ON TENSORS AND MULTIDIMENSIONAL MATRICES IN MULTIDIMENSIONAL PROBABILISTIC MODELING

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The analysis of the general concepts both of the tensor and multidimensional-matrix approaches in probabilistic modeling is performed and the relationships between the tensor and the multidimensional matrix are clarified. The matrix representation of the second order tensor definition known in the literature is generalized to the arbitrary order tensors. The theorems on the tensor nature of the covariance matrix and the multidimensional matrix of the arbitrary order moments of the random vector are proved. The theorem on the orthogonality of the transformation matrix of the arbitrary order multidimensional-matrix moment of the random vector provided the orthogonality of the random vector transformation is proved.

Keywords: multidimensional probabilistic modeling; linear vector space; tensor; multidimensional matrix; multidimensional-matrix representation of tensor; multidimensional-matrix probabilistic moments.

О ТЕНЗОРАХ И МНОГОМЕРНЫХ МАТРИЦАХ ВО МНОГОМЕРНОМ ВЕРОЯТНОСТНОМ МОДЕЛИРОВАНИИ

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Проведен анализ общих концепций как тензорного, так и многомерно-матричного подходов в вероятностном моделировании, а также установлены соотношения между тензорами и многомерными матрицами. Матричное представление определения тензоров второго порядка, известное в литературе, обобщено для тензоров произвольного порядка. Доказаны теоремы о происхождении тензоров для ковариационных матриц и многомерных матриц моментов произвольного порядка случайных векторов. Доказана теорема об ортогональности матрицы преобразования для многомерного матричного момента произвольного порядка случайного вектора при условии ортогональности преобразования случайного вектора.

Ключевые слова: многомерное вероятностное моделирование; линейное векторное пространство; тензор; многомерная матрица; многомерно-матричное представление тензора; многомерно-матричные вероятностные моменты.

Introduction

The distinct non-classical approaches are used today in multidimensional probabilistic modeling, such as matrix and multidimensional-matrix and, to a lesser extent, tensor and multiway approaches. In this article, the quite minor analysis of the multidimensional-matrix and tensor approaches is performed in order to reasonably compare their capabilities in multidimensional probabilistic modeling.

The matrix and tensor approaches are based on two independent areas of knowledge: the tensor analysis [1–8] and the matrix analysis [8–12]. The tensor analysis is developed for the tensors of the arbitrary order. The matrix analysis is limited to the two-dimensional matrices. The situation in the generalization of the two-dimensional matrices to the multidimensional case is as follows: on the one hand, the very well fundamentals of the theory of the multidimensional matrices there exist [13–15], on the other hand, the searches for other approaches to its development are known. These approaches assume the continuation of the development to the mathematical completion, in some cases [16–19], or lead the theory of the multidimensional matrix into the tensor analysis, in other cases [20–21].

The notions of a tensor and a matrix are clearly distinguished [8, 9]. It would be a confusion of concepts to identify a matrix with a tensor [6]. However, the situation is somewhat different in the literature related to the data analysis. Sometimes, the notion tensor is attracted to the multidimensional data analysis [20, 21]. So, in [20] it is noted that the multidimensional matrices and tensors are convenient mathematical tool for such an analysis, and a tensor is used instead of a multidimensional matrix. We will call this approach as the tensor approach. The tensor approach is reduced to accepting a tensor as a multidimensional matrix without taking into account the definition and properties of a tensor. We find in [21] that tensors are multidimensional generalizations of matrices. The illegality of such an approach is noted shortly in [15]. We want to emphasize by this article that the generalization of the matrix to the multidimensional case should be performed in the matrix analysis but not in the tensor analysis. The two-dimensional (usual) matrix should be the natural particular case of the multidimensional matrix. We shell consider the some questions of the tensor theory and multidimensional matrix theory and state the relationships between tensors and multidimensional matrices to achieve our goal.

1. Transformations of the coordinate systems

Tensor is an object in the linear finite-dimensional space. Linear n-dimensional space is defined by a set of n linearly independent elements (vectors) $e_2, e_2, ..., e_n$. This set is called the basis e_i, $i = 1,2,...,n$, of the n-dimensional space. Each point x in the n-dimensional space is represented in the following form:

$$x = x^1 e_1 + x^2 e_2 + ... + x^n e_n = \sum_{i=1}^{n} x^i e_i \,, \tag{1}$$

where $x^1, x^2, ..., x^n$ are real numbers which are called the coordinates of the point x. We will call x as the position vector of the point or simply vector x. The expression (1) is called the expansion of the vector x by the basis e_i. We will use the term "coordinate system x^i with the basis e_i" or simple "coordinate system x^i" along with the term basis e_i.

In tensor analysis, the so called Einstein summation convention is used: if an index is repeated in some term of the expression then the term must be summed with respect to that index for all admissible values of the index. For example, $x^i e_i$ is written instead of $\sum_{i=1}^{n} x^i e_i$, and $b_j = x^i_j e_i$ means the equality $b_j = \sum_{i=1}^{n} x^i_j e_i$.

The tensor definition is inextricably connected with the transformation of the basis (coordinate system). Let x^i be the initial coordinate system with the initial basis e_i and x^{*i} be the new coordinate system with the new basis e_{*i}. The reciprocal bases are introduced along with the initial bases: e^i (with the coordinate system x_i) is the reciprocal to the initial basis e_i and e^{*i} (with the coordinate system x_{*i}) is the reciprocal to the new basis e_{*i}. We will call the basis e^i reciprocal to the initial basis e_i as the initial reciprocal basis e^i and the basis e^{*i} reciprocal to the new basis e_{*i} as the new reciprocal basis e^{*i}. The reciprocal bases are the bases which are orthogonal to their caused bases, i.e. the following equalities hold

$$(e_i, e^j) = \delta_i{}^j = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \quad i, j = 1,2,...,n, \tag{2}$$

$$(e_{*i}, e^{*j}) = \delta_{*i}{}^{*j} = \begin{cases} 1, & *i = *j, \\ 0, & *i \neq *j, \end{cases} \quad i, j, *i, *j = 1,2,...,n, \tag{3}$$

where (e_i, e^j), (e_{*i}, e^{*j}) are the dot products of the vectors and $\delta_i{}^j$, $\delta_{*i}{}^{*j}$ is the Kronecker delta.

The notations of the bases and vector coordinates are given in the table.

The notations of the bases and vector coordinates

	Initial basis, coordinates	New basis, coordinates
	e_i, x^i – contravariant coordinates	e_{*i}, x^{*i}
Reciprocal basis, coordinates	e^i, x_i – covariant coordinates	e^{*i}, x_{*i}

The position vector of the point with respect to two bases is given by the expression

$$x = e_{*i} x^{*i} = e_j x^j . \qquad (4)$$

We find from (4) that

$$x^{*i} = (e^{*i}, e_j) x^j = \alpha^{*i}{}_j x^j , \qquad (5)$$

where

$$\alpha^{*i}{}_j = (e^{*i}, e_j) . \qquad (6)$$

The equality (5) defines the transformation of the initial coordinate system x^i to the new coordinate system x^{*i} as determined by the bases e_i, and e_{*i}. We find the expression (5) by taking the dot product of both sides of (4) with the new reciprocal basis e^{*i} and taking into account the equality (3): $(e^{*i}, (e_{*i} x^{*i})) = (e^{*i}, (e_j x^j))$, $(e^{*i}, e_{*i}) x^{*i} = (e^{*i}, e_j) x^j$, $x^{*i} = (e^{*i}, e_j) x^j$.

We also find from the equality $e_i x^i = e_{*j} x^{*j}$ that

$$x^i = (e^i, e_{*j}) x^{*j} = \alpha^i{}_{*j} x^{*j} , \qquad (7)$$

where

$$\alpha^i{}_{*j} = (e^i, e_{*j}) . \qquad (8)$$

The equality (7) defines the transformation of the new coordinate system x^{*i} to the initial coordinate system x^i. We find the expression (7) by taking the dot product of both sides of the equality $e_i x^i = e_{*j} x^{*j}$ with initial reciprocal basis e^i and taking into account the equality (2): $(e^i,(e_i x^i)) = (e^i,(e_{*j} x^{*j}))$, $(e^i, e_i) x^i = (e^i, e_{*j}) x^{*j}$, $x^i = (e^i, e_{*j}) x^{*j}$.

It is clear that the transformations $\alpha^{*i}{}_j = (e^{*i}, e_j)$ (6) and $\alpha^i{}_{*j} = (e^i, e_{*j})$ (8) are mutually inverse.

The expressions (5), (6), (7), (8) can be represented in the vector-matrix form. If one introduces the row-vectors $X^T = (x^1, x^2, ..., x^n)$, $X^{*T} = (x^{*1}, x^{*2}, ..., x^{*n})$ and the matrix $\Lambda^* = (\lambda^*_{i,j}) = (\alpha^{*i}{}_j)$ with elements $\alpha^{*i}{}_j = (e^{*i}, e_j)$ (6), then one gets instead of (5)

$$X^* = \Lambda^* X , \qquad (9)$$

where

$$\Lambda^* = (\lambda^*_{i,j}) = (\alpha^{*i}{}_j) = ((e^{*i}, e_j)) . \qquad (10)$$

If one introduces the matrix $\Lambda_* = (\lambda_{*,i,j}) = (\alpha^i{}_{*j})$ with elements $\alpha^i{}_{*j} = (e^i, e_{*j})$ (8) then one gets instead of (7)

$$X = \Lambda_* X^* ,$$

where

$$\Lambda_* = (\lambda_{*,i,j}) = (\alpha^i{}_{*j}) = ((e^i, e_{*j})) . \qquad (11)$$

It is clear that $(\Lambda^*)^{-1} = \Lambda_*$, $\Lambda_*^{-1} = \Lambda^*$, i.e.

$$\Lambda^* \Lambda_* = \Lambda_* \Lambda^* = I . \qquad (12)$$

The property (12) in element form looks like this

$$\alpha^{*i}_j \alpha^j_{*k} = \delta^i_k .$$

It is clear, that if $\Lambda^* = (\alpha^{*i}{}_j)$, then $\Lambda^{*T} = (\alpha_i{}^{*j})$, and if $\Lambda_* = (\alpha^i{}_{*j})$, then $\Lambda_*^T = (\alpha_{*i}{}^j)$. The mutually inverse transformations are following: $\alpha^{*i}{}_j$ and $\alpha^i{}_{*j}$, $\alpha_i{}^{*j}$ and $\alpha_{*i}{}^j$.

2. Transformations of the bases

The following transformation of the initial reciprocal basis e^j to the new reciprocal basis e^{*i} follows from (6):

$$e^{*i} = \alpha^{*i}{}_j e^j. \tag{13}$$

We get the expression (13) by multiplying the both sides of (6) by e^j and taking into account the equality (2): $\alpha^{*i}{}_j e^j = (e^{*i}, e_j)e^j$, $\alpha^{*i}{}_j e^j = e^{*i}$.

The following transformation of the new basis e_{*i} to initial basis e_i follows from the equality $\alpha_i{}^{*j} = (e_i, e^{*j})$:

$$e_i = \alpha_i{}^{*j} e_{*j}. \tag{14}$$

We get the expression (14) by multiplying the both sides of the equality $\alpha_i{}^{*j} = (e_i, e^{*j})$ by e_{*j} and taking into account the property (3): $\alpha_i{}^{*j} e_{*j} = (e_i, e^{*j})e_{*j}$, $\alpha_i{}^{*j} e_{*j} = e_i$.

We write also the following transformations as the inverse to the transformations (13), (14) respectively: the transformation of the new reciprocal basis e^{*i} to the initial reciprocal basis e^j:

$$e^i = \alpha^i{}_{*j} e^{*j},$$

and the transformation of the initial basis e_i to the new basis e_{*i}:

$$e_{*i} = \alpha_{*i}{}^j e_j. \tag{15}$$

3. Transformations of the vectors. Covariant and contravariant components

Any vector a in n-dimensional space can be represented by different expansion, for instance, by the initial reciprocal basis e^i and by the new reciprocal basis e^{*i},

$$a = e^j a_j = e^{*i} a_{*i}, \tag{16}$$

or by the initial basis e_i and by the new basis e_{*i}

$$a = e_j a^j = e_{*i} a^{*i}. \tag{17}$$

The components a_i of the vector a in the initial reciprocal basis e^i are called the *covariant* components of the vector a, and the components a^i of the vector a in the initial basis e_i are called the *contravariant* components of the vector a.

If we take the dot product of e_{*i} with both sides of (16) noting that $(e_{*i}, e^{*j}) = \delta_{*i}{}^{*j}$, we find the transformation of the initial reciprocal components a_i to the new reciprocal components e_{*i}, i.e. the transformation of the covariant components:

$$a_{*i} = (e_{*i}, e^j) a_j = \alpha_{*i}{}^j a_j. \tag{18}$$

The relationship (18) has the same form as the relationship (15) of the initial basis e_i to the new basis e_{*i}. Thus, the initial reciprocal components a_i transform to the new reciprocal components a_{*i} in the same fashion as the initial basis vectors e_i transform to the new basis vectors e_{*i}, and for this reason they are called *covariant* components [5].

Similarly by taking the dot product of both sides of (17) with e^{*i}, we get the transformation of the initial components a^i to the new components a^{*i}, i.e. the transformation of the contravariant components:

$$a^{*i} = (e^{*i}, e_j) a^j = \alpha^{*i}{}_j a^j. \tag{19}$$

The transformation (19) has the same form as the transformation (13) of the initial reciprocal basis e^j to the new reciprocal basis e^{*i}. Thus, the initial components a^i transform to the new components a^{*i} in the opposite fashion as the initial basis vectors e_i transform to the new basis vectors e_{*i}. Accordingly, the components a^i are called contravariant components of the vector.

4. The case of the orthogonal bases

If the initial basis e_i is orthogonal, then the initial reciprocal basis e^i is the same as the e_i [5], i.e. $e^i = e_i$. If the new basis e_{*i} is orthogonal too, then the new reciprocal basis e^{*i} is the same as the e_{*i}, i.e. $e^{*i} = e_{*i}$. In this case, the elements of the transition from the new coordinate system x^{*i} to the initial coordinate system x^i satisfy the equalities (see (8))

$$\alpha^i{}_{*j} = (e^i, e_{*j}) = (e_i, e_{*j}) = (e_i, e^{*j}) = (e^i, e^{*j}), \qquad (20)$$

and the elements of the transition from the initial coordinate system x^i to the new coordinate system x^{*i} satisfy the equalities (see (6))

$$\alpha^{*i}{}_j = (e^{*i}, e_j) = (e_{*i}, e_j) = (e_{*i}, e^j) = (e^{*i}, e^j). \qquad (21)$$

Note that the elements $\alpha^i{}_{*j}$ (20) and $\alpha^{*i}{}_j$ (21) represent the matrices Λ_* (11) and Λ^* (10) respectively. Comparing the expressions (20), (21) shows that $\Lambda^* = \Lambda_*^T$. Since $\Lambda^*\Lambda_* = I$ then $\Lambda_*\Lambda_*^T = I$ and $\Lambda_*^T = \Lambda_*^{-1}$. This means that the matrices Λ^* and Λ_* are orthogonal. The orthogonality property of the matrix Λ^* in tensor notation looks like this:

$$\alpha^{*i}{}_j \alpha^{*i}{}_k = \delta_{j,k}. \qquad (22)$$

5. Definition of a tensor

Definition of a tensor [5]. A tensor a of the order $p = r + s$ of the type (r, s) (r time covariant and s time contravariant) is the geometrical object which

1) is defined by n^{r+s} components $a^{k_1,\ldots,k_s}_{j_1,\ldots,j_r}$ in the initial basis e_i, $i = 1,2,\ldots,n$, of the real n-dimensional linear space L^n,

2) has such a property that its components $\bar{a}^{*l_1,\ldots,*l_s}_{*i_1,\ldots,*i_r}$ in the new basis e_{*i}, $*i = 1,2,\ldots,n$, are connected with the components $a^{k_1,\ldots,k_s}_{j_1,\ldots,j_r}$ in the initial basis e_i by the relations

$$\bar{a}^{*l_1,\ldots,*l_s}_{*i_1,\ldots,*i_r} = \alpha_{*i_1}{}^{j_1} \cdots \alpha_{*i_r}{}^{j_r} \alpha^{*l_1}{}_{k_1} \cdots \alpha^{*l_s}{}_{k_s} a^{k_1,\ldots,k_s}_{j_1,\ldots,j_r}, \qquad (23)$$

in which $\alpha_{*i}{}^{j}$ are elements of the transition from the initial basis e_i to the new basis e_{*i}, and $\alpha^{*l}{}_{k}$ are the elements of the inverse transition from the initial reciprocal basis e^{j} to the new reciprocal basis e^{*i}.

Note that the components $a^{k_1,\dots,k_s}_{j_1,\dots,j_r}$ of a tensor are the functions of the coordinates of the coordinate system in which they are considered. If a tensor is considered in the initial coordinate system x^i then its components are the functions of the variables x^i: $a^{k_1,\dots,k_s}_{j_1,\dots,j_r} = a^{k_1,\dots,k_s}_{j_1,\dots,j_r}(x^1, x^2 \dots, x^n)$. The components of the tensor in a new coordinate system x^{*i} are the functions of the variables x^{*i}: $\bar{a}^{*l_1,\dots,*l_s}_{*i_1,\dots,*i_r} = \bar{a}^{*l_1,\dots,*l_s}_{*i_1,\dots,*i_r}(x^{*i}, x^{*2},\dots, x^{*n})$. The definition (23) means that the equalities (23) hold for all values of the variables $x^1, x^2,\dots, x^n$ provided the coordinate system transformation.

If we suppose in the definition (23) $s = 0$ then we receive the following definition of the covariant tensor of the order r:

$$\bar{a}_{*i_1,\dots,*i_r} = \alpha_{*i_1}{}^{j_1} \cdots \alpha_{*i_r}{}^{j_r} a_{j_1,\dots,j_r}\,. \tag{24}$$

Supposing in the definition (23) $r = 0$ give the following definition of the contravariant tensor of the order s:

$$\bar{a}^{*l_1,\dots,*l_s} = \alpha^{*l_1}{}_{k_1} \cdots \alpha^{*l_s}{}_{k_s} a^{k_1,\dots,k_s}\,. \tag{25}$$

The separate definition is applied for the order zero tensor [7].

Definition of a tensor of the order zero. A tensor a of the order zero (a scalar) is the geometrical object which is defined in the initial coordinate system x^i by the scalar function $a(x^1, x^2,\dots, x^n)$ and in the new coordinate system x^{*i} by the scalar function $\bar{a}(x^{*1}, x^{*2},\dots, x^{*n})$ connected with the function $a(x^1, x^2,\dots, x^n)$ by the equality $\bar{a} = a$ for each point of the space.

Definition of the outer product of the tensors. The outer product of two tensors $a^{i_1,\dots,i_r}_{i'_1,\dots,i'_s}$ and $b^{k_1,\dots,k_p}_{k'_1,\dots,k'_q}$ is the tensor which defined by the following expression:

$$c^{i_1,\dots,i_r,k_1,\dots,k_p}_{i'_1,\dots,i'_s,k'_1,\dots,k'_q} = a^{i_1,\dots,i_r}_{i'_1,\dots,i'_s} b^{k_1,\dots,k_p}_{k'_1,\dots,k'_q}\,.$$

Definition of the inner product of the tensors. The inner product of two tensors is a contraction of the outer product with respect to two indices, each belonging to a component of the tensors.

Example 1. The simple example of the tensor of the order zero is the Euclidean distance in the Euclidean space with the orthogonal initial basis e_i and the orthogonal new basis e_{*i}. Indeed, let $(x^1, x^2, ..., x^n)$ and $(x'^1, x'^2, ..., x'^n)$ be two points in pointed space. Euclidean distance between these two points is defined by the formula (in tensor notation)

$$d_x^2 = (x^i - x'^i)(x^i - x'^i).$$

The new coordinates x^{*i} of the points in the new basis e_{*i} are defined as follows (see (5)):

$$x^{*i} = \alpha^{*i}{}_j x^j, \; x'^{*i} = \alpha^{*i}{}_j x'^j.$$

Therefore, we receive in the new coordinate system

$$d_{x^*}^2 = (x^{*i} - x'^{*i})(x^{*i} - x'^{*i}) = \alpha^{*i}{}_j (x^j - x'^j)\alpha^{*i}{}_k (x^k - x'^k) =$$

$$= \alpha^{*i}{}_j \alpha^{*i}{}_k (x^j - x'^j)(x^k - x'k).$$

Since $\alpha^{*i}{}_j \alpha^{*i}{}_k = \delta_{j,k}$ (see (22)) then

$$d_{x^*}^2 = \delta_{j,k}(x^j - x'^j)(x^k - x'^k) = (x^j - x'^j)(x^k - x'^k) = d_x^2.$$

We have the equality $d_{x^*}^2 = d_x^2$, so the Euclidean distance between two points is the tensor of the order zero in accordance with the definition of a tensor of the order zero.

Example 2. The vector is the tensor of the order one. Indeed, the transformation (18) of the covariant components of a vector is the definition (24) provided $r = 1$.

Example 3. The simple example of the tensor of the order two is the outer product of the two vectors. Let $(a^1, ..., a^n)$ and $(b^1, ..., b^n)$ be two vectors in the initial coordinate system x^i (with the basis e_i). The quantities $a^{i,j} = a^i b^j$ are the elements of the so called outer product of these vectors. The new components of the vectors in the new basis e_{*i} are defined by the formulae $a^{*i} = \alpha^{*i}{}_k a^k$, $b^{*j} = \alpha^{*j}{}_l b^l$. Then we have the following string:

$$a^{*i} b^{*j} = \bar{a}^{*i,*j} = \alpha^{*i}{}_k a^k \alpha^{*j}{}_l b^l = \alpha^{*i}{}_k \alpha^{*j}{}_l a^k b^l = \alpha^{*i}{}_k \alpha^{*j}{}_l a^{k,l}.$$

This expression is the definition (25) of the contravariant tensor of the order $s = 2$.

6. Multidimensional matrices

Definition of a multidimensional matrix. A multidimensional (p-dimensional) matrix is a system of numbers or variables $a_{i_1,i_2,...,i_p}$, $i_\alpha = 1,2,...,n_\alpha$, $\alpha = 1,2,...,p$, located at the points of the p-dimensional space defined by the coordinates $i_1, i_2,..., i_p$.

The p-dimensional matrix is denoted as

$$A = (a_{i_1,i_2,...,i_p}),\ i_\alpha = 1,2,...,n_\alpha,\ \alpha = 1,2,...,p, \qquad (26)$$

or $A = (a_{\bar{i}})$, where $\bar{i} = (i_1, i_2,..., i_p)$ is a multi-index, $i_\alpha = 1,2,...,n_\alpha$, $\alpha = 1,2,..., p$.

If $n_1 = n_2 = ...n_p = n$, then the matrix (26) is called a p-dimensional matrix of the order n (a hyper-square matrix). In this connection, the matrix (26) with distinct $n_1, n_2,...n_p$ could be called a hyper-rectangular matrix.

Thus, a zero-dimensional matrix is a scalar, a one-dimensional matrix is a vector and a two-dimensional matrix is an ordinary matrix in traditional notation.

Any p-dimensional matrix $A = (a_{i_1,i_2,...,i_p})$ can be represented in the form $A = (a_{l,s,c})$, where $l = (l_1, l_2,..., l_\kappa)$, $s = (s_1, s_2,..., s_\lambda)$, $c = (c_1,..., c_\mu)$ are multi-indexes, $\kappa + \lambda + \mu = p$. We will say that the matrix A has (κ, λ, μ)-structure and denote it $A_{(\kappa,\lambda,\mu)}$.

Transpose of a multidimensional matrix. The matrix $A^T = (a^T_{i_1,i_2,...,i_p})$ the elements $a^T_{i_1,i_2,...,i_p}$ of which are connected with the elements $a_{i_1,i_2,...,i_p}$ of the matrix $A = (a_{i_1,i_2,...,i_p})$ by the equalities

$$a^T_{i_1,i_2,...,i_p} = a_{i_{\alpha_1}, i_{\alpha_2},..., i_{\alpha_p}}, \qquad (27)$$

where $i_{\alpha_1}, i_{\alpha_2}, ..., i_{\alpha_p}$ is some permutation of the indices $i_1, i_2, ..., i_p$ is called the transposed according to the substitution

$$T = \begin{pmatrix} i_1, ..., & i_p \\ i_{\alpha_1}, ..., i_{\alpha_p} \end{pmatrix}$$

matrix A. For example, let

$$A = (a_{i,j,k}) = \begin{pmatrix} \overbrace{x}^{1,1,1} & \overbrace{2}^{1,1,2} & \overbrace{y}^{1,2,1} & \overbrace{xy}^{1,2,2} \\ \overbrace{7}^{2,1,1} & \overbrace{\beta}^{2,1,2} & \overbrace{z}^{2,2,1} & \overbrace{t}^{2,2,2} \end{pmatrix}$$

and $T = \begin{pmatrix} i, j, k \\ k, j, i \end{pmatrix}$. Then, in accordance with the formula (27), $a^T_{i,j,k} = a_{k,j,i}$, and

$$A^T = (a^T_{i,j,k}) = \begin{pmatrix} \overbrace{x}^{1,1,1} & \overbrace{7}^{1,1,2} & \overbrace{y}^{1,2,1} & \overbrace{z}^{1,2,2} \\ \overbrace{2}^{2,1,1} & \overbrace{\beta}^{2,1,2} & \overbrace{xy}^{2,2,1} & \overbrace{t}^{2,2,2} \end{pmatrix}.$$

Note, that the Matlab function ipermute.m performs a transpose of a multidimensional array in accordance with the definition (27).

The some standard substitutions are introduced in the work [15] which allow us to form various substitutions. They are substitutions of the types ‘onward’, ‘back’, ‘onward-back’.

The substitution on the p indices the lower string of which is formed from the upper string by the transfer of the r left indices to the right (onward) is called substitution of the type ‘onward’ and is denoted $B_{p,r}$ or simple B_r:

$$B_{p,r} = \begin{pmatrix} i_1, & i_2, & ..., & i_{p-r}, & i_{p-r+1}, & ..., & i_p \\ i_{r+1}, & i_{r+2}, & ..., & i_p, & i_1, & ..., & i_r \end{pmatrix}, \; p \geq r. \qquad (28)$$

The substitution on the p indices the lower string of which is formed from the upper string by the transfer of the r right indices to the left (back) is called substitution of the type ‘back’ and is denoted $H_{p,r}$ or simple H_r:

$$H_{p,r} = \begin{pmatrix} i_1, & i_2, & \dots, & i_r, & i_{r+1}, & \dots, & i_p \\ i_{p-r+1}, & i_{p-r+2}, & \dots, & i_p, & i_1, & \dots, & i_{p-r} \end{pmatrix}, \ p \geq r.$$

The substitution on the p indices the lower string of which is form from the upper string by the transfer of the r left indices to the right (onward) and the s right indices to the left (back) is called substitution of the type 'onward-back' and is denoted $B_r H_s$:

$$B_r H_s = \begin{pmatrix} i_1, & \dots, & i_r, & \dots, & i_{p-s+1}, & \dots, & i_p \\ i_{p-s+1}, & \dots, & i_p, & \dots, & i_1, & \dots, & i_r \end{pmatrix}, \ p \geq r+s.$$

Multiplication of two multidimensional matrices. If a p-dimensional matrix A is represented in the form of $A = (a_{i_1,i_2,\dots,i_p}) = (a_{l,s,c})$, where $l = (l_1, l_2, \dots, l_\kappa)$, $s = (s_1, s_2, \dots, s_\lambda)$, $c = (c_1, \dots, c_\mu)$ are multi-indices, $\kappa + \lambda + \mu = p$, and a q-dimensional matrix B is represented in the form of $B = (b_{j_1, j_2, \dots, j_q}) = (b_{c,s,m})$, where $m = (m_1, \dots, m_\nu)$ is a multi-index, $\lambda + \mu + \nu = q$, then the matrix $D = (d_{l,s,m})$ is called a (λ, μ)-folded product of the matrices A and B, if its elements are defined by the expression

$$d_{l,s,m} = \sum_c a_{l,s,c} b_{c,s,m} = \sum_{c_1} \sum_{c_2} \cdots \sum_{c_\mu} a_{l,s,c} b_{c,s,m} .$$

The (λ, μ)-folded product of the matrices A and B is denoted ${}^{\lambda,\mu}(AB)$. Thus,

$$D = {}^{\lambda,\mu}(AB) = (\sum_c a_{l,s,c} b_{c,s,m}) = (d_{l,s,m}).$$

In the case of the $(0,0)$-folded product we often omit the left upper indices and write AB instead of ${}^{0,0}(AB)$.

In the general case ${}^{\lambda,\mu}(AB) \neq {}^{\lambda,\mu}(BA)$.

The associative law of multiplication of the multidimensional matrices holds:

$${}^{\lambda',\mu'}({}^{\lambda,\mu}(AB)C) = {}^{\lambda,\mu}(A\,{}^{\lambda',\mu'}(BC)).$$

The distributive law of multiplication of the multidimensional matrices is as follows:

$$^{\lambda,\mu}(A(B+C)) = {}^{\lambda,\mu}(AB) + {}^{\lambda,\mu}(AC).$$

Degree of multidimensional matrix. The matrix $D = {}^{\lambda,\mu}(AA) = {}^{\lambda,\mu}A^2$ is called a (λ,μ)-folded square of the matrix A, and the matrix $D = {}^{\lambda,\mu}(A\,{}^{\lambda,\mu}(A \cdots {}^{\lambda,\mu}(AA))) = {}^{\lambda,\mu}A^k$ is called a (λ,μ)-folded k-th degree of the matrix A. If it is $(0,0)$-folded k-th degree of the matrix A, then we omit the left upper indices and write A^k instead of ${}^{0,0}A^k$.

Identity multidimensional matrix. The matrix $E(\lambda,\mu)$ is called a (λ,μ)-identity matrix if the equalities

$$^{\lambda,\mu}(AE(\lambda,\mu)) = {}^{\lambda,\mu}(E(\lambda,\mu)A) = A$$

are satisfied for any multidimensional matrix A. The matrix $E(\lambda,\mu)$ is $(\lambda+2\mu)$-dimensional matrix whose elements are defined by the formula

$$E(\lambda,\mu) = (e_{c,s,m}) = \left(\begin{cases} 1, & if \quad c = m, \\ 0, & if \quad c \neq m \end{cases} \right),$$

$$c = (c_1,...,c_\mu),\ s = (s_1,...,s_\lambda),\ m = (m_1,...,m_\mu). \tag{29}$$

7. Multidimensional-matrix representation of tensor

It follows from (25) that the definition of the contravariant second order tensor has the form

$$\bar{a}^{*i_1,*i_2} = \alpha^{*i_1}{}_{j_1}\alpha^{*i_2}{}_{j_2} a^{j_1,j_2}. \tag{30}$$

It is convenient to express a second order tensor in form of a matrix [3]. It allows using the matrix notation in the operations with tensors. Introducing the matrix $\Lambda^* = (\lambda^*_{i,j}) = (\alpha^{*i}{}_j)$ (10) and the matrices of the second order tensors $a = (a^{j_1,j_2})$, $\bar{a} = (\bar{a}^{*i_1,*i_2})$ allows us to obtain the following form of the representation of the definition of the second order tensor (30) [3]:

$$\bar{a} = \Lambda^* a \Lambda^{*T}. \tag{31}$$

Indeed, we have the following transformations:

$$\bar{a}^{*i_1,*i_2} = \sum_{j_1,j_2} \lambda^*_{*i_1,j_1}\lambda^*_{*i_2,j_2} a^{j_1,j_2} = \sum_{j_1,j_2} \lambda^*_{*i_1,j_1} a^{j_1,j_2} \lambda^*_{*i_2,j_2} = \sum_{j_1,j_2} \lambda^*_{*i_1,j_1} a^{j_1,j_2} \lambda^{*T}_{j_2,*i_2}$$

The last expression has the matrix form (31). Since $\Lambda^* = \Lambda_*^{-1}$, then the inverse to the (31) transformation has the following matrix form:

$$a = \Lambda_* \bar{a} \Lambda_*^T, \tag{32}$$

where Λ_* is the matrix $\Lambda_* = (\lambda_{*,i,j}) = (\alpha^i{}_{*j})$ (11).

The matrix representation is more convenient for the visual perception and computer calculations since the matrix algebra is very good represented in all programming systems.

It is noted in [3] that the matrix notation fails for tensors of higher order. However, this statement is refuted below. We give below the generalization of the expression (32) for the arbitrary order tensor in the framework of the theory of the multidimensional matrices. Let us turn for this to the tensor definition (25) in the case of the arbitrary bases e_i, e_{*i} and introduce apart the two-dimensional matrix $\Lambda^* = (\lambda^*_{i,j}) = (\alpha^{*i}{}_j)$ (10) also the s-dimensional matrices $a = (a^{k_1,\dots,k_s})$, $\bar{a} = (\bar{a}^{*l_1,\dots,*l_s})$ of tensors. Then we can write the definition (25) in compliance with the summation convention in terms of these matrices:

$$\bar{a}^{*l_1,\dots,*l_s} = \lambda^*_{*l_1,k_1} \cdots \lambda^*_{*l_s,k_s} a^{k_1,\dots,k_s}.$$

If we use the summation sign then the last expression takes the following form:

$$\bar{a}^{*l_1,\dots,*l_s} = \sum_{k_1=1}^{n} \cdots \sum_{k_s=1}^{n} \lambda^*_{*l_1,k_1} \cdots \lambda^*_{*l_s,k_s} a^{k_1,\dots,k_s} =$$

$$= \sum_{k_1=1}^{n} \cdots \sum_{k_s=1}^{n} z_{*l_1,k_1,*l_2,k_2,\dots,*l_s k_s} a^{k_1,\dots,k_s}, \tag{33}$$

where we introduce the $2s$-dimensional matrix

$$z = (z_{*l_1,k_1,*l_2,k_2,\dots,*l_s k_s}) = (\lambda^*_{*l_1,k_1} \cdots \lambda^*_{*l_s,k_s}). \tag{34}$$

The matrix z (34) is the $(0,0)$-folded s-th degree of the matrix $\Lambda^* = (\lambda^*_{i,j})$: $z = {}^{0,0}(\Lambda^*)^s$. On the other hand, we can write the following equation along with the equation (33) by introducing the matrix $\Lambda = (\lambda_{*l_1,\dots,*l_s,k_1,\dots,k_s})$:

$$\bar{a}^{*l_1,\dots,*l_s} = \sum_{k_1=1}^{n} \cdots \sum_{k_s=1}^{n} \lambda_{*l_1,\dots,*l_s,k_1,\dots,k_s} a^{k_1,\dots,k_s} . \tag{35}$$

If

$$z_{l_1,k_1,l_2,k_2,\dots,l_s k_s} = \lambda_{*l_1,\dots,*l_s,k_1,\dots,k_s} \tag{36}$$

then the expressions (33), (35) are equivalent. Taking into account (34), we will have instead of (36):

$$\lambda_{*l_1,\dots,*l_s,k_1,\dots,k_s} = \lambda^*_{*l_1,k_1} \cdots \lambda^*_{*l_s,k_s} . \tag{37}$$

The equality (36) means, that the matrices z and Λ are connected by transpose operation, namely

$$z = \Lambda^{T_{2s}} , \tag{38}$$

where $T_{2s} = \begin{pmatrix} l_1, k_1, l_2, k_2, \dots, l_s, k_s \\ l_1, l_2 \dots, l_s, k_1, k_2 \dots, k_s \end{pmatrix}$ is the transpose substitution on the $2s$ indexes, in which we use the index l instead of the index $*l$. In rank form this substitution is defined by following expression:

$$T_{2s} = \begin{pmatrix} 1,2,3,\dots,s, & s+1, s+2, \dots, 2s-1, 2s \\ 2,4,6,\dots,2s, & 1, \quad 3, \quad \dots, 2s-3, 2s-1 \end{pmatrix}. \tag{39}$$

It is follows from (38) that

$$\Lambda = z^{T_{2s}^{-1}} = ((\Lambda^*)^s)^{T_{2s}^{-1}} , \tag{40}$$

where T_{2s}^{-1} is substitution inverse to the substitution T_{2s}.

Thus, we received the following form for representation the tensor definition (25):

$$\bar{a} = {}^{0,s}(\Lambda a) = {}^{0,s}\left(((\Lambda^*)^s)^{T_{2s}^{-1}} a\right), \tag{41}$$

where Λ is the matrix (40), z is defined by the formula (34), $\Lambda^* = (\lambda^*_{i,j})$ is the matrix (10), T_{2s} is the transpose substitution (39), ${}^{0,s}(\Lambda a)$ is the $(0,s)$-folded product of the matrices Λ and a [15].

The known expression (32) is the particular case of the expression (41) provided $s=2$. We can write the following expression instead of (32):

$$\bar{a}={}^{0,2}(((\Lambda^*)^2)^{T_4^{-1}}a),$$

where $T_4=\begin{pmatrix}1,2,3,4\\2,4,1,3\end{pmatrix}$, $T_4^{-1}=\begin{pmatrix}1,2,3,4\\3,1,4,2\end{pmatrix}$.

8. Probabilistic applications

Let us prove the theorems related to the probabilistic applications.

The linear transformation of the random vector which reduces its covariance matrix to the diagonal form is considered in the principal components method [22]. The following theorem applies to such a transformation.

Theorem 1. The elements $R_{\xi,i,j}$ of the covariance matrix $R_\xi=(R_{\xi,i,j})$ of the random vector $\xi^T=(\xi^1,...,\xi^n)$ can be considered as the components of the second order tensor.

Proof. The covariance matrix of the random vector $\xi^T=(\xi^1,...,\xi^n)$ is defined in the initial basis e_i by the expression $R_\xi=E(\dot{\xi}\dot{\xi}^T)$, where $\dot{\xi}$ is the centered random vector, and E means the mathematical expectation. If we introduce the linear transformation $\dot{\eta}=\Lambda^*\dot{\xi}$ (9) with the transformation matrix Λ^* (10), then we get for the covariance matrix of the random vector η:

$$R_\eta=E(\dot{\eta}\dot{\eta}^T)=E\left(\Lambda^*\dot{\xi}(\Lambda^*\dot{\xi})^T\right)=E\left(\Lambda^*\dot{\xi}\dot{\xi}^T\Lambda^{*T}\right)=\Lambda^*R_\xi\Lambda^{*T}.$$

We can see that the covariance matrix is transformed in accordance with the transformation (31) of the second order tensor. Thus, the elements $R_{\xi,i,j}$ of the covariance matrix $R_\xi=(R_{\xi,i,j})$ of the random vector $\xi^T=(\xi^1,...,\xi^n)$ can be considered as the components of the second order tensor.

The following theorem is more general then theorem 1.

Theorem 2. If $\xi=(\xi^k)$, $k=1,2,...,n$, is the random vector in the n-dimensional Euclidean space with the initial basis e_i, and $\nu_s=E({}^{0,0}\xi^s)=(\nu_s^{k_1,...,k_s})$ is the s-th order multidimensional-matrix initial moment of the vector ξ [15] (s-dimensional matrix), then the elements $\nu_s^{k_1,...,k_s}$ of the matrix ν_s can be considered as the components of the s-th order tensor.

Proof. Let $\eta = (\eta^{*l})$, $l = 1,2,...,n$, be the random vector ξ in the new basis e_{*i}, $\Lambda^* = (\lambda^*_{i,j}) = (\alpha^{*i}{}_j)$ be the transformation matrix from initial basis e_i to the new basis e_{*i}, $\bar{\nu}_r = E({}^{0,0}\eta^r)$ be the initial moment of the order s of the vector $\eta = (\eta^{*l})$ in the new basis e_{*i}. Since $\eta^{*l} = \lambda^*_{*l,k}\xi^k = \alpha^{*l}{}_k\xi^k$, then

$$\bar{\nu}_s = E({}^{0,0}\eta^s) = (\bar{\nu}_s^{*l_1,...,*l_s}) = E\left(\left(\sum_{k_1=1}^{n}\cdots\sum_{k_s=1}^{n}\lambda_{*l_1,k_1}\xi^{k_1}\lambda_{*l_2,k_2}\xi^{k_2}\cdots\lambda_{*l_s,k_s}\xi^{k_s}\right)\right) =$$

$$= E\left(\left(\sum_{k_1=1}^{n}\cdots\sum_{k_s=1}^{n}\lambda_{*l_1,k_1}\lambda_{*l_2,k_2}\cdots\lambda_{*l_s,k_s}\xi^{k_1}\xi^{k_2}\cdots\xi^{k_s}\right)\right) =$$

$$= \left(\sum_{k_1=1}^{n}\cdots\sum_{k_s=1}^{n}\lambda_{*l_1,k_1}\lambda_{*l_2,k_2}\cdots\lambda_{*l_s,k_s}E(\xi^{k_1}\xi^{k_2}\cdots\xi^{k_s})\right) =$$

$$= \left(\sum_{k_1=1}^{n}\cdots\sum_{k_s=1}^{n}\lambda_{*l_1,k_1}\lambda_{*l_2,k_2}\cdots\lambda_{*l_s,k_s}\nu_s^{k_1,...,k_s}\right) = {}^{0,s}(((\Lambda^*)^s)^{T^{-1}_{2s}}\nu_s) = {}^{0,s}(\Lambda\nu_s). \quad (42)$$

We can see that the equality (41) holds. Theorem 2 is proved.

The following theorem defines the structure of the transformation matrix Λ of the s-th order initial moment ν_s in (42) provided the orthogonal transformation of the random vector.

Theorem 3. If the transition matrix $\Lambda^* = (\lambda^*_{i,j}) = (\alpha^{*i}{}_j)$ (10) from the initial coordinate system x^i to the new coordinate system x^{*i} is orthogonal, then the $2s$-dimensional matrix Λ in tensor definition (41) and in the s-th order initial moment ν_s transformation (42) is orthogonal too.

Proof. Orthogonality of the matrix $\Lambda^* = (\lambda^*_{i,j}) = (\alpha^{*i}{}_j)$ (10) means that

$$\sum_{k=1}^{n}\lambda^*_{i,k}\lambda^*_{j,k} = \sum_{k=1}^{n}\lambda^*_{k,i}\lambda^*_{k,j} = \delta_{i,j},$$

where $\delta_{i,j}$ is Kronecker delta, or in matrix form

$$\Lambda^*(\Lambda^*)^T = (\Lambda^*)^T\Lambda^* = I,$$

where I is identical matrix of the order n. The $2s$-dimensionality matrix $\Lambda = \Lambda_{(s,0,s)}$ is called $(s,0,s)$-orthogonal if the following equality holds [22]:

$$V = {}^{0,s}(\Lambda\Lambda^{\Pi}) = {}^{0,s}(\Lambda^{\Pi}\Lambda) = E(0,s),$$

where $\Pi = B_{2s,s}$ is the transpose substitution at the $2s$ indices of the type 'onward' (28) and $E(0,s)$ is the $(0,s)$-identical matrix (29) [15].

Let us rewrite (37) using other indices:

$$\Lambda = (\lambda_{j_1,j_2,\ldots,j_s,j_{s+1},j_{s+2},\ldots,j_{2s}}) = (\lambda^*_{j_1,j_{s+1}}\lambda^*_{j_2,j_{s+2}}\cdots\lambda^*_{j_s,j_{2s}}).$$

Then

$$\Lambda^{\Pi} = (\lambda^{\Pi}{}_{j_1,j_2,\ldots,j_s,j_{s+1},\ldots j_{s+2},\ldots,j_{2s}}) = (\lambda_{j_{s+1},j_{s+2},\ldots,j_{2s},j_1,j_2,\ldots,j_s}) =$$

$$= (\lambda^*_{j_{s+1},j_1}\lambda^*_{j_{s+2},j_2}\cdots\lambda^*_{j_{2s},j_s}).$$

Further,

$$V = {}^{0,s}(\Lambda\Lambda^{\Pi}) = \left(\sum_{k_{s+1}=1}^{n}\cdots\sum_{k_{2s}=1}^{n}\lambda_{j_1,j_2,\ldots,j_s,k_{s+1},k_{s+2},\ldots,k_{2s}}\lambda^{\Pi}_{k_{s+1},k_{s+2},\ldots,k_{2s},j_{s+1},j_{s+2},\ldots,j_{2s}}\right) =$$

$$= \left(v_{j_1,j_2,\ldots,j_s,j_{s+1},j_{s+2},\ldots,j_{2s}}\right) =$$

$$= \left(\sum_{k_{s+1}=1}^{n}\cdots\sum_{k_{2s}=1}^{n}\lambda^*_{j_1,k_{s+1}}\lambda^*_{j_2,k_{s+2}}\cdots\lambda^*_{j_s,k_{2s}}\lambda^*_{j_{s+1},k_{s+1}}\lambda^*_{j_{s+2},k_{s+2}}\cdots\lambda^*_{j_{2s},k_{2s}}\right) =$$

$$= \left(\sum_{k_{s+1}=1}^{n}\cdots\sum_{k_{2s}=1}^{n}(\lambda^*_{j_1,k_{s+1}}\lambda^*_{j_{s+1},k_{s+1}})(\lambda^*_{j_2,k_{s+2}}\lambda^*_{j_{s+2},k_{s+2}})\cdots(\lambda^*_{j_s,k_{2s}}\lambda^*_{j_{2s},k_{2s}})\right) =$$

$$= \left((\delta_{j_1,j_{s+1}})(\delta_{j_2,j_{s+2}})\cdots(\delta_{j_s,j_{2s}})\right) = \left(\begin{cases}1, & \text{if } j_1 = j_{s+1}, j_2 = j_{s+2},\ldots, j_s = j_{2s}, \\ 0 & \text{otherwhise.}\end{cases}\right) =$$

$$= E(0,s).$$

The proof is completed.

9. Relationship between a tensor and a multidimensional matrix

Let us list the signs characterizing the relationship between a tensor and a multidimensional matrix.

1. A tensor is not a matrix; a tensor is a set of scalars represented by an indexed variable. The work with tensors in the framework of the tensor analysis is the work with indexed variables, i.e. with scalars but not with matrix or multidimensional-matrix variables. A tensor (covariant for simplicity) is denoted $a_{i_1,...,j_r}$, while a multidimensional matrix is denoted $a=(a_{i_1,...,j_r})$ and is considered as a "hypercomplex number" [9].

2. All of tensor indices are written out in tensor notation explicitly. The tracking of the indices in tensor expressions is difficult with a large number of indices. "The writing out of the indices leads to cumbersome formulae with tensor notation" [20].

3. All of tensor indices run the values $1,2,...,n$, where n is the dimensionality of the space in which the tensor is defined, while the indices of a multidimensional matrix can run the arbitrary number of values. This means that a tensor can be represented only by the hyper-square matrix, and it is impossible to receive a hyper-rectangular matrix provided declaring a tensor as a matrix.

4. A tensor is the set of functions defined in the linear vector space L^n. Its component $a_{j_1^*,...,j_r^*}(x^1,x^2...,x^n)$ in the coordinate system x^i with the basis e_i provided the fixed values of the indices $j_1^*,...,j_r^*$ is the some characteristic of the mutual connections of the components with numbers $j_1^*,...,j_r^*$ of the vector $(x^1,x^2...,x^n)\in L^n$. A multidimensional matrix is not connected with a specific space. A multidimensional matrix can be a number matrix (constant) or a function of other multidimensional matrix, and it is impossible to assert in general case that the element $a_{j_1^*,...,j_r^*}$ of the matrix $a=(a_{j_1,...,j_r})$ is a characteristic of some mutual connections between the components with the numbers $j_1^*,...,j_r^*$.

5. The operations of the outer and inner product of the tensors do not allow realization of the (λ,μ)-folded product of the multidimensional matrices provided $\lambda\neq 0$.

6. It is not possible to represent a mixed (covariant and contravariant) tensor with a multidimensional matrix, since no way of ordering covariant and contravariant indices has been established.

7. The multidimensional-matrix notation can be used in tensor analysis, what was shown by representation of the definition of a covariant tensor in multidimensional-matrix notation.

8. Any mathematical object should develop in the framework of his theory: tensor in the framework of tensors, matrix in the framework of matrices. A multidimensional matrix should generalize the usual (two-dimensional) matrix inheriting or generalizing the methods of the theory of usual matrices. A tensor as a multidimensional matrix is not such a generalization. Tensor as multidimensional matrix transfers the matrix into the framework of the other theory. The use of the term tensor without taking into account its properties seems unacceptable.

Conclusion

So, the article analyzes two approaches used in the multidimensional probabilistic modeling: multidimensional-matrix and tensor approaches. As the result, the differences and interconnections of these approaches are revealed. In particular, the multidimensional-matrix interpretation (multidimensional-matrix representation) of the arbitrary order tensor, which is absent in the literature, is obtained. This opens the way for generalization of tensor concepts to the multidimensional-matrix spaces. The number of theorems establishing the connections between the multidimensional probabilistic concepts and tensors are proved. At the same time, the performed analysis shows the illegality of the formal using a tensor as a multidimensional matrix.

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