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DECOMPOSITION ALGORITHMS POTENTIALS FOR THE NON-HOMOGENEOUS GENERALIZED NETWORKED PROBLEMS OF LINEAR-FRACTIONAL PROGRAMMING

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Abstract: We use potentials for calculate a reduced costs in the increment of the objective function for the linear-fractional non-homogeneous flow programming optimization problem with additional constraints of general kind. The effective algorithm for solution of the system of potentials for a sparse matrix is considered.

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1. Introduction

We consider the linear-fractional non-homogeneous flow programming optimization problem with additional constraints of general kind:

$$f(x) = \frac{p(x)}{q(x)} = \frac{\sum\limits_{(i,j)\in U} \sum\limits_{k\in K(i,j)} p_{ij}^k x_{ij}^k + \beta}{\sum\limits_{(i,j)\in U} \sum\limits_{k\in K(i,j)} q_{ij}^k x_{ij}^k + \gamma} \longrightarrow \max, \qquad (1)$$

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$$\sum_{j \in I_i^+(U^k)} x_{ij}^k - \sum_{j \in I_i^-(U^k)} \mu_{ji}^k x_{ji}^k = a_i^k, i \in I^k, k \in K; \tag{2}$$

$$\sum_{(i,j) \in U} \sum_{k \in K(i,j)} \lambda_{ij}^{kp} x_{ij}^k = \alpha_p, p = \overline{1,l}; \tag{3}$$

$$\sum_{k \in K_0(i,j)} x_{ij}^k \leq d_{ij}^0, x_{ij}^k \geq 0, k \in K_0(i,j), (i,j) \in U_0; \tag{4}$$

$$0 \leq x_{ij}^k \leq d_{ij}^k, k \in K_1(i,j), (i,j) \in U; \tag{5}$$

$$x_{ij}^k \geq 0, k \in K(i,j) \setminus K_1(i,j), (i,j) \in U \setminus U_0, \tag{6}$$

where $G = (I, U)$ is a finite oriented connected multigraph (multinetwork) without multiple arcs and loops, I is a set of nodes and $U \subset I \times I$ is a set of multiarcs. The finite non-empty set $K = \{1, \ldots, |K|\}$ is the set of different products (commodities) transported through the multinetwork G. Let us denote a connected network corresponding to a certain type of flow $k \in K$: $G^k = (I^k, U^k)$, $I^k \subseteq I$, $U^k = \{(i,j)^k : (i,j) \in \widehat{U}^k\}$, $\widehat{U}^k \subseteq U$ – a set of arcs of the multinetwork G carrying the flow of type $k \in K$, $I^k = I(U^k), I(U^k) = \{i \in I : i \in I^k\}$ is the set of nodes used for transporting (producing/consuming/transiting) the k^{th} product. In order to distinguish the products, which can simultaneously pass through an multiarc $(i,j) \in U$, we introduce the set $K(i,j) = \{k \in K : (i,j)^k \in U^k\}$. Similarly, $K(i) = \{k \in K : i \in I^k\}$ is the set of products simultaneously transported through a node $i \in I$. Let's define a set U_0 as an arbitrary subset of multiarcs of the multinetwork G, $U_0 \subseteq U$. Each multiarc $(i,j) \in U_0$ has an aggregate capacity constraint for a total amount of transported products from a subset $K_0(i,j) \subseteq K(i,j), |K_0(i,j)| > 1$. For all multiarcs $(i,j) \in U$ we assume the amount of each product $k \in K(i,j)$ to be non-negative. For a set $K_1(i,j)$ are true the following conditions: $K_1(i,j) = K(i,j) \setminus K_0(i,j)$, if $(i,j) \in U_0$ and $K_1(i,j) \subseteq K(i,j)$, if $(i,j) \in U \setminus U_0$. Moreover, each multiarc $(i,j) \in U$ can be equipped with carrying capacities for products from a set $K_1(i,j)$, where $K_1(i,j) \subseteq K(i,j)$ is an arbitrary subset of products transported through the multiarc (i,j). $I_i^+(U^k) = \{j \in I^k : (i,j) \in U^k\}, I_i^-(U^k) = \{j \in I^k : (j,i) \in U^k\}$; x_{ij}^k – amount of the k^{th} product transported through an multiarc (i,j); d_{ij}^k – carrying capacity of an multiarc (i,j) for the k^{th} product; d_{ij}^0 – aggregate capacity of an multiarc $(i,j) \in U_0$ for a total amount of products $K_0(i,j)$; λ_{ij}^{kp} – weight of a unit of the k^{th} product transported through an multiarc (i,j) in the p^{th} additional constraint; μ_{ij}^k – a flow transformation coefficient for arc $(i,j)^k$, $\mu_{ij}^k \in]0,1]$; α_p – total weighted amount of products imposed by the p^{th} additional constraint; a_i^k – intensity of a node i for the k^{th} product, $p_{ij}^k, q_{ij}^k, \beta, \gamma \in \mathbf{R}$.

2. Sparse Systems for a Potentials

The formula of the increment of the objective function (1) for the extreme linear-fractional non-homogeneous problem of flow programming (1)-(6) with additional constraints has the following kind:

$$\Delta f = \frac{\sum\limits_{k\in K}\sum\limits_{(\tau,\rho)^k\in U_N^k}\widetilde{\Delta}^k(\tau,\rho)\Delta x_{\tau\rho}^k + \sum\limits_{(i,j)\in U^*} r_{ij}\Delta z_{ij}}{\sum\limits_{k\in K}\sum\limits_{(\tau,\rho)^k\in U^k\backslash U_L^k}\Delta_Q^k(\tau,\rho)\left(x_{\tau\rho}^k+\Delta x_{\tau\rho}^k\right)+Q+\gamma}, \tag{7}$$

where

$$\widetilde{\Delta}^k(\tau,\rho) = \Delta^k(\tau,\rho) - \sum_{p=1}^{l} r_p\Lambda_{\tau\rho}^{kp} - \sum_{(i,j)\in U^*} r_{ij}\delta_{ij}(B_{\tau\rho}^k), \tag{8}$$

$$\Delta^k(\tau,\rho) = \Delta_P^k(\tau,\rho) - f(x)\Delta_Q^k(\tau,\rho),$$

$$\Delta_P^k(\tau,\rho) = p_{\tau\rho}^k + \sum_{(i,j)^k\in U_L^k} p_{ij}^k\delta_{ij}^k(\tau,\rho),$$

$$\Delta_Q^k(\tau,\rho) = q_{\tau\rho}^k + \sum_{(i,j)^k\in U_L^k} q_{ij}^k\delta_{ij}^k(\tau,\rho),$$

$$\Lambda_{\tau\rho}^{kp} = \lambda_{\tau\rho}^{kp} + \sum_{(i,j)^k\in U_L^k}\lambda_{ij}^{kp}\delta_{ij}^k(\tau,\rho), (\tau,\rho)^k\in U^k\backslash U_L^k,\ p=\overline{1,l},$$

$$\delta_{ij}(B_{\tau\rho}^k) = \begin{cases} \delta_{ij}^k(\tau,\rho), k\in K_0(i,j), & 0, k\notin K_0(i,j), \\ (i,j)\in U_0, (\tau,\rho)^k\in U^k\backslash U_L^k, k\in K. & \end{cases}$$

$$r_p = \sum_{k\in K}\sum_{(\tau,\rho)^k\in U_B^k}\Delta^k(\tau,\rho)\nu_{t(\tau,\rho)^k,p},\ p=\overline{1,l},$$

$$r_{ij} = \sum_{k\in K}\sum_{(\tau,\rho)^k\in U_B^k}\Delta^k(\tau,\rho)\nu_{t(\tau,\rho)^k,l+\xi(i,j)},\ (i,j)\in U^*,$$

where $x = \left(x_{ij}^k, (i,j)\in U, k\in K(i,j)\right)$ be a multiflow of the problem (1)$-$(6) i. e. components of the vector x meet the conditions (2)$-$(6). Along with the multiflow x let us define support multiflow $\{x, U_S\}$ as a pair [1], containing of an arbitrary multiflow x and a support [1, 4] U_S of multigraph $G = \{I, U\}$ of the problem (1)-(6), $U_S = \{U_S^k, k \in K, U^*\}$, $U_S^k \subset U^k, k \in K$; $U^* \subseteq \overline{U}_0, \overline{U}_0 = \left\{(i,j) \in U_0 : |K_S^0(i,j)| > 1\right\}$, $K_S(i,j) = \{k \in K(i,j) : (i,j)^k \in U_S^k\}$, $(i,j) \in U$, $K_S^0(i,j) = K_S(i,j) \bigcap K_0(i,j), (i,j) \in U_0$ of the problem (1)−(6). A support U_S of multigraph $G = \{I, U\}$ of the problem (1)-(6) includes a support $U_L = \{U_L^k, k \in K\}$ for system (2) and the set $U_B = \{U_B^k, k \in K\}$ of bicycling arcs [1]-[4].

Let's consider some other multiflow

$$\overline{x} = \left(\overline{x}_{ij}^k = x_{ij}^k + \Delta x_{ij}^k : (i,j) \in U, k \in K(i,j)\right)$$

Then $\Delta x = \left(\Delta x_{ij}^k, (i,j) \in U, k \in K(i,j)\right)$ is the vector of flow increments along the multiarc $(i,j) \in U$,

$$z_{ij} = \sum_{k \in K_0(i,j)} x_{ij}^k, \quad \overline{z}_{ij} = \sum_{k \in K_0(i,j)} \overline{x}_{ij}^k,$$
$$\Delta z_{ij} = \overline{z}_{ij} - z_{ij} = \sum_{k \in K_0(i,j)} \Delta x_{ij}^k, \ (i,j) \in U_0, \tag{9}$$

$\delta^k(\tau,\rho) = (\delta_{ij}^k(\tau,\rho), \ (i,j)^k \in U^k)$ – characteristic vector, entailed by arc $(\tau,\rho)^k \in U^k \setminus U_L^k$ concerning a support U_L^k for system (2), $k \in K$ [2],

$$Q = \sum_{k \in K} \sum_{(i,j)^k \in U_L^k} q_{ij}^k \left(\widetilde{x}_{ij}^k - \sum_{(\tau,\rho)^k \in U^k \setminus U_L^k} \widetilde{x}_{\tau\rho}^k \delta_{ij}^k(\tau,\rho) \right), \tag{10}$$

where $\tilde{x}^k = (\tilde{x}_{ij}^k, (i,j)^k \in U^k)$ – is partial solution of the nonhomogeneous system (2) and $\delta^k(\tau,\rho) = (\delta_{ij}^k(\tau,\rho), (i,j)^k \in U^k)$, $(\tau,\rho)^k \in U^k \setminus U_L^k, k \in K$ is the system of characteristic vectors, entailed by an arc $(\tau,\rho)^k \in U^k \setminus U_L^k, k \in K$ for the fixed $k \in K$ [2, 4].

Remark. We use the partial solution

$$\widetilde{x}^k = (\widetilde{x}_{ij}^k, (i,j)^k \in U^k), k \in K$$

which is constructed to the following rules: non-supporting elements $(\tau,\rho)^k \in U^k \setminus U_L^k, k \in K$ are equal to zeros and supporting elements $(i,j)^k \in U_L^k, k \in K$ satisfy system (2).

For calculation of reduced costs (8) we will use the sparse system potentials. Let's write down system (11)-(13) of potentials r, u^k, $k \in K$:

$$r = (r_p : p = \overline{1, l}; r_{ij},\ (i,j) \in U^*),$$

$$u^k = (u_i^k,\ i \in I^k),\ k \in K,$$

for a support U_S [5] of the multigraph G for a problem (1)-(6):

$$u_i^k - \mu_{ij}^k u_j^k + \sum_{p=1}^{l} \lambda_{ij}^{kp} r_p = -\left(p_{ij}^k - f(x) q_{ij}^k\right), \quad (i,j) \in U \backslash U^*,\ k \in K_S(i,j); \tag{11}$$

$$u_i^k - \mu_{ij}^k u_j^k + \sum_{p=1}^{l} \lambda_{ij}^{kp} r_p + r_{ij} = -\left(p_{ij}^k - f(x) q_{ij}^k\right), \quad (i,j) \in U^*,\ k \in K_S^0(i,j); \tag{12}$$

$$u_i^k - \mu_{ij}^k u_j^k + \sum_{p=1}^{l} \lambda_{ij}^{kp} r_p = -\left(p_{ij}^k - f(x) q_{ij}^k\right),$$

$$(i,j) \in U \backslash U^*,\ k \in K_S(i,j) \ \ \mathit{or} \ \ (i,j) \in U^*,\ k \in K_S(i,j) \backslash K_S^0(i,j). \tag{13}$$

Consider the effective algorithm for solving sparse system of potentials which is based on principles of decomposition of sparse system (11)-(13).

Let's construct a matrix

$$D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}, D_1 = \left(\Lambda_{\tau\rho}^{kp}, p = \overline{1, l},\ t(\tau,\rho)^k = \overline{1, |U_B|}\right),$$

$$D_2 = \left(\delta_{ij}(B_{\tau\rho}^k), \xi(i,j) = \overline{1, |U^*|}, t(\tau,\rho)^k = \overline{1, |U_B|}\right),$$

$\xi = \xi(i,j)$ – a number of arc $(i,j) \in U^*, \xi \in \{1, 2, \ldots, |U^*|\}$, where

$$\Lambda_{\tau\rho}^{kp} = \lambda_{\tau\rho}^{kp} + \sum_{(i,j)^k \in U_L^k} \lambda_{ij}^{kp}\, \delta_{ij}^k(\tau,\rho), (\tau,\rho)^k \in U^k \backslash U_L^k.$$

$\delta^k(\tau,\rho) = (\delta_{ij}^k(\tau,\rho),\ (i,j)^k \in U^k)$ – characteristic vector, entailed by arc $(\tau,\rho)^k \in U^k \setminus U_L^k$ concerning a support U_L^k for system (2), $k \in K$, $\widetilde{t} = |U_B|$,

$U_B = \{U_B^k, k \in K\}$ [2, 4].
The vector

$$r = (r_p, p = \overline{1,l}; r_{ij}, (i,j) \in U^*),$$

we compute from the system

$$D'r = \omega \tag{14}$$

where $\omega = (\omega_t,\ t = \overline{1,\ \widetilde{t}})$,

$$\omega_t = -\sum_{(i,j)^k \in B^k_{\tau\rho}} \left(p^k_{ij} - f(x)q^k_{ij}\right) \delta^k_{ij}(\tau,\rho), t = t(\tau,\rho)^k,\ k \in K,$$

The system (14) has unique solution, as $det\ D \neq 0$ [4].

For each $k \in K$ let us put $u^k_i = 0$ for some $i \in I^k$. The other components of vectors $u^k = (u^k_i : i \in I^k), k \in K$ are uniquely determined by the system (15):

$$\begin{gathered}
u^k_i - \mu^k_{ij} u^k_j = -\sum_{p=1}^{l} \lambda^{kp}_{ij} r_p - \left(p^k_{ij} - f(x)q^k_{ij}\right), \\
(i,j) \in U \backslash U^*, k \in K_S(i,j), (i,j)^k \in U^k_L; \\
u^k_i - \mu^k_{ij} u^k_j = -\sum_{p=1}^{l} \lambda^{kp}_{ij} r_p - r_{ij} - \left(p^k_{ij} - f(x)q^k_{ij}\right), \\
(i,j) \in U^*,\ k \in K^0_S(i,j), (i,j)^k \in U^k_L; \\
u^k_i - \mu^k_{ij} u^k_j = -\sum_{p=1}^{l} \lambda^{kp}_{ij} r_p - \left(p^k_{ij} - f(x)q^k_{ij}\right), \\
(i,j) \in U^*, k \in K_S(i,j) \backslash K^0_S(i,j), (i,j)^k \in U^k_L.
\end{gathered} \tag{15}$$

The system (15) consists from $|K|$ independent subsystems. For calculation nonzero components of vector $u^k = (u^k_i : i \in I^k)$ for every independent subsystem (15) for fixed $k \in K$ we are able to take using $O(|I^k|)$ arithmetical operations, where $|I^k|$ – the number of nodes of the graph $G^k = (I^k, U^k)$. The algorithm with $O(n)$ computational complexity in the worst case is used for calculation nonzero component of everyone characteristic vector $\delta^k(\tau,\rho)$, where $n = |I^k|$ [2, 4].

We add to a vector $r = (r_p : p = \overline{1,l}; r_{ij}, (i,j) \in U^*)$ the following components $r_{ij} = 0, (i,j) \in U_0 \backslash U^*$. Let's receive a new vector

$$\widetilde{r} = (r_p : p = \overline{1,l}; r_{ij}, (i,j) \in U^*; r_{ij} = 0, (i,j) \in U_0 \backslash U^*).$$

A reduced costs $\widetilde{\Delta}_{ij}^{k}$ we calculate for the arcs $(i,j)^k \in U_N^k$, $U_N^k = U^k \backslash U_S^k$, $k \in K$ and also for the arcs $(i,j)^k$, $k \in K_S^0(i,j)$, $(i,j) \in U^*$, using the formulas:

$$\widetilde{\Delta}_{ij}^{k} = -\left(p_{ij}^{k} - f(x)q_{ij}^{k}\right) - (u_i^k - \mu_{ij}^k u_j^k + \sum_{p=1}^{l} \lambda_{ij}^{kp} r_p).$$

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