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CLASSICAL SOLUTIONS OF A MIXED PROBLEM FOR WAVE EQUATION WITH DISCONTINUOUS CAUCHY CONDITIONS IN A CURVILINEAR HALF-STRIP

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For a one-dimensional wave equation, we consider a mixed problem in a curvilinear half-strip. The initial conditions have a first-kind discontinuity at one point. The mixed problem models the problem of a longitudinal impact on a finite elastic rod with a movable boundary. We construct the solution using the method of characteristics in an explicit analytical form. For the problem in question, we prove the uniqueness of the solution and establish the conditions under which its classical solution exists.

Keywords: wave equation; mixed problem; method of characteristics; classical solution; matching conditions; conjugation conditions; discontinuous conditions; curvilinear domain.

Statement of the problem

In the curvilinear domain $Q = \{(t,x) : t \in (0,\infty) \wedge x \in (\gamma(t), l)\}$, where l is a positive real number, of two independent variables $(t,x) \in \overline{Q} \subset \mathrm{R}^2$, for the wave equation

$$\left(\partial_t^2 - a^2 \partial_x^2\right) u(t,x) = f(t,x), \quad (t,x) \in Q, \tag{1}$$

we consider the following mixed problem with the initial conditions

$$u(0,x) = \varphi(x), \quad \partial_t u(0,x) = \psi(x) + \begin{cases} 0, & x \in [0,l), \\ v, & x = l, \end{cases} \quad x \in [0,l], \tag{2}$$

and the boundary conditions

$$u\big(t,\gamma(t)\big)=\mu_1(t), \quad \big(\partial_t^2+b\partial_x\big)u(t,l)=\mu_2(t), \quad t\in[0,\infty), \qquad (3)$$

where a, v, and b are real numbers, $a>0$ for definiteness, f is a function given on the set $\overline{Q}$, φ and ψ are some real-valued functions defined on the segment $[0,l]$, and μ_1 and μ_2 are some real-valued functions defined on the half-line $[0,\infty)$. We also assume that

$$\gamma\in C^1\big([0,\infty)\big),\ D\gamma(t)\in(-a,a) \text{ for all } t\in[0,\infty),\ \lim_{t\to+\infty}\gamma(t)\pm at=\pm\infty, \quad (4)$$

and the curves $x=\gamma(t)$ and $x=l$ do not intersect.

Problem (1)–(3) models the following problem from the wave theory of longitudinal impact [1]. Suppose that an elastic finite homogeneous rod of constant cross-section, whose left moving boundary $x=\gamma(t)$ is fixed, is subjected at the initial moment $t=0$ to an impact at the end $x=l$ by a load that sticks to the rod. We assume that an external volumetric force acts on the rod and that both the displacements and the rate of change of the displacements of the rod at the initial moment are nonzero. In addition, we ignore the weight of the rod and any potential vertical displacements, and we assume that there are no shock waves within the rod. Under these conditions, the displacements $u(t,x)$ of the rod satisfy the mixed problem (1)–(3), where $a=\sqrt{E\rho^{-1}}$, $b=SEM^{-1}$, where $E>0$ is Young's modulus of the rod material, $\rho>0$ is the density of the rod material, $S>0$ is the cross-sectional area of the rod, $M>0$ is the mass of the impacting load, $-v$ is the velocity of the impacting load, $-\mu_1(t)$ is the external force acting on the end of the rod, $\mu_2(t)$ is the external force acting on the end of the rod, divided by the mass of the impacting load, and the function f is the external volumetric force divided by ρ.

Auxiliary functions

Consider the following functions:

$$\gamma_+:[0,\infty)\,\mathrm{e}\ t\mapsto\gamma(t)+at, \quad \gamma_-:[0,\infty)\,\mathrm{e}\ t\mapsto\gamma(t)-at. \quad (5)$$

We also need the inverse of the functions γ_+ and γ_-, which we will denote as Φ_+ and Φ_-. Specifically, we have:

$$\Phi_+\big(\gamma(t)+at\big)=t,\ \Phi_-\big(\gamma(t)-at\big)=t.$$

These inverse functions exist under the conditions specified in (4). According to the inverse function theorem, we derive the following formulas:

$$D\Phi_{\pm}(t) = \left(D\gamma\left(\Phi_{\pm}(t)\right) \pm a\right)^{-1}, \quad t \in [0, \infty), \tag{6}$$

$$D^2\Phi_{\pm}(t) = -D^2\gamma\left(\Phi_{\pm}(t)\right)\left(D\gamma\left(\Phi_{\pm}(t)\right) \pm a\right)^{-3}, \quad t \in [0, \infty), \tag{7}$$

It is important to note that the representations in (6) and (7), along with the conditions in (4), imply that Φ_+ is an increasing function and Φ_- is a decreasing function.

Main results

We partition the domain $\overline{Q}$ according to the following formulas:

$$Q^{(0,0)} = Q \cap \left\{(t, x) : x - at \in [0, l] \wedge x + at \in [0, l]\right\}, \tag{8}$$

$$Q^{(1,0)} = Q \cap \left\{(t, x) : x - at \in \left[\gamma_-(r_1), 0\right] \wedge x + at \in [0, l]\right\}, \tag{9}$$

$$Q^{(0,1)} = Q \cap \left\{(t, x) : x - at \in [0, l] \wedge x + at \in [l, l + al_1]\right\}, \tag{10}$$

$$Q^{(i,j)} = Q \cap \left\{(t, x) : x - at \in \left[\gamma_-(r_i), \gamma_-(r_{i-1})\right] \wedge x + at \in \left[l + al_{j-1}, l + al_j\right]\right\}, \tag{11}$$

where $r_0 = l_0 = 0$, $l_i = r_{i-1} + a^{-1}\left(l - \gamma\left(r_{i-1}\right)\right)$, $r_i = \Phi_+\left(l + al_{i-1}\right)$. From the geometric considerations and conditions (4), it is easy to show the correctness of the partitioning (8) – (11) of the domain Q.

In the general case, where $\psi \notin C^1([0, l])$, problem (1)–(3) has no solution in the class $C^2\left(\overline{Q}\right)$; in other words, problem (1)–(3) lacks a global classical solution defined on the set $\overline{Q}$. However, it is possible to define a classical solution on a smaller set $\overline{Q}$, Γ, where

$$\Gamma = \left(\bigcup_{i=0}^{\infty} \bigcup_{j=\max(i-1,0)}^{i+1} \partial Q^{(i,j)}\right), \quad \partial\overline{Q},$$

that will satisfy Eq. (1) on the set $\overline{Q}$, Γ in the standard sense with additional conjugation conditions on the set Γ.

Definition 1. A function u is a *classical solution* of problem (1)–(3) if it is representable in the form $u = u_1 + u_2$, where u_1 is a classical solution of problem (1)–(3) with $v = 0$ and u_2 satisfies Eq. (1) with $f \equiv 0$ in the domains $\overline{Q^{(i,j)}}$, $i \in \mathrm{N} \cup \{0\}$, $j \in \mathrm{N} \cup \{0\}$, $|i - j| \leq 1$, the initial conditions

$u_2(0,x)=\partial_t u_2(0,x)=0,\ \ x\in[0,l]$, boundary conditions (3) with $\mu_1=\mu_2\equiv 0$, and the following matching conditions

$$[(u_2)^+-(u_2)^-](t,x=\gamma_-(r_i)+at)=0, i\in\{0\}\cup\mathrm{N}, \tag{12}$$
$$[(u_2)^+-(u_2)^-](t,x=l+al_i-at)=0, i\in\{0\}\cup\mathrm{N}, \tag{13}$$
$$[(\partial_t u_2)^+-(\partial_t u_2)^-](t,x=l+al_0-at)=C^{(0)}=v, \tag{14}$$
$$[(\partial_t u_2)^+-(\partial_t u_2)^-](t,x=l+al_i-at)=C^{(i)}, i\in\mathrm{Even}[\mathrm{N}], \tag{15}$$
$$[(\partial_t u_2)^+-(\partial_t u_2)^-](t,x=l+al_i-at)=0, i\in\mathrm{Odd}[\mathrm{N}], \tag{16}$$

where $C^{(i)}$ are some constants determined from physical conditions,

$$\mathrm{Even}[\Omega]=\{x: x\in\Omega\wedge x\equiv 0\ (\mathrm{mod}\ 2)\},$$

and

$$\mathrm{Odd}[\Omega]=\{x: x\in\Omega\wedge x\equiv 1\ (\mathrm{mod}\ 2)\}.$$

Theorem 1. Let the smoothness conditions

$$f\in C^1(\bar{Q}),\quad \varphi\in C^2([0,l]),\quad \psi\in C^1([0,l]),$$
$$\mu_1\in C^2([0,\infty)),\quad \mu_2\in C([0,\infty)),\quad \gamma\in C^2([0,\infty))$$

be satisfied. Problem (1) – (3) has a unique solution in the sense of Definition 1 if and only if the following matching conditions

$$\mu_1(0)-\varphi(0)=0,$$
$$D\mu_1(0)-\psi(0)+D\gamma(0)D\varphi(0)=0,$$
$$D^2\mu_1(0)-\left(a^2+(D\gamma(0))^2\right)D^2\varphi(0)-$$
$$f(0,0)-2D\gamma(0)D\psi(0)-D^2\gamma(0)D\varphi(0)=0,$$
$$\mu_2(0)-f(0,l)-bD\varphi(0)-a^2D^2\varphi(l)=0$$

are satisfied.

Remark 1. The solution to problem (1) – (3) is determined in a non-unique way, i.e., we have to specify the constants $C^{(i)}$, $i\in\mathrm{Even}[\mathrm{N}]$.

Remark 2. According to Theorem 1, any choice of the constants $C^{(i)}$, $i\in\mathrm{Even}[\mathrm{N}]$, uniquely determines the solution.

Remark 3. Suppose $D\gamma(r_j)=0,\ j\in\mathrm{Even}[\mathrm{N}]$. Then, we can set $C^{(j)}=v$ and obtain a physically correct solution in Theorem 1.

In Remark 3, we introduced the term "physically correct solution." Let us explain it. We call a solution physically correct if it satisfies shock conditions that correspond to the original physical statements (see, for example, [2, p. 139]). In problem (1)–(3), we can use the property that shock waves propagate with the same speed in the elastic rods [3] as demonstrated in [4, p. 64–66] and [5], but here we must also consider the interaction of shock waves with the moving boundary.

Theorem 2. If we set

$$C^{(i)} = v\prod_{j=1}^{i/2} \frac{a+\gamma'\left(r_{2j-1}\right)}{a-\gamma'\left(r_{2j-1}\right)},$$

then a solution of problem (1) – (3) constructed in Theorem 1 is physically correct.

Conclusions

In the present report, we have obtained the necessary and sufficient conditions under which a unique classical solution of a mixed problem exists for the wave equation with discontinuous conditions in a curvilinear half-strip. We have proposed a method for constructing solutions to mixed problems for hyperbolic equations with discontinuous conditions in curvilinear domains.

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