By some transformations, equation (1) is reduced to the following four types of equations:

$$u_{tx} + A_1 u_t + A_2 u_x + \lambda f(u) = 0, \tag{2}$$

$$u_{tt} - u_{xx} + A_1 u_t + A_2 u_x + \lambda f(u) = 0, \tag{3}$$

$$u_{tt} + u_{xx} + A_1 u_t + A_2 u_x + \lambda f(u) = 0, \tag{4}$$

$$u_t + A_{22} u_{xx} + A_2 u_x + \lambda f(u) = 0. \tag{5}$$

Equation (2) is a hyperbolic nonlinear Klein–Gordon equation in the first form. Equation (3) is a hyperbolic nonlinear Klein–Gordon equation in the second form. Equation (4) is an elliptic nonlinear Klein–Gordon equation. The equation (5) is a nonlinear convection–diffusion equation.

The reduction for equation (1) will be carried out using symmetries and substitution of variables $u = y(z(t,x))$.

We will look for infinitesimal transformations for equation (1) in the form [2–5]

$$\widetilde{t} = t + \varepsilon\tau(t,x,u) + \ldots, \quad \widetilde{x} = x + \varepsilon\xi(t,x,u) + \ldots, \quad \widetilde{u} = u + \varepsilon\eta(t,x,u) + \ldots,$$

and $\widetilde{f}(\widetilde{u}) = f(u) + \varepsilon f^{(1)}(u)\eta + \ldots$

Then the group generator for equation (1) has the form

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}.$$

References

1. *Korzyuk V. I.* Equations of Mathematical Physics. Moscow, 2021.
2. *Hydon P. E.* Symmetry Methods for Differential Equations: A Beginner's Guide. Cambridge, 2000.
3. *Ibragimov N. H.* Transformation Groups Applied to Mathematical Physics. Dordrecht, 1985.
4. *Ovsiannikov L. V.* Group Analysis of Differential Equations. New York, 1982.
5. *Olver P. J.* Applications of Lie Groups to Differential Equations. Graduate Texts Math. V. 107. New York, 1993.

ON FRACTIONAL DERIVATIVE OF THE MITTAG-LEFFLER TYPE FUNCTIONS

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The report is devoted to the calculation of the fractional derivatives of the Mittag-Leffler function with respect to parameters.

We calculate the fractional derivative of the Mittag-Leffler function. We use both main types of fractional derivatives, namely, the Riemann–Liouville fractional derivative

$$\left({}^{RL}D^{\mu} f(\tau)\right)(t) := \frac{1}{\Gamma(n-\mu)}\frac{d^n}{dt^n}\int\limits_0^t \frac{f(\tau)\,d\tau}{(t-\tau)^{\mu-n+1}}, \quad n = [\mu]+1,$$

and the Dzherbashian–Caputo fractional derivative

$$\left({}^{DC}D^{\mu}f(\tau)\right)(t) := \frac{1}{\Gamma(n-\mu)}\int\limits_0^t \frac{f^{(n)}(\tau)\,d\tau}{(t-\tau)^{\mu-n+1}}, \quad n = [\mu]+1.$$

It is motivated by the recent approaches to the calculation of the derivatives of special functions with respect to parameters (see, e.g., [1]) and by the study of evolution equations in the form proposed in [2].

The Mittag-Leffler function is an entire function of the complex variable z defined by the following power series (see, e.g. [3]):

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},$$

for every $z, \beta \in \mathbb{C}$ and $\operatorname{Re}\alpha > 0$.

Calculation are based on a complex integral representation of the function $1/\Gamma(z)$ valid for *unrestricted* z (see, e.g., [3, Appendix A]):

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int\limits_{Ha_-}\frac{e^t}{t^z}\,dt, \quad z \in \mathbb{C},$$

with integration along so called Hankel path Ha_-.

$$\left({}^{RL}D^{\mu}_{0+,\alpha}\frac{1}{\Gamma(\alpha k+\beta)}\right)(\alpha) = k^{\mu}\frac{1}{2\pi i}\int\limits_{Ha_-} e^s s^{-\alpha k-\beta}(-\ln s)^{\mu-1}\,ds =: k^{\mu}c_k(\alpha,\beta),$$

$$\left({}^{RL}D^{\mu}_{0+,\beta}\frac{1}{\Gamma(\alpha k+\beta)}\right)(\beta) = \frac{1}{2\pi i}\int\limits_{Ha_-} e^s s^{-\alpha k-\beta}(-\ln s)^{\mu-1}\,ds = c_k(\alpha,\beta).$$

Final results is presented in the form of series:

$$\left({}^{RL}D^{\mu}_{0+,\alpha}E_{\alpha,\beta}\right)(\alpha) = \sum_{k=0}^{\infty} k^{\mu}c_k(\alpha,\beta)z^k,$$

$$\left({}^{RL}D^{\mu}_{0+,\beta}E_{\alpha,\beta}\right)(\alpha) = \sum_{k=0}^{\infty} c_k(\alpha,\beta)z^k.$$

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References

1. *Rogosin S. V., Giraldi F., Mainardi F.* On differentiation with respect to parameters of the functions of the Mittag-Leffler type // arXiv: 2408.05225.

2. *Gal C. G, Gal S. G., Goldstein J. A.* Evolution Equations With A Complex Spatial Variable. Singapore, 2014.

3. *Gorenflo R., Kilbas A., Mainardi F., Rogosin S.* Mittag-Leffler Functions, Related Topics and Applications. Berlin, 2020.