2, 4)$.

Let us consider the case $\beta \geq 1$. By the triangle inequality, $\delta^{-\frac{k}{2}} + \beta^{-\frac{k}{2}} > 1$. Hence $\delta^{-k} > \beta^{-k} - 2\beta^{-\frac{k}{2}} + 1$. Taking into account this inequality, we obtain

$$g'_\beta(\delta, \beta, -k) > 2(\delta+1)^2 + \beta^{-k}\left(\frac{k}{\beta}\delta(\beta+1)^2 + \frac{k}{\beta}(\beta+1)(\delta+1)^2 + 2(\beta+1)\right) -$$
$$-2\beta^{-\frac{k}{2}}\Big((\delta+1)^2 + 2(\beta+1)(\delta+1)\Big) := \omega(\delta, \beta, k).$$

Since $\delta \geq \beta$, we have $\beta^{-\frac{k}{2}} > \frac{1}{2}$, which implies $\beta^{-k} > \frac{1}{4}$. Moreover, $\beta^{-\frac{k}{2}} \leq 1$. It follows from the inequality $\beta^{-\frac{k}{2}} > \frac{1}{2}$ that $\beta < 2^{\frac{2}{k}} \leq 2$. So $\beta \in [1, 2]$.

We prove that $\omega(\delta, \beta, k) > 0$ for $k \in [2, 4]$, $\beta \in [1, 2]$, $\delta \geq \beta$. It is enough to show that $\omega'_\delta(\delta, \beta, k) > 0$ and $\omega(\beta, \beta, k) > 0$.

Firstly, we prove that $\omega'_\delta(\delta, \beta, k) > 0$:

$$\omega'_\delta(\delta, \beta, k) = 4(\delta+1) + \beta^{-k}(\beta+1)\left(\frac{k}{\beta}(\beta+1) + 2\frac{k}{\beta}(\delta+1)\right) - 4\beta^{-\frac{k}{2}}\Big((\delta+1) + (\beta+1)\Big) \geq$$
$$\geq \beta^{-k}(\beta+1)\left(\frac{k}{\beta}(\beta+1) + 2\frac{k}{\beta}(\delta+1) - 4\beta^{\frac{k}{2}}\right) \geq \beta^{-k}(\beta+1)\left(3k\left(1+\frac{1}{\beta}\right) - 4\beta^{\frac{k}{2}}\right) \geq$$
$$\geq \beta^{-k}(\beta+1)t(k),$$

where $t(k) = 4.5k - 4 \cdot 2^{\frac{k}{2}}$. Thus, it remains to prove that $t(k) > 0$:

$$t'(k_0) = 4.5 - 2 \cdot 2^{\frac{k_0}{2}}\log(2) = 0 \Rightarrow k_0 = 2\frac{\Big(\log(2.25) - \log\big(\log(2)\big)\Big)}{\log(2)} = 3.397\,38\ldots.$$

Since $t(k_0) = 2.3\ldots > 0$, $t(2) = 1 > 0$ and $t(4) = 2 > 0$, we get $t(k) > 0$.

Secondly, we will show that $\omega(\beta, \beta, k) > 0$:

$$\omega(\beta, \beta, k) = 2(\beta+1)^2 + \beta^{-k}\left(k(\beta+1)^2 + \frac{k}{\beta}(\beta+1)^3 + 2(\beta+1)\right) - 6\beta^{-\frac{k}{2}}(\beta+1)^2 =$$
$$= \beta^{-k}(\beta+1)\left((\beta+1)\left(2\beta^k - 6\beta^{\frac{k}{2}} + \frac{k}{\beta} + 2k\right) + 2\right) \geq \beta^{-k}(\beta+1)\Big((\beta+1)\varphi(\beta, k) + 2\Big),$$

where $\varphi(\beta, k) = 2\beta^k - 6\beta^{\frac{k}{2}} + 5$. Let $\beta^{\frac{k}{2}} = x$. Then the equality $\varphi(\beta, k) = 0$ implies $2x^2 - 6x + 5 = 0$. Since $\varphi(1, k) > 0$, we can deduce that for any $k \in [2, 4]$, $\beta \in [1, 2]$ the function $\varphi(\beta, k)$ takes only positive values.

So $g'_\beta(\delta, \beta, -k) > 0$ for $\delta^{-\frac{k}{2}} + \beta^{-\frac{k}{2}} > 1, \delta \geq \beta \geq 1.$

Let us consider the case $\beta < 1$. We have

$$g'_\beta(\delta, \beta, -k) > \beta^{-k}\left(\frac{k}{\beta}\delta(\beta+1)^2 + \frac{k}{\beta}(\beta+1)(\delta+1)^2 - 2\delta(\beta+1) - (\delta+1)^2\right) + (\delta+1)^2 - 2(\beta+1)(\delta+1).$$

Since $\beta^{-k} > 1$ and a multiplier for β^{-k} is non-negative for $\beta < 1$, $k \in [2, 4)$, we obtain

$$g'_\beta(\delta, \beta, -k) > \frac{k}{\beta}\delta(\beta+1)^2 + \frac{k}{\beta}(\beta+1)(\delta+1)^2 - 2\delta(\beta+1) - (\delta+1)^2 + (\delta+1)^2 - 2(\beta+1)(\delta+1) \geq$$

$$\geq 2\delta(\beta+1)^2 + 4(\delta+1)^2 - 2\delta(\beta+1) - 2(\beta+1)(\delta+1) > 0.$$

Define new variables $\tilde{\delta} = \frac{1}{\delta}$, $\tilde{\beta} = \frac{1}{\beta}$ and function $\tilde{g}(\tilde{\delta}, \tilde{\beta}, k) := g(\delta, \beta, -k)$. Then

$$\tilde{g}(\tilde{\delta}, \tilde{\beta}, k) = \frac{\tilde{\delta}^k\left(\tilde{\beta}(\tilde{\delta}+1)^2 + (\tilde{\beta}+1)^2(\tilde{\delta}+1)\tilde{\delta}\right) - \tilde{\beta}^k\left(\tilde{\delta}(\tilde{\beta}+1)^2 + (\tilde{\beta}+1)(\tilde{\delta}+1)^2\tilde{\beta}\right)}{\tilde{\delta}^2\tilde{\beta}^2} + \frac{(\tilde{\beta}+1)(\tilde{\delta}+1)(\tilde{\beta}-\tilde{\delta})}{\tilde{\delta}^2\tilde{\beta}^2}.$$

We will prove that for any $k \geq 2$ there holds $\tilde{g}(\tilde{\delta}, f(\tilde{\delta}, k), k) < 0$, where $f(\tilde{\delta}, k) = \left(1 - \tilde{\delta}^{\frac{k}{2}}\right)^{\frac{2}{k}}$, $\tilde{\delta} \in \left(0, \left(\frac{1}{2}\right)^{\frac{2}{k}}\right)$. It is clear that the last statement is equivalent to inequality $h(\tilde{\delta}, k) < 0$, where $h(\tilde{\delta}, k) = \tilde{g}(\tilde{\delta}, f(\tilde{\delta}, k), k)\tilde{\delta}^2 f(\tilde{\delta}, k)^2$.

Taking into account the inequality $f(\tilde{\delta}, k) > \tilde{\delta}$ valid for $\tilde{\delta} \in \left(0, \left(\frac{1}{2}\right)^{\frac{2}{k}}\right)$, we obtain

$$h(\tilde{\delta}, k) = \tilde{\delta}^k\left(\tilde{\delta} - f(\tilde{\delta}, k)\right)\left(\tilde{\delta} + f(\tilde{\delta}, k) + 2f(\tilde{\delta}, k)\tilde{\delta}\right) +$$

$$+ 2\tilde{\delta}^{\frac{k}{2}}\left(\left(1 + f(\tilde{\delta}, k)\right)^2\tilde{\delta} + \left(1 + f(\tilde{\delta}, k)\right)(1 + \tilde{\delta})^2 f(\tilde{\delta}, k)\right) - \tilde{\delta}\left(1 + f(\tilde{\delta}, k)\right)^2(2 + \tilde{\delta}) <$$

$$< \left(1 + f(\tilde{\delta}, k)\right)\left(2\tilde{\delta}^{\frac{k}{2}}\left(\left(1 + f(\tilde{\delta}, k)\right)\tilde{\delta} + (1 + \tilde{\delta})^2 f(\tilde{\delta}, k)\right) - \tilde{\delta}\left(1 + f(\tilde{\delta}, k)\right)(2 + \tilde{\delta})\right) =$$

$$= \tilde{\delta}\left(1 + f(\tilde{\delta}, k)\right)\left(f(\tilde{\delta}, k)\left(2\tilde{\delta}^{\frac{k}{2}-1}(1 + 3\tilde{\delta}) - 2 + \tilde{\delta}\left(2\tilde{\delta}^{\frac{k}{2}} - 1\right)\right) + 2\tilde{\delta}^{\frac{k}{2}} - 2 - \tilde{\delta}\right) < \tilde{\delta}\left(1 + f(\tilde{\delta}, k)\right)\tilde{h}(\tilde{\delta}, k),$$

where $\tilde{h}(\tilde{\delta}, k) = 2f(\tilde{\delta}, k)\left(\tilde{\delta}^{\frac{k}{2}-1}(1 + 3\tilde{\delta}) - 1\right) + 2\tilde{\delta}^{\frac{k}{2}} - 2 - \tilde{\delta}$. It is sufficient to prove that $\tilde{h}(\tilde{\delta}, k) < 0$.

The required statement is true for $m = p \in \mathbb{N}$ if for any $k \in [2p, 2p + 2]$ and $\tilde{\delta} \in \left(0, \left(\frac{1}{2}\right)^{\frac{2}{k}}\right)$ there holds the inequality $\tilde{h}(\tilde{\delta}, k) < 0$. We will prove that by induction on $m \in \mathbb{N}$.

Let $m = 1$. Then $k \in [2, 4]$ and $\tilde{\delta} \in \left(0, \left(\frac{1}{2}\right)^{\frac{2}{k}}\right)$.

We prove that for any fixed $k \geq 2$ there holds $f(\tilde{\delta}, k) \leq 2\left(\frac{1}{2}\right)^{\frac{2}{k}} - \tilde{\delta}$. Indeed, since $f''_{\tilde{\delta}}(\tilde{\delta}, k) = \left(1 - \frac{k}{2}\right)\tilde{\delta}^{\frac{k}{2}-2} \times \times \left(1 - \tilde{\delta}^{\frac{k}{2}}\right)^{\frac{2}{k}-2} \leq 0$, we see that $f(\tilde{\delta}, k)$ is a concave function, and the line $\tilde{\beta} = 2\left(\frac{1}{2}\right)^{\frac{2}{k}} - \tilde{\delta}$ is a tangent line to the function $f(\tilde{\delta}, k)$ at the point $\tilde{\delta} = \left(\frac{1}{2}\right)^{\frac{2}{k}}$.

Let us consider the case $\tilde{\delta} \in \left[\frac{1}{2}, \left(\frac{1}{2}\right)^{\frac{2}{k}}\right)$, $k \in (2, 4]$. Since $\tilde{\delta}^{\frac{k}{2}-1}(1 + 3\tilde{\delta}) - 1 \geq \tilde{\delta} + 3\tilde{\delta}^2 - 1 \geq \frac{1}{4} > 0$, we obtain the estimate

$$\tilde{h}(\tilde{\delta}, k) \leq 2\left(2\left(\frac{1}{2}\right)^{\frac{2}{k}} - \tilde{\delta}\right)\left(\tilde{\delta}^{\frac{k}{2}-1}(1 + 3\tilde{\delta}) - 1\right) + 2\tilde{\delta}^{\frac{k}{2}} - 2 - \tilde{\delta}.$$

Let us denote the right-hand side of this inequality by $s(\tilde{\delta}, k)$. Then we obtain

$$s(\tilde{\delta}, k) = -6\tilde{\delta}^{\frac{k}{2}+1} + 12\alpha(k)\tilde{\delta}^{\frac{k}{2}} + 4\alpha(k)\tilde{\delta}^{\frac{k}{2}-1} + \tilde{\delta} - 2(1 + 2\alpha(k)),$$

where $\alpha(k) = \left(\frac{1}{2}\right)^{\frac{2}{k}}$. We note that $s(\alpha(k), k) = 0$. So it is sufficient to show that $s'_{\tilde{\delta}}(\tilde{\delta}, k) > 0$ for $\tilde{\delta} \in (0, \alpha(k))$, $k \geq 2$:

$$s'_{\tilde{\delta}}(\tilde{\delta}, k) = -3(k + 2)\tilde{\delta}^{\frac{k}{2}} + 6\alpha(k)k\tilde{\delta}^{\frac{k}{2}-1} + 2\alpha(k)(k - 2)\tilde{\delta}^{\frac{k}{2}-2} + 1 >$$
$$> -3(k + 2)\tilde{\delta}^{\frac{k}{2}} + 6\tilde{\delta}k\tilde{\delta}^{\frac{k}{2}-1} + 2\alpha(k)(k - 2)\tilde{\delta}^{\frac{k}{2}-2} + 1 =$$
$$= 3(k - 2)\tilde{\delta}^{\frac{k}{2}} + 2\alpha(k)(k - 2)\tilde{\delta}^{\frac{k}{2}-2} + 1 > 0.$$

Let us consider the case $\tilde{\delta} \in \left(0, \frac{1}{2}\right)$, $k \in [2, 4]$. We will prove the required inequality for $k = 2$. In this case, the function $\tilde{h}(\tilde{\delta}, 2)$ is a parabola with vertex at the point $\tilde{\delta}_0 = \frac{7}{12}$ and branches directed below. Then

$$\tilde{h}(\tilde{\delta}, 2) = -6\tilde{\delta}^2 + 7\tilde{\delta} - 2 < -6\left(\frac{1}{2}\right)^2 + 7\left(\frac{1}{2}\right) - 2 = 0.$$

Let $k \in (2, 4]$. We fix an arbitrary $\tilde{\delta} \in \left(0, \frac{1}{2}\right)$. There are two alternatives:

1) $\tilde{\delta}^{\frac{k}{2}-1}(1 + 3\tilde{\delta}) - 1 \geq 0$. As it was shown before, we have

$$\tilde{h}(\tilde{\delta}, k) \leq s(\tilde{\delta}, k) < 0;$$

2) $\tilde{\delta}^{\frac{k}{2}-1}(1 + 3\tilde{\delta}) - 1 < 0$. Then

$$\tilde{h}(\tilde{\delta}, k) \leq 2(1 - \tilde{\delta})\left(\tilde{\delta}^{\frac{k}{2}-1}(1 + 3\tilde{\delta}) - 1\right) + 2\tilde{\delta}^{\frac{k}{2}} - 2 - \tilde{\delta} \leq \tilde{h}(\tilde{\delta}, 2) < 0.$$

We suppose that the required statement is valid for $m = p - 1 \in \mathbb{N}$, which means $\tilde{h}(\tilde{\delta}, k) < 0$ for any $k \in [2p - 2, 2p]$ and $\tilde{\delta} \in \left(0, \left(\frac{1}{2}\right)^{\frac{2}{k}}\right)$. Let us show that the statement is valid for $m = p$.

We examine the case $\tilde{\delta} \in \left[\left(\frac{1}{2}\right)^{\frac{1}{p}}, \left(\frac{1}{2}\right)^{\frac{2}{k}}\right)$, $k \in (2p, 2p + 2]$. Let us prove that $\tilde{\delta}^{\frac{k}{2}-1}(1 + 3\tilde{\delta}) - 1 > 0$:

$$\tilde{\delta}^{\frac{k}{2}-1}\left(1+3\tilde{\delta}\right)-1 \geq \left(\left(\frac{1}{2}\right)^{\frac{1}{p}}\right)^{p}\left(1+3\left(\frac{1}{2}\right)^{\frac{1}{p}}\right)-1=\frac{1}{2}\left(3\left(\frac{1}{2}\right)^{\frac{1}{p}}-1\right)>\left[2<3^{p}\right]>0.$$

Then we obtain

$$\tilde{h}\left(\tilde{\delta}, k\right) \leq s\left(\tilde{\delta}, k\right)<0.$$

Let us examine the case $\tilde{\delta} \in \left(0, \left(\frac{1}{2}\right)^{\frac{1}{p}}\right)$, $k \in \left[2p, 2p+2\right]$. We fix an arbitrary $\tilde{\delta} \in \left(0, \left(\frac{1}{2}\right)^{\frac{1}{p}}\right)$. As it was shown for $k \in \left(2, 4\right]$, there are two possible alternatives. In the first alternative the needed inequality is obvious, so we consider the second alternative, i. e. $\tilde{\delta}^{\frac{k}{2}-1}\left(1+3\tilde{\delta}\right)-1<0$. Then

$$\tilde{h}\left(\tilde{\delta}, k\right) \leq 2\left(1-\tilde{\delta}^{p}\right)^{\frac{1}{p}}\left(\tilde{\delta}^{\frac{k}{2}-1}\left(1+3\tilde{\delta}\right)-1\right)+2\tilde{\delta}^{\frac{k}{2}}-2-\tilde{\delta} \leq \tilde{h}\left(\tilde{\delta}, 2p\right)<0.$$

On the plain $O\tilde{\delta}\tilde{\beta}$, we consider a domain bounded by the lines $\tilde{\delta}=0$, $\tilde{\beta}=\tilde{\delta}$ and the curve $\tilde{\beta}=f\left(\tilde{\delta}, k\right)$ (fig. 3). As it was already shown, we have $g'_{\beta}\left(\delta, \beta, -k\right)>0$ in this domain, which means $\tilde{g}'_{\tilde{\beta}}\left(\tilde{\delta}, \tilde{\beta}, k\right)<0$. Also it was shown that $\tilde{g}\left(\tilde{\delta}, f\left(\tilde{\delta}, k\right), k\right)<0$ for $\tilde{\delta} \in \left(0, \left(\frac{1}{2}\right)^{\frac{2}{k}}\right)$. Since $\tilde{g}\left(\tilde{\delta}, \tilde{\delta}, k\right)=0$, we obtain $\tilde{g}\left(\tilde{\delta}, \tilde{\beta}, k\right)<0$ in the mentioned domain (coloured in red in fig. 3) that contradicts properties 1 and 2.

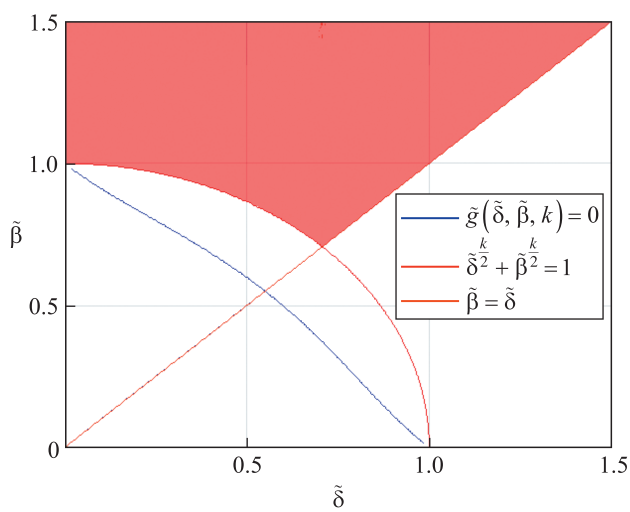

Fig. 3. Illustration to the proof of proposition 9 ($k = 4$)

Proposition 10. *If property 2 holds, then $c = b = 1 > a$.*

P r o o f. The needed statement follows from proposition 8.

Proposition 11. *Property 2 holds if and only if $n < -2$.*

P r o o f. By proposition 10, we have $c = b = 1$, $a < 1$. Let $k=\frac{2}{n}$. Similarly to the proof of proposition 3, we deduce that property 2 is valid for some $n < 0$ if and only if the equation

$$\delta^{k}=\frac{8\delta+4}{\delta^{2}+6\delta+5}$$

is solvable for $\delta \in \left(1, \infty\right)$.

Let us consider the function

$$k(\delta) = \frac{\log(8\delta + 4) - \log(\delta^2 + 6\delta + 5)}{\log \delta}, \ \delta \in (1, \infty).$$

We have $k(1+0) = 0$, $k(\infty) = -1$. Since $1 < \dfrac{\delta^2 + 6\delta + 5}{8\delta + 4} < \delta$, the value set of the continuous function $k(\delta)$, $\delta \in (1, \infty)$, is the interval $(-1, 0)$.

Proposition 12. *For any $n < -2$ there exists a triangle with sides a, b, c such that $c > 1 = b > a$ and $l_{a,n} = l_{c,n}$.*

P r o o f. Let us fix an arbitrary $n_1 < -2$, we set $k_1 = \dfrac{2}{n_1}$. Take $n_0 \in (n_1, -2)$. By proposition 11, there exists a triangle with sides *a*, *b*, *c* such that $c_0 = b_0 = 1 > a_0$ and $g(\delta_0, \beta_0, k_0) = 0$, $k_0 = \dfrac{2}{n_0}$. We have $g(\delta_0, \beta_0, k_1) > 0$ since the function $k \to g(\delta_0, \beta_0, k)$ is increasing. Also for any sufficiently small $\varepsilon > 0$ we get $g(\delta_1, \beta_1, k_1) < < g(\delta_1, \beta_1, 0) = 0$, where $\beta_1 = \beta_0 - \varepsilon$; $\delta_1 = \dfrac{1}{\beta_1}$. By the intermediate values theorem, one can find the point (δ_2, β_2), which belongs to the segment with endpoints (δ_0, β_0), (δ_1, β_1) such that $g(\delta_2, \beta_2, k_1) = 0$. It remains to check that the numbers $a_1 = \delta_2^{\frac{k_1}{2}}$, $b_1 = 1$, $c_1 = \beta_2^{\frac{k_1}{2}}$ define a triangle. Taking a sufficiently small $\varepsilon \in (0, 1)$, we obtain

$$c_1 = \beta_2^{\frac{k_1}{2}} < \beta_1^{\frac{k_1}{2}} = (1-\varepsilon)^{\frac{k_1}{2}} < (1-\varepsilon)^{\frac{k_0}{2}} < 1 + \delta_0^{\frac{k_0}{2}} <$$
$$< 1 + \delta_0^{\frac{k_1}{2}} < 1 + \delta_2^{\frac{k_1}{2}} = b_1 + a_1,$$

as it is required.

Remark 3. By propositions 9 and 12, without loss of generality we may assume that $n \in [-2, -1)$ while investigating property 1.

Proposition 13. *For any $n \in [-2, -1)$ property 1 holds.*

P r o o f. Let us consider an equation $L(\delta, \beta, k) = 0$ with the function L defined after proposition 4. We introduce new variables $\tilde{\delta} = \dfrac{1}{\delta}$, $\tilde{\beta} = \dfrac{1}{\beta}$, $\tilde{k} = -k$. Then the equation $L(\delta, \beta, k) = 0$ is equivalent to the equation $G(\tilde{\delta}, \tilde{\beta}, \tilde{k}) = 0$, where

$$G(\delta, \beta, k) = \frac{\delta^k - \beta^k}{\beta - \delta}\left(\beta(\delta+1)^2 + (\beta+1)^2 \delta(\delta+1)\right) - \beta^k(\delta + \beta + 2\delta\beta) + (\beta+1)(\delta+1).$$

It follows from proposition 8 that $\tilde{\delta}\tilde{\beta} \le 1$. Moreover, $\tilde{k} \in [1, 2)$.

Let us prove that for any $\tilde{k} \in [1, 2)$ the equation $G(\tilde{\delta}, \tilde{\beta}, \tilde{k}) = 0$ has a solution in the domain $0 < \tilde{\delta} < \tilde{\beta} < 1$ such that there exists a triangle with sides $a = \delta^{\frac{k}{2}} = \tilde{\delta}^{\frac{\tilde{k}}{2}}$, $b = 1$, $c = \beta^{\frac{k}{2}} = \tilde{\beta}^{\frac{\tilde{k}}{2}}$. Clearly, the existence of such triangle is defined by the inequality $\tilde{\delta}^{\frac{\tilde{k}}{2}} + \tilde{\beta}^{\frac{\tilde{k}}{2}} > 1$ (fig. 4).

We fix an arbitrary $\tilde{k} \in [1, 2)$. Let us consider the function $\beta = f(\delta) = \left(1 - \delta^{\frac{\tilde{k}}{2}}\right)^{\frac{2}{\tilde{k}}}$ and the implicit function $\beta = F(\delta, \tilde{k})$ defined by the equation $G(\delta, \beta, \tilde{k}) = 0$ in a neighbourhood of the point δ = 0, β = 1. Indeed, such implicit function exists in some one-sided neighbourhood of the point $\delta = 0$, $\beta = 1$, since $G(0, 1, \tilde{k}) = 0$ and $G'_\beta(0, 1, \tilde{k}) \ne 0$. As $f(0) = F(0, \tilde{k}) = 1$, it is enough to show that $f'(0) < F'_\delta(0, \tilde{k})$.

By straightforward computations, we obtain $f'(0) = -\infty$, $F'_\delta(0, \tilde{k}) = -\dfrac{4}{\tilde{k}}$ for $\tilde{k} \in (1, 2)$ and $F'_\delta(0, \tilde{k}) = -3.5$ for $\tilde{k} = 1$.

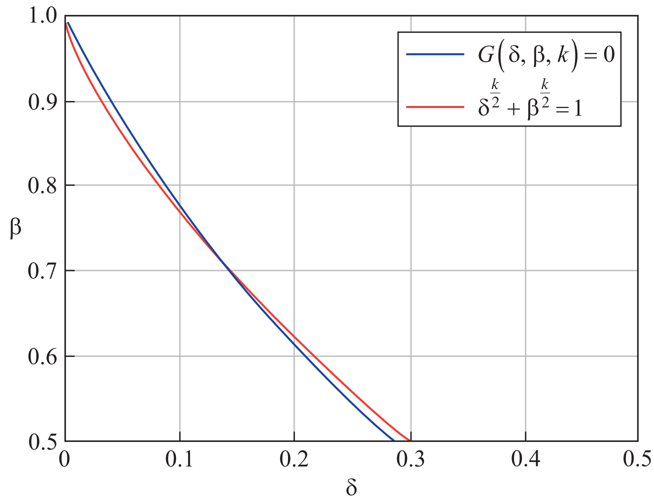

Fig. 4. Illustration to the proof of proposition 13 ($k = 1.5$)

Remark 4. Applying the same arguments as were used in proposition 7, one can obtain that properties 2 and 3 are equivalent for $n < 0$. Hence, the validity of theorem 1 for $n < 0$ follows from propositions 9, 11–13.

Proof of theorem 2

We consider number-theoretic properties of the so called critical triangle, i. e. a unique (up to homothety) non-equilateral triangle with three equal n-lines $l_{a,n} = l_{b,n} = l_{c,n}$ for minimal possible $n \geq 0$. As it was shown in proposition 3, such triangle is isosceles and has sides $a = b = 1$ and $c = \beta_0^{\frac{k_0}{2}}$, where $\beta_0 = 0.139\,697\,7\ldots$ and $k_0 = 0.068\,626\,2\ldots$, i. e. $c = 0.934\,692\,6\ldots$.

Proposition 14. *The critical triangle can not be straightedge-and-compass constructed.*

P r o o f. It is sufficient to prove that the number $c = \beta_0^{k_0}$ is transcendental. We assume the contrary, i. e. that $\beta_0^{k_0}$ is an algebraic number. Since

$$\beta_0^{k_0} = \frac{8\beta_0 + 4}{\beta_0^2 + 6\beta_0 + 5},$$

the right-hand side is an algebraic number as well. Hence, β_0 is also an algebraic number. As it was shown in proposition 3, the number β_0 is a solution of the equation

$$\frac{2\beta(2+\beta)(1-\beta)}{(2\beta+1)(\beta+1)(\beta+5)} \log\beta = \log\frac{8\beta+4}{(\beta+1)(\beta+5)}.$$

The numbers $\beta = \beta_0$, $\gamma = \frac{8\beta_0 + 4}{(\beta_0+1)(\beta_0+5)}$, $\eta = \frac{2\beta_0(2+\beta_0)(1-\beta_0)}{(2\beta_0+1)(\beta_0+1)(\beta_0+5)}$ are algebraic. Moreover, $\log\gamma = \eta\log\beta$. Hence $\gamma = \beta^{\eta}$. By Gelfond – Schneider theorem [8], the number β^{η} can not be algebraic if η is an irrational algebraic number. So the number η is rational. Hence, the degree of the number β does not exceed 3.

We consider the case in which β is a rational number. Clearly, the number

$$\gamma\beta = \beta^{\eta+1} = \beta^{\frac{11\beta^2 + 20\beta + 5}{(2\beta+1)(\beta+1)(\beta+5)}}$$

is also rational. Let $\beta = \frac{m}{n}$, where $(m, n) = 1$, then

$$\frac{11\beta^2 + 20\beta + 5}{(2\beta+1)(\beta+1)(\beta+5)} = \frac{11m^2n + 20mn^2 + 5n^3}{2m^3 + 13m^2n + 16mn^2 + 5n^3}.$$

We estimate the GCD:

$$d = \left(11m^2n + 20mn^2 + 5n^3,\ 2m^3 + 13m^2n + 16mn^2 + 5n^3\right) =$$

$$= \left(2m\left(m^2 + mn - 2n^2\right),\ n\left(11m^2 + 20mn + 5n^2\right)\right) \leq$$

$$\leq 10\left(m^2 + mn - 2n^2,\ 11m^2 + 20mn + 5n^2\right) =$$
$$= 10\left(m^2 + mn - 2n^2,\ 9mn + 27n^2\right) \leq 90\left(m^2 + mn - 2n^2,\ m + 3n\right) =$$
$$= 90\left(m + 3n,\ -2mn - 2n^2\right) \leq 180\left(m + 3n,\ m + n\right) = 180\left(2n,\ m + n\right) \leq 360.$$

Let $\beta^{\frac{11\beta^2+20\beta+5}{(2\beta+1)(\beta+1)(\beta+5)}} = \frac{m_1}{n_1}$, where $(m_1, n_1) = 1$. Then

$$m^{\frac{11m^2n+20mn^2+5n^3}{d}} n_1^{\frac{2m^3+13m^2n+16mn^2+5n^3}{d}} = n^{\frac{11m^2n+20mn^2+5n^3}{d}} m_1^{\frac{2m^3+13m^2n+16mn^2+5n^3}{d}}.$$

Since m, n and m_1, n_1 are coprime, the numbers m, m_1 have the same prime divisors. The same holds for n and n_1. Consequently, there exist positive integer numbers q, r such that

$$m = q^{\frac{2m^3+13m^2n+16mn^2+5n^3}{d}},\ n = r^{\frac{2m^3+13m^2n+16mn^2+5n^3}{d}}.$$

Since $d \leq 360$, we get $n^{360} \geq n^d \geq r^{5n^3}$. Clearly, $n > 1$, then $r > 1$. This implies $n^{72} \geq 2^{n^3}$. Clearly, $n \leq 5$, which is impossible since $\beta < 0.14$. Thus, the case $\deg(\beta) = 1$ is impossible.

We consider the case $\deg(\beta) = 3$. Let $\frac{m}{n} = \eta + 1$. Then the polynomial

$$f(x) = x^3 + \frac{13m - 11n}{2m}x^2 + \frac{8m - 10n}{m}x + \frac{5m - 5n}{2m}$$

is a minimal polynomial of the number β, and $Nm_K(\beta) = \frac{5m - 5n}{2m}$, where $K = \mathbb{Q}(\beta)$ is a cubic field. Since $\gamma\beta \in \mathbb{Q}(\beta)$, we have $\gamma_1 := \beta^{\eta+1} \in \mathbb{Q}(\beta)$. Since $\beta^m = \gamma_1^n$, we get $\left(Nm_K(\beta)\right)^m = \left(Nm_K(\gamma_1)\right)^n$. So $\left(Nm_K(\beta)\right)^{\eta+1} \in \mathbb{Q}$. Therefore, there exist $r,\ s \in \mathbb{N}$, $(r, s) = 1$, such that $\left(\frac{5m - 5n}{2m}\right)^{\frac{m}{n}} = \frac{r}{s}$. Then $(5m - 5n)^m s^n = r^n (2m)^m$. Let $d = (5m - 5n, 2m)$. This implies $\frac{5m - 5n}{d} = q^n$ for some $q \in \mathbb{N}$. Since $5m > \frac{5m - 5n}{d} = q^n$, we obtain $q = 1$. Since $d \leq 10$, we get $m - n \leq dq^n \leq 10$, which is impossible for $n > 100$. For $n \leq 100$ we use brute force and can make sure that there are no admissible n, m.

It remains to consider the case of $\deg(\beta_1) = 2$. Let $h(x) \in \mathbb{Q}[x]$ be a minimal polynomial for β. Then the polynomial $h(x)$ divides the annihilating polynomial $f(x)$ defined above. This means the polynomial $f(x)$ has a rational root. Denote this root by $\frac{r}{s}$, where $r \in \mathbb{Z}$ and $s \in \mathbb{N}$, $(r, s) = 1$. We have

$$2mr^3 + (13m - 11n)r^2 s + (16m - 20n)rs^2 + (5m - 5n)s^3 = 0.$$

Clearly, $s|2m$. Let $K = \mathbb{Q}(\beta)$ be a quadratic field. According to Vieta's formulas, we have $Nm_K(\beta) = \frac{(5m - 5n)s}{2m|r|}$. Similarly to the previous case, $\gamma_1 := \beta^{\eta+1} \in K$. Then there exist numbers $p,\ q \in \mathbb{N}$, $(p, q) = 1$, such that $\left((5m - 5n)s\right)^m q^n = p^n \left(2m|r|\right)^m$. Let $d = \left((5m - 5n)s,\ 2m|r|\right)$, then one can find $t \in \mathbb{N}$ such that $\frac{(5m - 5n)s}{d} = t^n$. Taking into account $2m \geq s$ and $n > \frac{m}{2}$, we obtain $10m^2 \geq t^n > t^{\frac{m}{2}}$, i. e. $100m^4 > t^m$. If $t \geq 2$, then $m \leq 30$ (applying the brute force over m, n, we make sure that the considered case is impossible). Hence $t = 1$ and $d = (5m - 5n)s$. Similarly, we prove that there exists $t_1 \in \mathbb{N}$ such that $\frac{2m|r|}{d} = t_1^n$. Since $r|5m - 5n$, we have $|r| \leq 5m$. Thus, $10m^2 > t_1^n$. Analogously to previous arguments, we obtain $t_1 = 1$. We have just proved that $d = (5m - 5n)s = 2m|r|$. Applying Vieta's formulas one more time for the roots x_i, x_j of the polynomial $f(x)$, we see that

$\left|x_i x_j\right| = 1$. Searching numerically for the roots of the polynomial $f(x)$ with enough accuracy, we see that the condition $\left|x_i x_j\right| = 1$ can not hold. The obtained contradiction finishes the proof of this proposition.

Corollary 2. *The number* $N_0 = 29.143\,359\ldots$ *is transcendental.*

P r o o f. We suppose the opposite is true, i. e. N_0 is an algebraic number. Then $k_0 = \frac{2}{N_0}$ is also an algebraic number. Since $k_0 = \frac{2\beta_0(2+\beta_0)(1-\beta_0)}{(2\beta_0+1)(\beta_0+1)(\beta_0+5)}$, we obtain that β_0 is algebraic number. Hence $\beta_0^{k_0} = \frac{8\beta_0+4}{(\beta_0+1)(\beta_0+5)}$ is also algebraic number, which contradicts proposition 14.

Proposition 15. *The number* $\beta_1 = 0.304\,55\ldots$ *is irrational.*

P r o o f. We suppose that the number β_1 is rational. Then the number

$$k_1 = \frac{3\beta_1 + 1 - \sqrt{\beta_1^2 + 6\beta_1 + 1 - 16\beta_1^3}}{2(\beta_1+1)(2\beta_1+1)}$$

is algebraic. Since the number

$$\beta_1^{-k_1} = \frac{\beta_1 - 1 + \sqrt{\beta_1^2 + 6\beta_1 + 1 - 16\beta_1^3}}{2\beta_1(\beta_1+1)}$$

is algebraic as well, by Hilbert's seventh problem, we obtain that the number k_1 should be rational. Let $\beta_1 = \frac{m}{n}$, where $(m, n) = 1$, and $k_1 = \frac{p}{q}$, where $(p, q) = 1$. Then the equation

$$\beta_1^{-k_1} = \frac{2}{\beta_1+1} - \frac{k_1(2\beta_1+1)}{\beta_1}$$

can be written as

$$\left(\frac{n}{m}\right)^{\frac{p}{q}} = \frac{2mnq - p(2m+n)(m+n)}{qm(m+n)}.$$

We obtain

$$n^p\left(qm(m+n)\right)^q = m^p\left(2mnq - p(2m+n)(m+n)\right)^q.$$

Since $(m, n) = 1$, $(p, q) = 1$, we get $n = r^q$ for some $r \in \mathbb{N}$. Then we have $m = s^q$ for some $s \in \mathbb{N}$. Substituting $n = r^q$, $m = s^q$ into the equation and extracting the root of the degree q from both parts, we get $r^p\left(qs^q\left(s^q + r^q\right)\right) = s^p\left(2r^q s^q q - p\left(2s^q + r^q\right)\left(s^q + r^q\right)\right)$. Since $(r, s) = 1$ and $q > p$, we have $r^p \left| p\left(2s^q + r^q\right) \times \right.$ $\times \left(s^q + r^q\right)$. Then we obtain $r^p \mid 2p$. Hence $2p \geq r^p$. We conclude that $r = 1$ or $(r, p) \in \{(2, 1), (2, 2)\}$. If $r = 1$, then $\beta_1 \in \mathbb{N}$, which is impossible. If $r = 2$, then $s = 1$, but the inequality $0.29 < 0.5^q < 0.31$ does not have solutions in positive integers. The obtained contradiction proves that $\beta_1 \not\in \mathbb{Q}$.

Proposition 16. *The number* $N_1 = 24.506\,13\ldots$ *is transcendental.*

P r o o f. We suppose the opposite, i. e. N_1 is algebraic. Then $k_1 = \frac{2}{N_1}$ is also algebraic number. As well we suppose that k_1 is irrational number. The number β_1 is also algebraic. Then, by the seventh Hilbert problem, the number $\beta_1^{k_1}$ is transcendental, that is impossible since

$$\beta_1^{-k_1} = \frac{\beta_1 - 1 + \sqrt{\beta_1^2 + 6\beta_1 + 1 - 16\beta_1^3}}{2\beta_1(\beta_1+1)},$$

and the right-hand side is an algebraic number. Hence, k_1 is rational number. Let $k_1 = \frac{m}{n}$, where $(m, n) = 1$. It follows from the relation

$$k_1 = \frac{3\beta_1 + 1 - \sqrt{\beta_1^2 + 6\beta_1 + 1 - 16\beta_1^3}}{2(\beta_1+1)(2\beta_1+1)}$$

that

$$\left(\left(3\beta_1+1\right)n-2m\left(\beta_1+1\right)\left(2\beta_1+1\right)\right)^2=\left(\beta_1^2+6\beta_1+1-16\beta_1^3\right)n^2.$$

Then

$$\left(8\beta_1^2+16\beta_1^3\right)n^2-4mn\left(\beta_1+1\right)\left(2\beta_1+1\right)\left(3\beta_1+1\right)+4m^2\left(\beta_1+1\right)^2\left(2\beta_1+1\right)^2=0.$$

Dividing by $4\left(1+2\beta_1\right)$, we obtain

$$2n^2\beta_1^2-mn\left(\beta_1+1\right)\left(3\beta_1+1\right)+m^2\left(\beta_1+1\right)^2\left(2\beta_1+1\right)=0.$$

Finally,

$$2m^2\beta_1^3+\left(5m^2-3mn+2n^2\right)\beta_1^2+\left(4m^2-4mn\right)\beta_1+m^2-mn=0.$$

So $\deg\left(\beta_1\right)\le 3$. By proposition 15, the case of $\deg\left(\beta_1\right)=1$ is not possible.

Let us consider the case $\deg\left(\beta_1\right)=3$. Then the polynomial

$$f\left(x\right)=x^3+\frac{5m^2-3mn+2n^2}{2m^2}x^2+\frac{2m-2n}{m}x+\frac{m-n}{2m}$$

is a minimal polynomial of the number β_1, and $Nm_K\left(\beta_1\right)=\dfrac{n-m}{2m}$, where $K=\mathbb{Q}\left(\beta_1\right)$ is a cubic field. Since

$$\beta_1^{-k_1}=\frac{2}{\beta_1+1}-\frac{k\left(2\beta_1+1\right)}{\beta_1}\in\mathbb{Q}\left(\beta_1\right),$$

we get $\gamma_1:=\beta_1^{k_1}\in\mathbb{Q}\left(\beta_1\right)$. Since $\beta_1^m=\gamma_1^n$, we have $\left(Nm_K\left(\beta_1\right)\right)^m=\left(Nm_K\left(\gamma_1\right)\right)^n$. Hence $\left(Nm_K\left(\beta_1\right)\right)^{k_1}\in\mathbb{Q}$. Thus, there exist $r,\ s\in\mathbb{N}$, $\left(r,\ s\right)=1$, such that

$$\left(\frac{n-m}{2m}\right)^{\frac{m}{n}}=\frac{r}{s}.$$

Then $\left(n-m\right)^m s^n=r^n\left(2m\right)^m$. Let $d=\left(n-m,\ 2m\right)$. We obtain that $\dfrac{n-m}{d}=q^n$ for some $q\in\mathbb{N}$. Since $n>\dfrac{n-m}{d}=$ $=q^n$, there holds $q=1$. Since $d\le 2$, we obtain that $n-m\le dq^n\le 2$, which is impossible for positive integer n, m since $k_1<0.1$.

It remains to consider the case $\deg\left(\beta_1\right)=2$. Let $h\left(x\right)\in\mathbb{Q}\left[x\right]$ be a minimal polynomial for β_1. Then the polynomial $h\left(x\right)$ divides the annihilating polynomial $f\left(x\right)$ defined above. That means the polynomial $f\left(x\right)$ has a rational root. Denote this root by $\dfrac{r}{s}$, where $r\in\mathbb{Z}$ and $s\in\mathbb{N}$, $\left(r,\ s\right)=1$. We have

$$2m^2r^3+\left(5m^2-3mn+2n^2\right)r^2s+\left(4m^2-4mn\right)rs^2+\left(m^2-mn\right)s^3=0.$$

Clearly, $s\mid 2m^2$. Let $K=\mathbb{Q}\left(\beta_1\right)$ be a quadratic field. According to Vieta's formulas, $Nm_K\left(\beta_1\right)=\dfrac{\left(n-m\right)s}{2m\left|r\right|}$. Similarly to the previous case, $\gamma_1:=\beta_1^{k_1}\in K$. Then there exist $p,\ q\in\mathbb{N}$, $\left(p,\ q\right)=1$, such that $\left(\left(n-m\right)s\right)^m q^n=p^n\left(2m\left|r\right|\right)^m$. Let $d=\left(\left(n-m\right)s,\ 2m\left|r\right|\right)$, then one can find $t\in\mathbb{N}$ such that $\dfrac{\left(n-m\right)s}{d}=t^n$. Taking into account $2n^2>2m^2\ge s$, we get $2n^3>t^n$. If $t\ge 2$, then $n\le 11$ (checking by brute force all remaining m, n, we see that this case is not possible). Hence $t=1$ and $d=\left(n-m\right)s$. Analogously, there exists $t_1\in\mathbb{N}$ such that $\dfrac{2m\left|r\right|}{d}=t_1^n$. Since $r\mid m^2-mn$, we get $\left|r\right|\le n^2$. So $2n^3>t_1^n$. That is why $t_1=1$. We have proved that $d=\left(n-m\right)s=2m\left|r\right|$. Applying Vieta's formulas, we see that for some roots x_i, x_j of the polynomial $f\left(x\right)$ there exists $\left|x_i x_j\right|=1$. Finding numerically the roots of the polynomial $f\left(x\right)$ with sufficient accuracy, we can see that the condition $\left|x_i x_j\right|=1$ can not hold. The obtained contradiction finishes the proof.

Corollary 3. *The number* $\beta_1 = 0.304\,55\ldots$ *is transcendental.*

P r o o f. Indeed, if the number β_1 is algebraic, then the number k_1 is also algebraic, which contradicts to proposition 16.

Remark 5. Validity of theorem 2 follows from propositions 14 and 16.

Conclusions

As it is shown in the paper [5], the equality of external n-lines $l_{a,n}^{ext} = l_{c,n}^{ext}$ in terms of variables δ, β, k can be written as $\tilde{g}(\delta, \beta, k) = 0$, where

$$\tilde{g}(\delta, \beta, k) = \delta^k \left(-\beta(\delta-1)^2 + (\beta-1)^2(\delta-1)\right) -$$
$$- \beta^k \left(-\delta(\beta-1)^2 + (\beta-1)(\delta-1)^2\right) + (\beta-1)(\delta-1)(\delta-\beta).$$

Despite apparent similarity of equations of surfaces $z = g(\delta, \beta, k)$ and $z = \tilde{g}(\delta, \beta, k)$, these surfaces have fundamentally different geometry. As it is shown in the work [5], for any $k \neq 0$ in any sufficiently small neighbourhood of the point $(\delta_0, \beta_0) = (1, 1)$ there exist points (δ, β) such that $\delta \neq \beta$ and $\tilde{g}(\delta, \beta, k) = 0$, that is why for any $n \neq 0$ there exist non-isosceles triangles with two equal external n-lines. These different behaviour of internal and external n-lines can be explained in the following way. In case of external n-lines the saddle point $(\delta_0, \beta_0) = (1, 1)$ of the family of surfaces $\tilde{L}(\delta, \beta, k) = \dfrac{\tilde{g}(\delta, \beta, k)}{\delta - \beta}$ becomes invariant when the parameter $k \neq 0$ changes. For the family of surfaces $z = L(\delta, \beta, k)$, which describes the equality of internal n-lines the saddle point $(\delta_0, \beta_0) = (1, 1)$ moves toward the limit point $(\beta_1, \beta_1) = (0.304\,55\ldots, 0.304\,55\ldots)$ when $k > 0$ increases up to the critical value $k = \dfrac{2}{N_1}$; as soon as k achieves this critical value, the function $L(\delta, \beta, k) = \dfrac{g(\delta, \beta, k)}{\delta - \beta}$ becomes strictly positive in the domain $\delta > 0$, $\beta > 0$. For negative values of k situation is different: for any $k < 0$ in a neighbourhood of the point $\beta = 1$, $\delta = \infty$ there exist solutions of the equation $L(\delta, \beta, k) = 0$, but for $k \in [-2, 0)$ these solutions do not define a triangle, since the needed triangle inequality $\beta^{\frac{k}{2}} + \delta^{\frac{k}{2}} > 1$ fails.

Partially, for $n = 1$ we get a new proof of the Steiner – Lehmus theorem, as well as the known counterexample of Bottema that the Steiner – Lehmus theorem fails for external bisectors (Bottema's triangle has angles 12°, 36°, 132° and two equal external bisectors [5]).

In the article [5], it is proved that for any real n the equalities $l_{a,n} = l_{c,n}$, $l_{a,-n} = l_{c,-n}$ imply that a triangle is isosceles ($a = c$), the same property holds for pairs of external n- and $(-n)$-lines. Results of this paper show that both the equalities $l_{a,n} = l_{c,n}$, $l_{a,-n} = l_{c,-n}$ are substantial for a triangle to be isosceles in case of internal n-lines. Indeed, one can find a non-isosceles triangle, which has two equal antisymmedians $l_{a,-2} = l_{c,-2}$. Clearly, such triangle can not have two equal symmedians $l_{a,2}$, $l_{c,2}$, otherwise it would be isosceles. In other words, the equality of symmedians $l_{a,2} = l_{c,2}$ implies that $a = c$ and, consequently, it implies the equality of antisymmedians $l_{a,-2} = l_{c,-2}$. The opposite is not true: the equality of two antisymmedians does not imply the equality of two symmedians. Nevertheless, the equality of two antibisectors $l_{a,-1} = l_{c,-1}$ implies the equality of two bisectors $l_{a,1} = l_{c,1}$. But the equality of three antisymmedians $l_{a,-2} = l_{b,-2} = l_{c,-2}$ implies that a triangle is equilateral.

Also it is interesting to consider the same generalisation of the Steiner – Lehmus theorem in other geometries, such as a weak geometry that does not depend on the fifth Euclid's postulate.

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