ON CALCULATING THE MOMENTS OF SOLUTIONS TO ONE CLASS OF LINEAR SKOROHOD SDE ON POISSON SPACE

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The exact solutions of a class of linear Skorohod SDE driven by Poisson process with first-order chaos in coefficients and initial condition are received. Application to evaluation of moments of solution are considered.

Keywords: stochastic differential equation of Skorokhod; Poisson process; Skorokhod integral; solution; moments; numerical results.

О ВЫЧИСЛЕНИИ МОМЕНТОВ РЕШЕНИЙ В ОДНОМ КЛАССЕ ЛИНЕЙНЫХ СТОХАСТИЧЕСКИХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ СКОРОХОДА НА ПРОСТРАНСТВЕ ПУАССОНА

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Получены точные решения в классе линейных стохастических дифференциальных уравнений типа Скорохода, порождаемых процессом Пуассона с хаосом первого порядка в коэффициентах и начальных условиях. Рассмотрены применения для оценивания моментов решений.

Ключевые слова: стохастическое дифференциальное уравнение Скорохода; пуассоновский процесс; интеграл Скорохода; решение; моменты; численные результаты.

1. Introduction

In [1]-[2] a solution of the linear stochastic differential equation of Skorokhod with random coefficients and initial condition with a leading Poisson process was received. However, it contains as an unknown parameter a family of transformations of the probabilistic space of the leading random process, determined by the solution of an integral stochastic equation. This report considers the special case when the solution to this integral equation can be found in an explicit form. The first three moments of the solution of the initial SDEs are evaluated and numerical example is concidered. An analogy with the case of linear Skorohod SDE with the leading Wiener process [3]-[4] is used.

2. Main results

In this article we consider the linear stochastic differential equation of Skorohod with random coefficients an initial condition and with a leading Poisson process within framework of the following model (anticipative stochastic calculus on Poisson space) adopted in [5]. The space B of sequences $\omega=(\omega_1,\omega_2,\ldots),\ \omega_k\in R,$ with norm $\|\omega\|_B=\sup\limits_{k\geq 1}\frac{|\omega_k|}{k}$ and probability measure P is considered as a probability space, such that the functionals $\tau_k: R\to R, \tau_k(\omega)=\omega_k,\ k\geq 1,$ are independent identically distributed exponential random variables. The Poisson process $N_t,\ t\in R,$ is specified as the $N_t=\sum\limits_{k\geq 0}1_{[T_k,\infty)}(t),\ T_k=\sum\limits_{i=1}^{k}\tau_i,\ k\geq 1,$ is the time of the k-th jump of the process. The space of smooth functionals $S=\left\{f_n(\tau_1,\ldots,\tau_n): f_n\in C_c^\infty\left(R_+^n\right)\right\},$ operator $D: L_2(B)\to L_2(B)\otimes H$ is defined by the extension of operator $DF=\left(\partial_k f_n(\tau_1,\ldots,\tau_n)\right),\ F\in S,\ \partial_k F_n(x_1,\ldots,x_n)=\frac{\partial}{\partial x_k}f_n(x_1,\ldots,x_n),$ where H is the space of square summable sequences, and operator $\tilde{D}: S\to L_2(B\times[0,1]),\ \ \tilde{D}F=-\sum\limits_{k=1}^{\infty}1_{[T_{k-1},T_k)}(t)\partial_k f_n(\tau_1\ldots,\tau_n)$ and its corresponding extension on $L_2(B)$. The Skorokhod integral is defined as operator $\delta: L^1(B\times[0,1])\to L^1(B),$ conjugate to operator $\tilde{D}: L_2(B)\to L^1(B\times[0,1])$. For the Skorokhod integral over a centered Poisson process, notation $\delta(F)=\int\limits_0^1 F_s\delta\tilde{N}_s$ is also used.

In [1]-[2] for equation

$$X_t=X_0-\int\limits_0^t\sigma_s X_s\delta\tilde{N}_s+\int\limits_0^t b_s X_s ds,\ t\in[0,1], \tag{1}$$

where σ_s, b_s are the anticipate random functions, $X_0\in L^\infty(B),$ a solution is obtained in the form

$$X_t=X_0(\phi_{0,t})\exp\left\{\int\limits_0^t\tilde{D}_s\sigma_s(\phi_{s,t})ds+\int\limits_0^t\sigma_s(\phi_{s,t})ds+\int\limits_0^t\sigma_s(\phi_{s,t})ds\right\}\times$$

$$\times\prod_{0\leq T_k\leq t}\left(1-\sigma_{T_k}\left(\phi_{T_k,t}\right)\right), \tag{2}$$

where transformation $\phi_{s,t} = \phi_{s,t}(\omega)$ is found as a solution to the integral stochastic equation

$$\phi_{s,t}(\omega) = \omega - \left(\int_s^t i_r(e_k)\sigma_r(\phi_{r,t}\omega)dr \right)_{k\geq 0}, \quad 0 \leq s < t \leq 1, \tag{3}$$

$i_r(u) = \sum_{k=1}^{\infty} u_k 1_{[T_{k-1},T_k)}(t)$, $u = (u_k)$, (e_k), $k = 1, 2, \ldots,$ – denote the canonical basis of $l_2(N)$.

In the general case, it is not possible to obtain an explicit solution to (3). The question of obtaining any approximations of (3) for their use in (1) has not yet been considered, due to the difficulty of obtaining satisfactory estimates of the accuracy suitable for use in approximating the solution in (2). Therefore, it is of interest and of some importance to obtain explicit solutions for specific coefficient functions of equation (1). Using an analogy with the linear Skorohod SDE with the leading Wiener process [3]-[4] our work examines special case of equation (3), for which the solution (2) of equation can be found explicitly

$$X_t(\omega) = X_0(\omega) - \int_0^t \left(a(s) + \lambda T_1(\omega)\right) X_s \delta\tilde{N}_s(\omega) + \int_0^t b_s X_s(\omega)ds, \; t \in [0,1], \tag{4}$$

where the derivative $a'(s) \in L_2[0,1]$. In this case the equation (3) have the form

$$\phi_{s,t}\omega = \omega - \left(\int_s^t i_r(e_k)\left(a(r) + \lambda T_1(\phi_{r,t}\omega)\right)dr \right)_{k\geq 1}, \quad 0 \leq s < t \leq 1,$$

or in coordinate form of the space B:

$$\left(\phi_{s,t}\omega\right)_k = \omega_k - \int_s^t 1_{[T_{k-1}(\omega),T_k(\omega))}(r)\left(a(r) + \lambda T_1\left(\phi_{r,t}\omega\right)_1\right)dr, \; k \geq 1, \, 0 \leq s < t \leq 1. \tag{5}$$

The solution of (5) for $k = 1$ is $\left(\phi_{s,t}\omega\right)_1 = \omega_1$ under $s \geq T_1(\omega)$ and

$$\left(\phi_{s,t}\omega\right)_1 = \left(\omega_1 - \int_s^{t\wedge T_1(\omega)} a(r)dr \right) \exp\left\{\lambda\left(s - t \wedge T_1(\omega)\right)\right\}$$

under $s < T_1(\omega)$. Substituting obtained $\left(\phi_{s,t}\omega\right)_1$ into (5) we get explicit expression for $\left(\phi_{s,t}\omega\right)_k$ $k \geq 2$.

In the special case $\sigma_s(\omega)=\nu+\lambda T_1(\omega)$, $\nu>0$, the solution of (5) is $(\phi_{s,t}\omega)_1=\omega_1$, if $s\ge T_1(\omega)$, and

$$\left(\phi_{s,t}\omega\right)_1=\left(\omega_1-\nu\left(t\wedge T_1(\omega)-s\right)\right)\exp\left\{\lambda\left(s-t\wedge T_1(\omega)\right)\right\},$$

if $s<T_1(\omega)$. So for $T_{k-1}(\omega)\le s<T_k$, $k\ge 2$,

$$\left(\phi_{s,t}\omega\right)_k=\omega_k-\nu\left(t\wedge T_k-T_{k-1}\right)-\left(\omega_1-\nu\left(t\wedge T_1(\omega)\right)\right)\times$$

$$\times\frac{1}{\lambda}\left(\exp\left\{\lambda\left(t\wedge T_k(\omega)\right)\right\}\exp\left\{\lambda T_{k-1}(\omega)\right\}\right)\exp\left\{-\lambda\left(t\wedge T_1(\omega)\right)\right\}\times$$

$$\times\nu\exp\left\{\lambda\left(t\wedge T_k(\omega)\right)\right\}\left(\left(T_k(\omega)-1\right)e^{\lambda T_k(\omega)}-\left(T_{k-1}(\omega)-1\right)e^{\lambda T_{k-1}(\omega)}\right).$$

Numerical results are received for $\sigma_s(\omega)=\lambda T_1(\omega)$. In this case $\left(\phi_{s,t}\omega\right)_1=\exp\left\{\lambda\left(s-t\wedge T_1(\omega)\right)\right\}$ and

$$\left(\phi_{s,t}\omega\right)_k=\omega_k-\omega_1 e^{-t\wedge T_1(\omega)}\frac{1}{\lambda}\left(e^{-\lambda\left(t\wedge T_k(\omega)\right)}-e^{-\lambda\left(s\wedge T_{k-1}(\omega)\right)}\right),\ k\ge 2,$$

for $s<T_1(\omega)$. For $s\ge T_1(\omega)$ we have $\left(\phi_{s,t}\omega\right)_1=\omega_1$ and $\left(\phi_{s,t}\omega\right)_k=0$, if $s\ge T_1(\omega)$.

$$\left(\phi_{s,t}\omega\right)_k=\omega_k-\omega_1\left(t\wedge T_k(\omega)-s\vee T_{k-1}(\omega)\right),k\ge 2.$$

Further,

$$\tilde{D}_s\sigma_s\left(\phi_{s,t}\omega\right)=-1_{[0,T_1)}e^{\lambda(s-t\wedge\omega_1)}\left(1-\lambda\omega_1\partial_1\left(t\wedge\omega_1\right)\right),$$

if $s<T_1(\omega)$ and $\tilde{D}_s\sigma_s\left(\phi_{s,t}\omega\right)=0$, if $s\ge T_1(\omega)$.

At last, $X_0\left(\phi_{0,t}\right)=\left(\phi_{0,t}\omega\right)_1=\omega_1\exp\left\{-\lambda\left(t\wedge T_1(\omega)\right)\right\}$,

$$\prod_{0\le T_k\le t}\left(1-\sigma_{T_k}\left(\phi_{T_k,t}\right)\right)=\prod_{k=1}^{N_t}\left(1-\sigma_{T_k}\left(\phi_{T_k,t}\right)\right)=\prod_{k=1}^{N_t}\left(1-\lambda\left(\phi_{T_k,t}\right)_1\right)=\prod_{k=1}^{N_t}\left(1-\lambda\omega_1\right),$$

and

$$X_t=\omega_1\exp\left\{-\lambda\left(t\wedge\omega_1\right)\right\}\exp\left\{-\left(e^{\omega_1}-1\right)\left(1-\lambda\omega_1\partial_1\left(t\wedge\omega_1\right)\right)+\right.$$

$$\left.+\frac{1}{\lambda}\omega_1\exp\left\{-\lambda\left(t\wedge\omega_1\right)\right\}\left(e^{\omega_1}-1\right)\right\}\left(1-\lambda\omega_1\right)^{N_t}.$$

Received formulas are used for evaluation of moments of the solutions, and some numerical are presented.

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