Журнал Белорусского государственного университета. Математика. Информатика. 2025;1:14–22
Journal of the Belarusian State University. Mathematics and Informatics. 2025;1:14–22

УДК 512.542

ФОРМАЦИИ КОНЕЧНЫХ ГРУПП ЗА ПОЛИНОМИАЛЬНОЕ ВРЕМЯ: $\mathfrak{F}$-РАДИКАЛ И $\mathfrak{F}$-ДЛИНА

В. И. МУРАШКО[1)]

[1)]*Гомельский государственный университет им. Франциска Скорины, ул. Советская, 104, 246028, г. Гомель, Беларусь*

Аннотация. Для композиционной формации Фиттинга $\mathfrak{F}$ конечных групп предложен алгоритм вычисления $\mathfrak{F}$-радикала конечной группы перестановок степени n, работающий за полиномиальное время от n. Показано, как можно вычислить $\mathfrak{F}$-радикал в случае, когда $\mathfrak{F}$ является примитивной насыщенной формацией разрешимых конечных групп. Представлены алгоритмы вычисления различных длин, связанных с конечной группой, включающих обобщенную высоту Фиттинга и не p-разрешимую длину, которые для группы перестановок степени n работают за полиномиальное время от n.

Ключевые слова: конечная группа; вычисления в группах перестановок; композиционная формация; формация Фиттинга; $\mathfrak{F}$-радикал; $\mathfrak{F}$-длина; полиномиальный алгоритм.

Благодарность. Работа выполнена при финансовой поддержке Белорусского республиканского фонда фундаментальных исследований (Ф23РНФ-237).

FORMATIONS OF FINITE GROUPS IN POLYNOMIAL TIME: THE $\mathfrak{F}$-RADICAL AND THE $\mathfrak{F}$-LENGTH

V. I. MURASHKA[a]

[a]*Francisk Skorina Gomel State University, 104 Savieckaja Street, Gomiel 246028, Belarus*

Abstract. For a Baer-local (composition) Fitting formation $\mathfrak{F}$ of finite groups the algorithm for the computation of the $\mathfrak{F}$-radical of a permutation finite group which runs in polynomial time from its degree is herein suggested. It is shown how one can compute the $\mathfrak{F}$-radical in case when $\mathfrak{F}$ is a primitive saturated formation of soluble finite groups. The algorithms for the computation of different lengths associated with a finite group (the generalised Fitting height, the non-p-soluble length and etc.) are presented. In the case of a permutation group these algorithms run in polynomial time from its degree.

Keywords: finite group; permutation group computation; Baer-local formation; Fitting formation; $\mathfrak{F}$-radical; $\mathfrak{F}$-length; polynomial time algorithm.

Acknowledgements. This work is supported by the Belarusian Republican Foundation for Fundamental Research (F23RNF-237).

Образец цитирования:
Мурашко ВИ. Формации конечных групп за полиномиальное время: $\mathfrak{F}$-радикал и $\mathfrak{F}$-длина. *Журнал Белорусского государственного университета. Математика. Информатика.* 2025; 1:14–22 (на англ.).
EDN: DFRMNH

For citation:
Murashka VI. Formations of finite groups in polynomial time: the $\mathfrak{F}$-radical and the $\mathfrak{F}$-length. *Journal of the Belarusian State University. Mathematics and Informatics.* 2025;1:14–22.
EDN: DFRMNH

Автор:
Вячеслав Игоревич Мурашко – кандидат физико-математических наук, доцент; доцент кафедры алгебры и геометрии факультета математики и технологий программирования, ведущий научный сотрудник научно-исследовательского сектора.

Author:
Viachaslau I. Murashka, PhD (physics and mathematics), docent; associate professor at the department of algebra and geometry, faculty of mathematics and technologies of programming, and leading researcher of research sector.
mvimath@yandex.ru
https://orcid.org/0000-0002-0769-2519

Introduction and the main result

All groups considered here are finite. Fitting showed that the product of two normal nilpotent subgroups is again nilpotent, i. e. in every group there exists the greatest normal nilpotent subgroup which is called the Fitting subgroup. Recall that a class of groups is a collection $\mathfrak{X}$ of groups with the property that if $G \in \mathfrak{X}$ and if $H \simeq G$, then $H \in \mathfrak{X}$. A group $G \in \mathfrak{X}$ is called an $\mathfrak{X}$-group. The greatest normal $\mathfrak{X}$-subgroup of a group G is called the $\mathfrak{X}$-radical and is denoted by $G_{\mathfrak{X}}$. It always exists in case when $\mathfrak{X}$ is a Fitting class. From the fundamental result of R. A. Bryce and J. Cossey [1] it follows that the hereditary non-empty Fitting class of soluble groups is a primitive local (saturated) formation. The similar result for the quotient group closed Fitting classes of metanilpotent groups was obtained in [2].

The computational theory of formations, Fitting and Schunk classes was discussed in [3–5]. The algorithm for computing the $\mathfrak{X}$-radical (of a soluble group) was presented only in [3]. Note that as was mentioned in [3] the suggested there algorithm (even when $\mathfrak{X}$ is the class of all nilpotent groups) for a permutation group of degree $3n$ may require to check for nilpotency 2^n subgroups. The main idea of that algorithm was to extend the $\mathfrak{X}$-radical from the given member of chief series to the next one. It was written in [3, p. 507]: «Note that for many Fitting classes $\mathfrak{F}$, more efficient algorithms for $\mathfrak{F}$-radicals can be obtained from the theoretical knowledge about $\mathfrak{F}$». Nevertheless, the methods how one can obtain such algorithms were not presented. The aim of this paper is to suggest an algorithm for the computation of the $\mathfrak{F}$-radical (which runs in polynomial time for permutation groups) for a Baer-local (composition) formation $\mathfrak{F}$.

Recall that the Fitting subgroup $\mathrm{F}(G)$, the p-nilpotent radical $\mathrm{O}_{p', p}(G)$, the soluble radical $R(G)$ of a group G are just the intersection of centralisers of all chief factors, all divisible by p chief factors and all non-abelian chief factors respectively. Recall that for a chief factor H/K of G the subgroup $HC_G(H/K)$ is called the inneriser of H/K. By its definition the generalised Fitting subgroup (the quasinilpotent radical) $\mathrm{F}^*(G)$ is the intersection of innerisers of all chief factors of G. The first three radicals are associated with local Fitting formations and the last one is associated with a Baer-local Fitting formation.

Note that L. A. Shemetkov [6] obtained similar characterisations of the $\mathfrak{F}$-radical for a Baer-local Fitting formation $\mathfrak{F}$ using the generalisation of the centraliser of a chief factor. We cannot use the results from [6] directly for three reasons. Formations $\mathfrak{F}$ in [6] are defined with the integrated Baer-local function, i. e. $f(H/K) \subseteq \mathfrak{F}$ for all chief factors H/K. In some applications (such as primitive saturated formations) the functions will not necessarily satisfy this condition. The characterisation of $G_{\mathfrak{F}}$ is obtained only in case $f(H/K) \neq \varnothing$ for all chief factors H/K of G. Also the constructive description of the generalised centraliser was not presented in [6].

Let f be a function which assigns to every simple group J a possibly empty formation $f(J)$. Now extend the domain of f. If G is the direct product of simple groups isomorphic to J, then we say that G has type J and let $f(G) = f(J)$. If J is a cyclic group of order p, then let $f(p) = f(J)$. Such functions f are called Baer functions. A formation $\mathfrak{F}$ is called Baer-local [7, chap. IV, definitions 4.9] (or composition (see [6] or [8, p. 4])) if for some Baer function f

$$\mathfrak{F} = \left(G \middle| G/C_G(H/K) \in f(H/K) \text{ for every chief factor } H/K \text{ of } G\right).$$

Such a formation $\mathfrak{F}$ is denoted by $BLF(f)$. The main result of this paper is the following theorem.

Theorem 1. *Let $\mathfrak{F}$ be a Baer-local Fitting formation defined by f such that $f(J)$ is a Fitting formation for any simple group J. Assume that $(G/K)_{f(J)}$ can be computed in polynomial time in n for every permutation group G of degree n, its normal subgroup K and a simple group J. Then $(G/K)_{\mathfrak{F}}$ can be computed in polynomial time in n for every permutation group G of degree n and its normal subgroup K.*

Note that if $f(J) = \varnothing$, then $G_{f(J)}$ is not defined for any group G. A Baer-local formation defined by the Baer function f is called local if $f(J) = \bigcap_{p \in \pi(J)} f(p)$ for every simple group J.

Corollary. *Let $\mathfrak{F}$ be a local Fitting formation defined by f such that $f(p)$ is a Fitting formation for all prime p. Assume that $(G/K)_{f(p)}$ can be computed in polynomial time in n for every permutation group G of degree n, its normal subgroup K and prime p. Then $(G/K)_{\mathfrak{F}}$ can be computed in polynomial time in n for every permutation group G of degree n and its normal subgroup K.*

Preliminaries

Recall that a formation is a class of groups $\mathfrak{F}$ that is closed under taking epimorphic images (i. e. from $G \in \mathfrak{F}$ and $N \trianglelefteq G$ it follows that $G/N \in \mathfrak{F}$) and subdirect products (i. e. from $G/N_1 \in \mathfrak{F}$ and $G/N_2 \in \mathfrak{F}$ it follows that

$G/(N_1 \cap N_2) \in \mathfrak{F}$). A class of groups $\mathfrak{F}$ is called a Fitting class if it is normally hereditary (i. e. from $N \trianglelefteq G \in \mathfrak{F}$ it follows that $N \in \mathfrak{F}$) and N_0-closed (i. e. from N_1, $N_2 \trianglelefteq G$ and N_1, $N_2 \in \mathfrak{F}$ it follows that $N_1 N_2 \in \mathfrak{F}$). If $\mathfrak{F}$ is a Fitting class and a formation, then $\mathfrak{F}$ is called the Fitting formation.

For a class of groups $\mathfrak{X}$ recall that $E\mathfrak{X}$ denotes the class of groups with a normal series whose factors are $\mathfrak{X}$-groups. If $\mathfrak{F}$ is a Fitting formation, then $E\mathfrak{F}$ is also a Fitting formation and is closed by extensions. Here $\mathrm{O}_\Sigma(G)$ denotes the greatest normal subgroup of G all whose composition factors belong to Σ for a class of simple groups Σ.

Recall that S_n denotes the symmetric group of degree n. We use standard computational conventions of abstract finite groups equipped with polynomial-time procedures to compute products and inverses of elements [9, chap. 2]. For both input and output, groups are specified by generators. We will consider only $G = \langle S \rangle \leq S_n$ with $|S| \leq n^2$. If it is necessary, Sims algorithm [16, parts 4.1 and 4.2] can be used to arrange that $|S| \leq n^2$. Quotient groups are specified by generators of a group and its normal subgroup. Note [10] that there exists a permutation group of degree n such that it has a quotient with no faithful representations of degree less than $2^{n/4}$.

For the rest of the paper n is used to denote the degree of the input permutations. A polynomial-time algorithm is an algorithm whose running time is upper-bounded by some polynomial function of n.

We need the following well known basic tools in our proofs (see, for example, [9] or [11]). Note that (1)–(4) are obtained with the help of the classification of finite simple groups.

Theorem 2. *Given normal subgroups A and B of a permutation group G of degree n with $A \leq B$, in polynomial time one can solve the following problems:*

(1) Find the centraliser $C_{G/A}(B/A)$ of B/A in G/A [11, P6(i)].

(2) Find a chief series for G containing A and B [11, P11].

(3) Test if G/A is simple [9, P10(i)]; if it is not, find a proper normal subgroup N/A of G/A [9, P10(ii)]; if it is, find the name of G/A [11, P12]. In particular, find a type of a chief factor.

(4) Find $\mathrm{O}_\Sigma(G/A)$ for a class Σ of simple groups [11, P16(i)].

(5) Given $H \leq G$, find $H \cap A$ [11, P4(i)].

(6) Find the derived subgroup $(G/A)'$ of G/A [11, P10(ii)].

(7) Find the order $|G/A|$ of G/A [11, P1].

The following lemma restricts the length of a chief series of a permutation group.

Lemma 1 [12]. *If $G \leq S_n$, then the length of every subgroup chain in G is at the most $2n - 3$ for $n \geq 2$.*

Proof of theorem 1

In this section $\mathfrak{F}$ is a Baer-local Fitting formation defined by f such that $f(J)$ is a Fitting formation for any simple group J. The idea of the theorem proof is to obtain the $\mathfrak{F}$-radical as the intersection of generalised centralisers of chief factors in the sense of the following definition.

Definition 1. For a chief factor H/K of G with $f(H/K) \neq \varnothing$ let $C_{G,f}(H/K)$ be defined by (*a*) $C_G(H/K) \subseteq C_{G,f}(H/K)$ and (*b*) $C_{G,f}(H/K)/C_G(H/K) = (G/C_G(H/K))_{f(H/K)}$.

Lemma 2. *Let B/A be a chief factor of a group G with $f(B/A) \neq \varnothing$ and N be a normal subgroup of G with $N \leq A$. Then*

$$C_{G/N,f}((B/N)/(A/N)) = C_{G,f}(B/A)/N.$$

P r o o f. Note that

$$C_{G/N}((B/N)/(A/N)) = \{gN \in G/N \mid [gN, g_i N] \in A/N \text{ for all } g_i N \in B/N\} =$$
$$= \{g \in G \mid [g, g_i]N \in A/N \text{ for all } g_i \in B\}/N = C_G(B/A)/N.$$

Now

$$(G/N)/C_{G/N}((B/N)/(A/N)) = (G/N)/(C_G(B/A)/N) \simeq G/C_G(B/A).$$

This isomorphism induces the isomorphism between (B/A)-radicals of the left and the right hands parts. It means that if $F/C_G(B/A) = (G/C_G(B/A))_{f(B/A)}$, then

$$(F/N)/(C_G(B/A)/N) = (F/N)/C_{G/N}((B/N)/(A/N)) = ((G/N)/C_{G/N}((B/N)/(A/N)))_{f(B/A)}.$$

Thus $C_{G/N,\,f}\big((B/N)/(A/N)\big)=C_{G,\,f}(B/A)/N$. The proof is completed.

Lemma 3. *Let H/K be a chief factor of a group G with $f(H/K)\neq\varnothing$. Then $G_{\mathfrak{F}}\leq C_{G,\,f}(H/K)$.*

P r o o f. If $H/K\notin\mathfrak{F}$, then from $G_{\mathfrak{F}}K/K\in\mathfrak{F}$ it follows that $H/K\cap G_{\mathfrak{F}}K/K=K/K$. Therefore $HG_{\mathfrak{F}}/K=$ $=(H/K)\times G_{\mathfrak{F}}K/K$. It means that $G_{\mathfrak{F}}\leq C_G(H/K)\leq C_{G,\,f}(H/K)$.

Assume that $H/K\in\mathfrak{F}$ and $p\in\pi(H/K)$. Since $\mathfrak{F}$ is N_0-closed, we see that $T/K=HG_{\mathfrak{F}}/K\in\mathfrak{F}$. For a subgroup L of G with $K\subseteq L$ denote L/K by $\bar{L}$. Now $\bar{T}/C_{\bar{T}}(\bar{M}/\bar{N})\in f(\bar{M}/\bar{N})=f(H/K)$ for every chief factor $\bar{M}/\bar{N}$ of $\bar{T}$ below $\bar{H}$.

If H/K is abelian, then H/K is a p-group for some prime p. Let $\bar{K}=\bar{H}_0\trianglelefteq\bar{H}_1\trianglelefteq\ldots\trianglelefteq\bar{H}_m=\bar{H}$ be a part of chief series of $\bar{T}$. Then $\bar{T}/C_{\bar{T}}(\bar{H}_i/\bar{H}_{i-1})\in f(p)=f(H/K)$. Let $\bar{C}=\bigcap_{i=1}^{m}C_{\bar{T}}(\bar{H}_i/\bar{H}_{i-1})$. Now $\bar{C}/C_{\bar{T}}(\bar{H})$ is a p-group by [7, chap. A, corollary 12.4(a)]. Since $f(p)$ is a formation, $\bar{T}/\bar{C}=(T/K)/(C/K)\in f(p)$. Thus $(T/K)/C_{T/K}(H/K)\in\mathfrak{N}_p f(p)$. Note that

$$(T/K)/C_{T/K}(H/K)\simeq(T/K)C_{G/K}(H/K)/C_{G/K}(H/K)\trianglelefteq(G/K)/C_{G/K}(H/K).$$

Since H/K is a chief factor of G/K, we see that $\mathrm{O}_p\big((G/K)/C_{G/K}(H/K)\big)\simeq 1$ by [7, chap. A, lemma 13.6]. Hence $\mathrm{O}_p\big((T/K)C_{G/K}(H/K)/C_{G/K}(H/K)\big)\simeq 1$. Thus

$$(T/K)/C_{T/K}(H/K)\simeq(T/K)C_{G/K}(H/K)/C_{G/K}(H/K)\in f(p)=f(H/K).$$

If H/K is non-abelian, then H/K is the direct product of minimal normal subgroups H_i/K of T/K by [7, chap. A, lemma 4.14]. Since $f(p)$ is a formation, from $C_{T/K}(H/K)=\bigcap_i C_{T/K}(H_i/K)$ it follows that $(T/K)/C_{T/K}(H/K)\in$ $\in f(H/K)$. Since $f(H/K)$ is a Fitting formation, we see that

$$(T/K)/C_{T/K}(H/K)=(T/K)/\big(C_T(H/K)/K\big)\simeq T/C_T(H/K)\simeq$$
$$\simeq TC_G(H/K)/C_G(H/K)\subseteq\big(G/C_G(H/K)\big)_{f(H/K)}=C_{G,\,f}(H/K)/C_G(H/K).$$

Therefore $G_{\mathfrak{F}}\leq T\leq C_{G,\,f}(H/K)$. The proof is completed.

Theorem 3. *Let $1=G_0\trianglelefteq G_1\trianglelefteq\ldots\trianglelefteq G_m=G$ be a chief series of a group G. Then*

$$G_{\mathfrak{F}}=\left(\bigcap_{i=1,\,f(G_i/G_{i-1})\neq\varnothing}^{m}C_{G,\,f}(G_i/G_{i-1})\right)_{E\mathfrak{F}}.$$

P r o o f. We assume that every intersection of empty collection of subgroups of G coincides with G. Let

$$D=\bigcap_{i=1,\,f(G_i/G_{i-1})\neq\varnothing}^{m}C_{G,\,f}(G_i/G_{i-1}).$$

From lemma 3 it follows that $G_{\mathfrak{F}}\subseteq D$. Note that $G_{\mathfrak{F}}\in E\mathfrak{F}$. Thus $G_{\mathfrak{F}}\leq D_{E\mathfrak{F}}$. Let $1=D_0\trianglelefteq D_1\trianglelefteq\ldots\trianglelefteq D_l=D_{E\mathfrak{F}}$ be a part of chief series of G below $D_{E\mathfrak{F}}$. Then by the Jordan – Hölder theorem there is $\rho:\{1,\ldots,l\}\to\{1,\ldots,m\}$ such that D_i/D_{i-1} is G-isomorphic to $G_{\rho(i)}/G_{\rho(i)-1}$ for all $i\in\{1,\ldots,l\}$. Now $C_G\big(G_{\rho(i)}/G_{\rho(i)-1}\big)=C_G(D_i/D_{i-1})$ for all $i\in\{1,\ldots,l\}$. Hence $C_{G,\,f}\big(G_{\rho(i)}/G_{\rho(i)-1}\big)=C_{G,\,f}(D_i/D_{i-1})$ by definition 1. Note that

$$D_{E\mathfrak{F}}/C_{D_{E\mathfrak{F}}}(D_i/D_{i-1})\simeq D_{E\mathfrak{F}}C_G(D_i/D_{i-1})/C_G(D_i/D_{i-1})=D_{E\mathfrak{F}}C_G\big(G_{\rho(i)}/G_{\rho(i)-1}\big)/C_G\big(G_{\rho(i)}/G_{\rho(i)-1}\big)\trianglelefteq$$
$$\trianglelefteq C_{G,\,f}\big(G_{\rho(i)}/G_{\rho(i)-1}\big)/C_G\big(G_{\rho(i)}/G_{\rho(i)-1}\big)\in f\big(G_{\rho(i)}/G_{\rho(i)-1}\big)=f(D_i/D_{i-1}).$$

Since $f(D_i/D_{i-1})$ is normally hereditary we see that $D_{E\mathfrak{F}}/C_{D_{E\mathfrak{F}}}(D_i/D_{i-1})\in f(D_i/D_{i-1})$. Since $f(D_i/D_{i-1})$ is closed under taking quotients, we see that

$$D_{E\mathfrak{F}}/C_{D_{E\mathfrak{F}}}(H/K)\in f(D_i/D_{i-1})=f(H/K)$$

for all chief factors H/K of $D_{E\mathfrak{F}}$ between D_{i-1} and D_i for all $i \in \{1, \ldots, l\}$. From the Jordan – Hölder theorem it follows that $D_{E\mathfrak{F}}/C_{D_{E\mathfrak{F}}}(H/K) \in f(H/K)$ for all chief factors H/K of $D_{E\mathfrak{F}}$. Therefore $D_{E\mathfrak{F}} \in \mathfrak{F}$. Thus $D_{E\mathfrak{F}} = G_{\mathfrak{F}}$. The proof is completed.

Lemma 4. *Let B/A be a chief factor of G. Then $C_{G,\, f}(B/A)$ can be computed in polynomial time (in the assumptions of theorem 1).*

P r o o f. Note that $T/A = C_{G/A}(B/A)$ can be computed in polynomial time by theorem 2(1) and $T = C_G(B/A)$. Now $G/C_G(B/A) = G/T$. By our assumption $(G/T)_{f(B/A)}$ can be computed in polynomial time. The proof is completed.

Lemma 5. *A chief factor H/K of a group G belongs to $\mathfrak{F}$ iff $(H/K)'_{f(H/K)} = (H/K)'$.*

P r o o f. Let H/K be a non-abelian chief factor of G of type J. We claim that $H/K \in \mathfrak{F}$ iff $(H/K)_{f(J)} = H/K$. Note that H/K is a direct product of groups isomorphic to J. If $(H/K)_{f(J)} = H/K$, then since $f(J)$ is normally hereditary, $J \in f(J)$. Now $J \in \mathfrak{F}$ by the definition of the Baer-local formation. Since $\mathfrak{F}$ is N_0-closed, $H/K \in \mathfrak{F}$. Assume now that $H/K \in \mathfrak{F}$. Hence $J \in \mathfrak{F}$. Since J is non-abelian, $J \simeq J/C_G(J) \in f(J)$. Therefore $H/K = (H/K)_{f(J)}$ as a direct product of $f(J)$-groups.

Let H/K be an abelian chief factor. We claim that $H/K \in \mathfrak{F}$ iff $f(H/K) \neq \varnothing$. It is clear that if $H/K \in \mathfrak{F}$, then $f(H/K) \neq \varnothing$. Assume that $f(H/K) \neq \varnothing$. Hence $1 \in f(H/K)$. Now $(H/K)/C_{H/K}(U/V) \simeq 1 \in f(H/K) = f(U/V)$ for any chief factor U/V of H/K. Thus $H/K \in \mathfrak{F}$ by the definition of the Baer-local formation.

Note that the two above mentioned cases are equivalent to $H/K \in \mathfrak{F}$ iff

$$(H/K)'_{f(H/K)} = (H/K)'.$$

The proof is completed.

Lemma 6. *In the assumptions of theorem 1 we can compute $(G/K)_{E\mathfrak{F}}$ in polynomial time.*

P r o o f. If H/K is a chief factor of G, then we can check if $H/K \in \mathfrak{F}$ in polynomial time by our assumptions, lemma 5 and theorem 2(6).

We can compute a part of the chief series $K = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_m = G$ of a group G in polynomial time by theorem 2(2). Define F_i by $F_i/K = (G_i/K)_{E\mathfrak{F}}$. Note that $F_i \cap G_{i-1} = F_{i-1}$ and $F_0 = K$. Now

$$F_i/F_{i-1} = F_i/(F_i \cap G_{i-1}) \simeq F_iG_{i-1}/G_{i-1} \trianglelefteq G_i/G_{i-1}.$$

Therefore F_i/F_{i-1} is isomorphic to either 1 or G_i/G_{i-1}. If $G_i/G_{i-1} \notin \mathfrak{F}$, then $F_i = F_{i-1}$. Assume that $G_i/G_{i-1} \in \mathfrak{F}$. If G_i/G_{i-1} is a group of type J, then F_i/F_{i-1} is also such a group. Since $E\mathfrak{F}$ is closed under extensions, we see that $F_i/F_{i-1} = \mathrm{O}_{(J)}(G_i/F_{i-1})$. Hence F_i can be computed in polynomial time by statements (3) and (4) of theorem 2.

Algorithm 1. EFRADICAL $(G, K, \mathfrak{F})$.
Result. $(G/K)_{E\mathfrak{F}}$.
Data. A normal subgroup K of a group G, $\mathfrak{F} = BLF(f)$.
Compute a chief series $K = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_m = G$;
$F \leftarrow K$;
for $i \in \{1, \ldots, m\}$ **do**
  if $(G_i/G_{i-1})'_{f(G_i/G_{i-1})} = (G_i/G_{i-1})'$ **then**
   $J \leftarrow$ type of G_i/G_{i-1};
   $F_1/F \leftarrow \mathrm{O}_{(J)}(G_i/F)$;
   $F \leftarrow F_1$;
return F/K.

P r o o f o f t h e o r e m 1. Let $K = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_m = G$ be a part of chief series of G (it can be computed in polynomial time by theorem 2(2)). Now by lemma 2

$$I/K = \bigcap_{i=1,\, f((G_i/K)/(G_{i-1}/K)) \neq \varnothing}^{m} C_{G/K,\, f}\big((G_i/K)/(G_{i-1}/K)\big) = \bigcap_{i=1,\, f(G_i/G_{i-1}) \neq \varnothing}^{m} C_{G,\, f}(G_i/G_{i-1})/K.$$

Hence this subgroup can be computed in a polynomial time by theorem 2(5) and lemma 4. Using lemma 6 we can compute $R/K = (I/K)_{E\mathfrak{F}}$. Now $(G/K)_{\mathfrak{F}} = (I/K)_{E\mathfrak{F}}$ by theorem 3.

Algorithm 2. FRADICAL$(G, K, \mathfrak{F})$.
Result. $(G/K)_{\mathfrak{F}}$.
Data. A normal subgroup K of a group G, $\mathfrak{F} = BLF(f)$.
Compute a chief series $K = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_m = G$ of G;
$T \leftarrow G$;
for $i \in \{1, \ldots, m\}$ **do**
  if $f(G_i/G_{i-1}) \neq \varnothing$ **then**
    $C/G_{i-1} \leftarrow C_{G/G_{i-1}}(G_i/G_{i-1})$;
    $F/C \leftarrow (G/C)_{f(G/G_{i-1})}$;
    $T/K \leftarrow T/K \cap F/K$;
return EFRADICAL$(T, K, \mathfrak{F})$.

Applications

The $\mathfrak{F}$-radical for a primitive saturated formation. Let $\mathscr{F}_0$ denote the family consisting of the empty set, the formation of groups of order one, and the formation of all soluble groups, and then, for $i > 0$, define $\mathscr{F}_i$ inductively by $\mathfrak{F} \in \mathscr{F}_i$, if either $\mathfrak{F} \in \mathscr{F}_{i-1}$ or $\mathfrak{F}$ is a local formation, with local definition f such that $f(p) \in \mathscr{F}_{i-1}$ for all prime p. Finally let $\mathscr{F}$ be the family comprising all formations $\mathfrak{F}$ such that $\mathfrak{F} = \bigcup_j \mathfrak{F}_j$ with each $\mathfrak{F}_j \in \bigcup_i \mathscr{F}_i$ and $\mathfrak{F}_j \subseteq \mathfrak{F}_{j+1}$. Formations from $\mathscr{F}$ are called primitive [7, chap. VII, definition 3.1]. As was mentioned in [13] if $G \in \mathfrak{F} \in \mathscr{F}$ and the nilpotent length of G is less than m, then there exists $\mathfrak{H} \in \mathscr{F}_m$ with $G \in \mathfrak{H}$.

Theorem 4. *Let m be a natural number. If $\mathfrak{F} \in \mathscr{F}_m$ and $K \trianglelefteq G \leq S_n$, then $(G/K)_{\mathfrak{F}}$ can be computed in polynomial time (in n).*

P r o o f. It is clear that if $\mathfrak{H} \in \mathscr{F}_0$, then $(G/K)_{\mathfrak{H}}$ can be computed in polynomial time by theorem 2(4). Assume that we can compute $(G/K)_{\mathfrak{H}}$ in polynomial time for every $\mathfrak{H} \in \mathscr{F}_{i-1}$. Let prove that we can do so for every $\mathfrak{H} \in \mathscr{F}_i$. If $\mathfrak{H} \in \mathscr{F}_i \setminus \mathscr{F}_{i-1}$, then the values $f(p)$ of local definition f of $\mathfrak{H}$ are in $\mathscr{F}_{i-1}$. By our assumption we can compute the $f(p)$-radicals of G/K in polynomial time for every $K \trianglelefteq G \leq S_n$. Hence we can compute the $\mathfrak{H}$-radical of G/K in polynomial time in n for every $K \trianglelefteq G \leq S_n$ and every $\mathfrak{H} \in \mathscr{F}_i$ by theorem 1. Thus $(G/K)_{\mathfrak{F}}$ can be computed in polynomial time for every $K \trianglelefteq G \leq S_n$.

The $\mathfrak{F}$-length. The lengths of a group associated with some of its radical (the nilpotent length, the p-length and etc.) play an important role in the theory of groups. Some approaches how one could introduce the $\mathfrak{F}$-length were suggested in [14, paragraph 5; 15]. Using the ideas of those approaches we introduce the following definition.

Definition 2. For Fitting classes $\mathfrak{F}$ and $\mathfrak{H}$ the $\mathfrak{F}$, $\mathfrak{H}$-length $l_{\mathfrak{F},\mathfrak{H}}(G)$ of a group G is equal to m if $1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_{2m+1} = G$ where $G_i/G_{i-1} = (G/G_{i-1})_{\mathfrak{H}}$ for odd i and $G_i/G_{i-1} = (G/G_{i-1})_{\mathfrak{F}} \neq 1$ for even i. Note that the $\mathfrak{F}$, $\mathfrak{H}$-length is defined only in case $G \in E(\mathfrak{F} \cup \mathfrak{H})$.

Remark 1. In some cases of this definition $\mathfrak{F} \cap \mathfrak{H} = \mathfrak{E}$ is the class of all identity groups. If $\mathfrak{H} = \mathfrak{E}$ then the $\mathfrak{F}$, $\mathfrak{H}$-length is just the $\mathfrak{F}$-length and it will be denoted by $l_{\mathfrak{F}}(G)$ for a group G.

Remark 2. If $\mathfrak{F} = \mathfrak{N}$ is the class of all nilpotent groups, then $l_{\mathfrak{N}}(G) = h(G)$ is the nilpotent length of G and defined only for soluble groups G. If $\mathfrak{F} = \mathfrak{G}_p$ is the class of all p-groups and $\mathfrak{H} = \mathfrak{G}_{p'}$ is the class of all p'-groups, then $l_{\mathfrak{G}_p, \mathfrak{G}_{p'}}(G) = l_p(G)$ is the p-length of G and defined only for p-soluble groups G.

Recall that E. I. Khukhro and P. Shumyatsky suggested the interesting generalisations of these lengths.

Definition 3 [16; 17]. (1) The generalised Fitting height $h^*(G)$ of a group G is the least number h such that $\mathrm{F}^*_{(h)}(G) = G$ where $\mathrm{F}^*_{(0)}(G) = 1$, and $\mathrm{F}^*_{(i+1)}(G)$ is the inverse image of the generalised Fitting subgroup $\mathrm{F}^*(G/\mathrm{F}^*_{(i)}(G))$.

(2) Let p be a prime, $1 = G_0 \leq G_1 \leq \ldots \leq G_{2h+1} = G$ be the shortest normal series in which for i odd the factor G_{i+1}/G_i is p-soluble (possibly trivial), and for i even the factor G_{i+1}/G_i is a (non-empty) direct product of non-abelian simple groups. Then $h = \lambda_p(G)$ is called the non-p-soluble length of a group G.

(3) $\lambda_2(G) = \lambda(G)$ is the non-soluble length of a group G.

Remark 3. If $\mathfrak{F}=\mathfrak{N}^*$ is the class of all quasinilpotent groups, then $l_{\mathfrak{N}^*}(G)=h^*(G)$. If $\mathfrak{H}=\mathfrak{S}^p$ is the class of all p-soluble groups, then from [18, lemma 2.7] it follows that $l_{\mathfrak{N}^*,\,\mathfrak{S}^p}(G)=\lambda_p(G)$.

Note that for Fitting classes $\mathfrak{F}$ and $\mathfrak{H}$ the class $\mathfrak{K}=\mathfrak{H}\lozenge\mathfrak{F}=\left(G\,|\,G/G_{\mathfrak{H}}\in\mathfrak{F}\right)$ is also a Fitting class and $G_{\mathfrak{K}}/G_{\mathfrak{H}}=\left(G/G_{\mathfrak{H}}\right)_{\mathfrak{F}}$ by [7, chap. IX, theorem 1.12]. Hence the $\mathfrak{F}$, $\mathfrak{H}$-length can be defined by $\mathfrak{K}$ in the following way: if $G_{\mathfrak{K}}\simeq G_{\mathfrak{H}}\neq G$, then $l_{\mathfrak{F},\,\mathfrak{H}}(G)=\infty$, else if $G\in\mathfrak{H}$, then $l_{\mathfrak{F},\,\mathfrak{H}}(G)=0$, otherwise $l_{\mathfrak{F},\,\mathfrak{H}}(G)=l_{\mathfrak{F},\,\mathfrak{H}}(G/G_{\mathfrak{K}})+1$.

Theorem 5. *Let $\mathfrak{F}$ and $\mathfrak{H}$ be Fitting classes. Assume that $(G/K)_{\mathfrak{F}}$ and $(G/K)_{\mathfrak{H}}$ can be computed in polynomial time for any $K\trianglelefteq G\leq S_n$. Then $l_{\mathfrak{F},\,\mathfrak{H}}(G/K)$ can be computed in polynomial time for any $K\trianglelefteq G\leq S_n$.*

P r o o f. From the statement of the theorem it follows that $(G/K)_{\mathfrak{K}}=R/K$ can be computed in polynomial time for any $K\trianglelefteq G\leq S_n$ by $R_0/K=(G/K)_{\mathfrak{H}}$ and $R/R_0=(G/R_0)_{\mathfrak{F}}$. Now from theorem 2 we can compute $l_{\mathfrak{F},\,\mathfrak{H}}(G/K)$ in polynomial time by NAIVEFLENGTH.

Algorithm 3. NAIVEFLENGTH$(G, K, \mathfrak{F}, \mathfrak{H})$.
Result. $l_{\mathfrak{F},\,\mathfrak{H}}(G/K)$.
Data. A normal subgroup K of a group G.
if $\left|(G/K)_{\mathfrak{K}}\right|=\left|(G/K)_{\mathfrak{H}}\right|\neq|G/K|$ **then return** ∞;
if $|G/K|=\left|(G/K)_{\mathfrak{H}}\right|$ **then return** 0;
else
  $R/K\leftarrow(G/K)_{\mathfrak{K}}$;
  return NAIVEFLENGTH$(G, R, \mathfrak{F}, \mathfrak{H})+1$.

We use $R_p(G)$ to denote the p-soluble radical of a group G. Let $\overline{\mathrm{F}}_p^*(G)$ be the inverse image of $\mathrm{F}^*\left(G/R_p(G)\right)$.

Lemma 7. *For any $K\trianglelefteq G\leq S_n$ one can compute* $\mathrm{F}(G/K)$, $\mathrm{O}_{p',\,p}(G/K)$, $\mathrm{F}^*(G/K)$ *and* $\overline{\mathrm{F}}_p^*(G/K)$ *in polynomial time.*

P r o o f. Recall that the generalised Fitting subgroup of a group is its $\mathfrak{N}^*$-radical. This class is a Baer-local formation defined by h where $h(J)=1$ if J is abelian and $h(J)=D_0(J)$ otherwise [7, chap. IX, lemma 2.6]. Hence $C_{G,\,h}(H/K)=C_G(H/K)=HC_G(H/K)$ if H/K is abelian. Note that if H/K is non-abelian (of type J), then $G/C_G(H/K)$ has the unique minimal normal subgroup which is isomorphic to H/K and hence coincides with the $h(J)$-radical of $G/C_G(H/K)$. Thus $C_{G,\,h}(H/K)=HC_G(H/K)$ in this case. From $C_G(H/K)/K=C_{G/K}(H/K)$ it follows that $C_G(H/K)$ can be computed in polynomial time by theorem 2(1). Now the generating set of $HC_G(H/K)$ is the joining of generating sets of H and $C_G(H/K)$. Thus $HC_G(H/K)$ can be computed in polynomial time. Hence from the proof of theorem 1 it follows that $\mathrm{F}^*(G/K)$ can be computed in polynomial time. Note that $R_p(G/K)$ can be computed in polynomial time by theorem 2(4). Therefore we can compute $\overline{\mathrm{F}}_p^*(G/K)$ in polynomial time.

Recall that the classes of all nilpotent and p-nilpotent groups can be locally defined by f_1 and f_2 respectively where $f_1(q)=1$ for all prime q and $f_2(p)=1$ for all prime $q\neq p$ and $f_2(q)=\mathfrak{G}$, where $\mathfrak{G}$ is the class of all groups. Hence $\mathrm{F}(G/K)$ and $\mathrm{O}_{p',\,p}(G/K)$ can be computed in polynomial time by theorem 1. The proof is completed.

From theorems 2 and 5, and lemma 7 the following result directly follows.

Theorem 6. *If $K\trianglelefteq G\leq S_n$ and p is a prime, then $h(G/K)$, $l_p(G/K)$, $h^*(G/K)$, $\lambda_p(G/K)$, $\lambda(G/K)$ can be computed in a polynomial time.*

The main problem of NAIVEFLENGTH is that we need to recompute the $\mathfrak{K}$-radicals in quotient groups. Note that if $\mathfrak{K}$ is a Baer-local formation and we know $C_{G,\,f}(H/K)$ for every chief factor H/K from some chief series of G, then we can compute the $\mathfrak{K}$-radical in every quotient group of G.

Theorem 7. *Let $\mathfrak{H}$ and $\mathfrak{F}$ be Fitting classes such that $\mathfrak{K}=\mathfrak{H}\lozenge\mathfrak{F}$ is a Baer-local Fitting formation defined by f where $f(J)$ is a Fitting formation for every simple group J. Assume that for any $K\trianglelefteq G\leq S_n$ one in polynomial time can check if $G/K\in\mathfrak{H}$ and compute $(G/K)_{f(J)}$ for every simple group J. Then $l_{\mathfrak{F},\,\mathfrak{H}}(G/K)$ can be computed in polynomial time for any $K\trianglelefteq G\leq S_n$.*

P r o o f. Let $K=G_0\trianglelefteq G_1\trianglelefteq\ldots\trianglelefteq G_m=G$ be a part of chief series of G. Then $G/K\in E\mathfrak{K}$ iff $G_i/G_{i-1}\in\mathfrak{K}$ for every i iff $\left(G_i/G_{i-1}\right)'_{f(G_i/G_{i-1})}=\left(G_i/G_{i-1}\right)'$ for every i by lemma 5. Note that if $G/K\in E\mathfrak{K}$, then every its normal subgroup also belongs to $E\mathfrak{K}$. Hence we don't need to use EFRADICAL in the computation of the $\mathfrak{K}$-radical in every quotient group of G.

If $K \le F \trianglelefteq G \le S_n$, then $G_iF/G_{i-1}F \simeq G_i/\left(G_i \cap G_{i-1}F\right) = G_i/\left(G_{i-1}\left(G_i \cap F\right)\right)$ is G-isomorphic to either 1 or G_i/G_{i-1}. Hence $F = KF = G_0F \trianglelefteq G_1F \trianglelefteq \ldots \trianglelefteq G_mF = G$ after we remove the repetitive terms will become the part of chief series of G above F. Note that $C_{G,f}\left(M/N\right) = C_{G,f}\left(L/T\right)$ for any G-isomorphic chief factors M/N and L/T of G. So if we know $C_{G,f}\left(G_i/G_{i-1}\right)$ and $FG_i/FG_{i-1} \ne 1$ (it can be checked by theorem 2), then we know $C_{G,f}\left(FG_i/FG_{i-1}\right)$. Now we can compute $\left(G/F\right)_{\mathfrak{K}}$ using lemma 2 and theorem 3. By the statement of theorem we can check if $\left(G/F\right)_{\mathfrak{K}} \in \mathfrak{H}$, i. e. $\left(G/F\right)_{\mathfrak{K}}/\left(G/F\right)_{\mathfrak{H}} \simeq 1$. Therefore $l_{\mathfrak{F},\mathfrak{H}}\left(G/K\right)$ can be computed in polynomial time for any $K \trianglelefteq G \le S_n$.

Algorithm 4. FLENGTH$\left(G, K, \mathfrak{F}, \mathfrak{H}\right)$.
Result. $l_{\mathfrak{F},\mathfrak{H}}\left(G/K\right)$.
Data. A normal subgroup K of a group G.
Compute a chief series $K = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_m = G$ of G;
if $G/K \in \mathfrak{H}$ **then**
    return 0;
else
    $l \leftarrow 1$;
$L \leftarrow []$, $T \leftarrow G$;
for $i \in \{1, \ldots, m\}$ **do**
    if $\left(G_i/G_{i-1}\right)'_{f\left(G_i/G_{i-1}\right)} = \left(G_i/G_{i-1}\right)'$ **then**
        $C/G_{i-1} \leftarrow C_{G/G_{i-1}}\left(G_i/G_{i-1}\right)$;
        $F/C \leftarrow \left(G/C\right)_{f\left(G/G_{i-1}\right)}$;
        add$\left(L, \left(G_{i-1}, G_i, F\right)\right)$;
        $T/K \leftarrow T/K \cap F/K$;
    else
        return ∞;
if $T/K \in \mathfrak{H}$ **then**
    return ∞;
while $G/T \notin \mathfrak{H}$ **do**
    $l \leftarrow l+1$;
    $F \leftarrow G$;
    for $R \in L$ **do**
        if $TR[1] = TR[2]$ **then**
            remove$\left(L, R\right)$;
        else
            $F \leftarrow F \cap R[3]$;
    if $F/T \in \mathfrak{H}$ **then**
        return ∞;
    $T \leftarrow F$;
return l.

Conclusions

This work belongs to a series of works [5; 19] dedicated to the computational recognition of formations and the computation of associated with them subgroups in every group. The key object of those works was the notion of a chief factor function. Recall [5] that a function f that associates 0 or 1 with every chief factor H/K of a group G is called a chief factor function if $\mathrm{f}\left(H/K, G\right) = \mathrm{f}\left(M/N, G\right)$ whenever H/K and M/N are G-isomorphic chief factors of G, and $\mathrm{f}\left(H/K, G\right) = \mathrm{f}\left(\left(H/N\right)/\left(K/N\right), G/N\right)$ for every $N \trianglelefteq G$ with $N \le K$. With every such function one can associate the formation $\mathcal{C}\left(\mathrm{f}\right)$ of groups G such that $G \simeq 1$ or $\mathrm{f}\left(H/K, G\right) = 1$ for every chief factor H/K of the group G. The family of such formations includes all Baer-local formations. In [5; 19] the algorithms for the computation of the $\mathfrak{F}$-residual and the $\mathfrak{F}$-hypercenter where $\mathfrak{F} = \mathcal{C}(\mathrm{f})$ were suggested respectively. Therefore the following questions seem interesting.

Question 1. Find conditions $(*)$ (or prove that such conditions do not exist) on a chief factor functions f such that $\mathcal{C}(\mathrm{f})$ is a Fitting formation if f satisfies $(*)$ and for each Fitting formation of the form $\mathcal{C}(\mathrm{f})$ there is a chief factor function f_1 which satisfies $(*)$ and $\mathcal{C}(\mathrm{f}) = \mathcal{C}(\mathrm{f}_1)$.

Question 2. If f is a chief factor function, conditions $(*)$ from question 1 exist and $\mathrm{f}(H/K, G)$ can be computed in polynomial time for every chief factor H/K of a group $G \leq S_n$, then can one compute $(G/K)_{\mathcal{C}(\mathrm{f})}$ in polynomial time for every $K \trianglelefteq G \leq S_n$?

The algorithms constructed in the paper are purely theoretical. They are based on known algorithms about permutation groups (see theorem 2). Not all of these algorithms were fully implemented in computer algebra systems. That is why the question of implementation of the constructed algorithms remains open.

In [20] Fitting formations were used in the study of sublattices of the lattice of all subgroups. The results of this paper can be used to compute the sublattices described in [20]. One of the modern directions in the theory of formations is the study of σ-local and Baer σ-local formations [21; 22]. These formations are defined with the help of the generalisations of p-nilpotent radical. The results of this paper can be used to compute those radicals and hence to check if a given group belongs to such a formation.

References

1. Bryce RA, Cossey J. Subgroup closed Fitting classes are formations. *Mathematical Proceedings of the Cambridge Philosophical Society.* 1982;91(2):225–258. DOI: 10.1017/S0305004100059272.

2. Berger TR, Bryce RA, Cossey J. Quotient closed metanilpotent Fitting classes. *Journal of the Australian Mathematical Society.* 1985;38(2):157–163. DOI: 10.1017/S1446788700023004.

3. Höfling B. Computing projectors, injectors, residuals and radicals of finite soluble groups. *Journal of Symbolic Computation.* 2001;32(5):499–511. DOI: 10.1006/jsco.2001.0477.

4. Eick B, Wright CRB. Computing subgroups by exhibition in finite solvable groups. *Journal of Symbolic Computation.* 2002; 33(2):129–143. DOI: 10.1006/jsco.2000.0503.

5. Murashka VI. Formations of finite groups in polynomial time: $\mathfrak{F}$-residuals and $\mathfrak{F}$-subnormality. *Journal of Symbolic Computation.* 2024;122:102271. DOI: 10.1016/j.jsc.2023.102271.

6. Shemetkov LA. Composition formations and radicals of finite groups. *Ukrainian Mathematical Journal.* 1988;40:318–322. DOI: 10.1007/BF01061312.

7. Doerk K, Hawkes T. *Finite soluble groups*. Berlin: Walter de Gruyter; 1992. 893 p. (De Gruyter expositions in mathematics; volume 4). DOI: 10.1515/9783110870138.

8. Guo W. *Structure theory for canonical classes of finite groups*. Berlin: Springer; 2015. 359 p. DOI: 10.1007/978-3-662-45747-4.

9. Seress Á. *Permutation group algorithms*. Cambridge: Cambridge University Press; 2003. 264 p. DOI: 10.1017/CBO978051 1546549.

10. Neumann PM. Some algorithms for computing with finite permutation groups. In: Robertson EF, Campbell CM, editors. *Proceedings of groups – St. Andrews 1985*. Cambridge: Cambridge University Press; 1987. p. 59–92 (London Mathematical Society Lecture Note Series; volume 121). DOI: 10.1017/CBO9780511600647.006.

11. Kantor WM, Luks EM. Computing in quotient groups. In: Ortiz H, editor. *Proceedings of the twenty-second annual ACM symposium on theory of computing; 1990 May 13–17; Baltimore, Maryland, USA*. New York: Association for Computing Machinery; 1990. p. 524–534. DOI: 10.1145/100216.100290.

12. Babai L. On the length of subgroup chains in the symmetric group. *Communications in Algebra*. 1986;14(9):1729–1736. DOI: 10.1080/00927878608823393.

13. Bryce RA, Cossey J. Fitting formations of finite soluble groups. *Mathematische Zeitschrift.* 1972;127(3):217–223. DOI: 10.1007/BF01114925.

14. Shemetkov LA. *Formations of finite groups*. Moscow: Nauka; 1978. 272 p. Russian.

15. Makan AR. The Fitting length of a finite soluble group and the number of conjugacy classes of its maximal metanilpotent subgroups. *Canadian Mathematical Bulletin.* 1973;16(2):233–237. DOI: 10.4153/CMB-1973-040-3.

16. Khukhro EI, Shumyatsky P. Nonsoluble and non-p-soluble length of finite groups. *Israel Journal of Mathematics.* 2015;207(2): 507–525. DOI: 10.1007/s11856-015-1180-x.

17. Khukhro EI, Shumyatsky P. On the length of finite factorized groups. *Annali di Matematica Pura ed Applicata.* 2015;194(6): 1775–1780. DOI: 10.1007/s10231-014-0443-1.

18. Murashka VI, Vasil'ev AF. On the lengths of mutually permutable products of finite groups. *Acta Mathematica Hungarica.* 2023;170(1):412–429. DOI: 10.1007/s10474-023-01346-2.

19. Murashka VI. Formations of finite groups in polynomial time II: the $\mathfrak{F}$-hypercenter and its generalizations. *Trudy Instituta matematiki i mekhaniki UrO RAN*. 2025;31(1):154–165. DOI: 10.21538/0134-4889-2025-31-1-154-165.

20. Skiba AN. On some classes of sublattices of the subgroup lattice. *Journal of the Belarusian State University. Mathematics and Informatics.* 2019;3:35–47. DOI: 10.33581/2520-6508-2019-3-35-47.

21. Safonov VG, Safonova IN, Skiba AN. On Baer-σ-local formations of finite groups. *Communications in Algebra*. 2020;48(9): 4002–4012. DOI: 10.1080/00927872.2020.1753760.

22. Safonova IN, Safonov VG. On some properties of the lattice of totally σ-local formations of finite groups. *Journal of the Belarusian State University. Mathematics and Informatics*. 2020;3:6–16. DOI: 10.33581/2520-6508-2020-3-6-16.

Received 25.10.2024 / revised 02.03.2025 / accepted 02.03.2025.