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CLASSICAL SOLUTION OF AN INITIAL-BOUNDARY VALUE PROBLEM WITH A MIXED BOUNDARY CONDITION FOR A MILDLY QUASILINEAR WAVE EQUATION

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We consider an initial-boundary value problem for a mildly quasilinear wave equation in the first quadrant in which we pose the Cauchy conditions on the spatial half-line and a mixed boundary condition on the time half-line. We use the method of characteristics to construct the solution in an implicit analytical form as a solution of some integro-differential equations. The solvability of these equations is studied, as well as the dependence on the initial data and the smoothness of their solutions. For the problem in question, the uniqueness of the solution is proved, and the conditions under which its classical solution exists are established.

Keywords: initial-boundary value problem; mildly quasilinear wave equation; classical solution; mixed boundary condition; matching conditions; method of characteristics.

Mathematics Subject Classification (2020): Primary 35L71, Secondary 35A09.

КЛАССИЧЕСКОЕ РЕШЕНИЕ НАЧАЛЬНО-КРАЕВОЙ ЗАДАЧИ СО СМЕШАННЫМ ГРАНИЧНЫМ УСЛОВИЕМ ДЛЯ СЛАБО КВАЗИЛИНЕЙНОГО ВОЛНОВОГО УРАВНЕНИЯ

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Рассматривается начальная задача в первом квадранте для слабо квазинейного волнового уравнения, в которой на пространственной полупрямой задаются условия Коши, а на временной полупрямой — смешанное граничное условие. Методом характеристик строится решение в неявном аналитическом виде как решение некоторых интегро-дифференциальных уравнений. Изучается разрешимость этих уравнений, а также зависимость от начальных данных и гладкость их решений. Для рассматриваемой задачи доказывается единственность решения и устанавливаются условия, при которых существует классическое решение.

Ключевые слова: начально-краевая задача; слабо квазилинейное волновое уравнение; классическое решение; смешанное граничное условие; условия согласования; метод характеристик.

1 Statement of the problem

In the domain $Q = (0,\infty) \times (0,\infty)$ of two independent variables $(t,x) \in \overline{Q} \subset \mathbb{R}$ for the nonlinear equation of the form

$$\Box_a u(t,x) = \mathcal{N}[u](t,x) := f(t,x,u(t,x),\partial_t u(t,x),\partial_x u(t,x)), \tag{1}$$

we consider a mixed problem with the Cauchy conditions

$$u(0,x) = \varphi(x), \quad \partial_t u(0,x) = \psi(x), \quad x \in [0,\infty), \tag{2}$$

and the Zaremba boundary condition

$$u(t,0) = \mu_1(t), \quad t \in [0,t_*), \tag{3}$$

$$\partial_x u(t,0) = \mu_2(t), \quad t \in [t_*,\infty), \tag{4}$$

In the formulas (1) – (4) we have used the following notation: $\Box_a = \partial_t^2 - a^2\partial_x^2$ is the d'Alembert operator ($a > 0$ for definiteness), t_* is a positive real number, f is a function given on the set $\overline{Q} \times \mathbb{R}^3, \varphi$ and ψ are functions given on the half-line $[0,\infty)$, μ_1 is a function given on the segment $[0,t_*)$, and μ_2 is a function given on the half-line $[t_*,\infty)$.

2 Integro-differential equations

We divide the domain Q by the characteristic $x - at = 0$ into two subdomains

$$Q^{(j)} = \{(t, x) \colon (-1)^j (at - x) > 0\}, \quad j = 1, 2.$$

In the closure $\overline{Q^{(j)}}$ of each of the subdomains $Q^{(j)}$ we consider the integro-differential equation

$$u^{(j)}(t, x) = g^{(1,j)}(x - at) + g^{(2)}(x + at) - \frac{1}{4a^2} \int\limits_0^{x-at} dy \int\limits_{(-1)^j(at-x)}^{x+at} \mathcal{N}[u^{(j)}]\left(\frac{z - y}{2a}, \frac{z + y}{2}\right) dz,$$
$$(t, x) \in \overline{Q^{(j)}}, \quad j = 1, 2, \quad (5)$$

where $g^{(2)}$, $g^{(1,1)}$, and $g^{(1,2)}$ are some functions, the first two of them given on the nonnegative half-line and the last one, on the nonpositive half-line.

On the closure $\overline{Q}$ of the domain Q, we define a function u by the following formula

$$u(t, x) = u^{(j)}(t, x), \quad (t, x) \in \overline{Q^{(j)}}, \quad j = 1, 2, \quad (6)$$

with the solution $u^{(j)}$ of Eq. (5) on the closure $\overline{Q^{(j)}}$ of the domain $Q^{(j)}$.

One cdn prove a theorem about the equivalence of the solution of the differential equation (1) and the integro-differential equation (5).

Theorem 2.1. *Let the condition $f \in C^1(\overline{Q} \times \mathbb{R})$ be satisfied. The function u belongs to the class $C^2(\overline{Q})$ and satisfies Eq. (1) if and only if for each $j = 1, 2$ it is a continuous differentiable solution of Eq. (5) in which the functions $g^{(1,1)}, g^{(1,2)}$, and $g^{(2)}$ are in the classes $C^2([0, \infty))$, $C^2((-\infty, 0])$, and $C^2([0, \infty))$, respectively, and the following matching conditions are satisfied:*

$$g^{(1,1)}(0) - g^{(1,2)}(0) = 0 \quad (7)$$
$$Dg^{(1,1)}(0) - Dg^{(1,2)}(0) = 0 \quad (8)$$
$$D^2 g^{(1,1)}(0) - D^2 g^{(1,2)}(0) + \frac{1}{a^2}\Big(f(0, 0, g^{(2)}(0) + g^{(1,1)}(0),$$
$$a^2 Dg^{(2)}(0) - a^2 Dg^{(1,1)}(0), Dg^{(2)}(0) + Dg^{(1,1)}(0))\Big) = 0, \quad (9)$$

where D is the ordinary derivative operator.

Proof. 1. Let the function $u \in C^2(\overline{Q})$ satisfy Eq. (1). Making the linear nondegenerate change of the independent variables $\xi = x - at, \eta = x + at$ and denoting $u(t, x) = v(\xi, \eta)$, we obtain the new differential equation

$$\partial_\xi \partial_\eta v(\xi, \eta) = -\frac{1}{4a^2} \mathcal{N}[u]\left(\frac{\eta - \xi}{2a}, \frac{\eta + \xi}{2}\right).$$

Let us integrate it twice to obtain the equation

$$v(\xi, \eta) = g^{(1,j)}(\xi) + g^{(2)}(\eta)$$
$$- \frac{1}{4a^2} \int\limits_0^\xi dy \int\limits_{|\xi|}^\eta \mathcal{N}[u]\left(\frac{z - y}{2a}, \frac{z + y}{2}\right) dz, \quad \left(\frac{\eta - \xi}{2a}, \frac{\eta + \xi}{2}\right) \in \overline{Q^{(j)}}. \quad (10)$$

Equations (10) are Eqs. (5). This also implies that the functions $g^{(1,1)}$, $g^{(1,2)}$, and $g^{(2)}$ belong to the classes $C^2([0, \infty)), C^2((-\infty, 0])$, and $C^2([0, \infty))$, respectively. In addition, the continuity conditions of the function and its partial derivatives are to the second order inclusive satisfied up, i.e.,

$$\partial_t^k \partial_x^p u^{(1)}(t, x = at) = \partial_t^k \partial_x^p u^{(2)}(t, x = at), \quad 0 \le k + p \le 2. \quad (11)$$

Repeating the reasoning of the paper [1], we derive conditions (7) – (9) based on equalities (11) and representations (5).

2. Let us assume that the continuous differentiable function u is represented in the form (5) and (6), where $g^{(1,1)} \in C^2([0, \infty))$, $g^{(1,2)} \in C^2((-\infty, 0])$, and $g^{(2)} \in C^2([0, \infty)$, and the matching conditions (7) – (9) are satisfied. Due to the smoothness condition $f \in C^1(\overline{Q} \times \mathbb{R})$ and the fact that the functions

$g^{(1,1)}$, $g^{(1,2)}$, and $g^{(2)}$ are twice continuously differentiable, similar to [2], we conclude that the function u belongs to the classes $C^2(\overline{Q^{(1)}})$ and $C^2(\overline{Q^{(2)}})$. We substitute the representations (5) into Eq. (1) and verify that the function u satisfies this equation in $\overline{Q^{(1)}}$ and $\overline{Q^{(2)}}$. In this case, for the function u to belong to the class $C^2(\overline{Q})$, it is sufficient that the values of the functions $u^{(j)}$, $j = 1, 2$, the values of their first derivatives, and the values of their second derivatives coincide with each other on the characteristic $x = at$, i.e., that the equalities (11) hold. The latter is equivalent to the validity of the conditions (7) – (9), as can be easily derived by following the argument in the reverse order to that in item 1 of the proof, based on the representations (5). However, these arguments present some difficulties, so we will outline them.

Let us consider the value of the difference between the quantities $u^{(1)}(t,x)$ and $u^{(2)}(t,x)$ on the characteristic $\gamma = \{(t,x) : x = at\}$

$$u^{(1)}(t,x) - u^{(2)}(t,x) = g^{(1,1)}(0) - g^{(1,2)}(0)$$

It implies the condition (7) for $k = p = 0$.

Let us calculate the first and second derivatives of the functions $u^{(j)}$, $j = 1, 2$, on the characteristic $\gamma = \{(t,x) \mid x = at\}$. We have

$$\begin{aligned}\partial_t u^{(1)}\Big(\frac{x}{2a},\frac{x}{2}\Big) - \partial_t u^{(2)}\Big(\frac{x}{2a},\frac{x}{2}\Big) &= -a\left(\partial_x u^{(1)}\Big(\frac{x}{2a},\frac{x}{2}\Big) - \partial_x u^{(2)}\Big(\frac{x}{2a},\frac{x}{2}\Big)\right)\\ &= aDg^{(1,2)}(0) - aDg^{(1,1)}(0) - \frac{1}{4a}\int_0^x \left(\mathcal{N}[u^{(2)}]\Big(\frac{z}{2a},\frac{z}{2}\Big) - \mathcal{N}[u^{(1)}]\Big(\frac{z}{2a},\frac{z}{2}\Big)\right) dz, \quad (12)\end{aligned}$$

which generally does not mean that the condition (8) is true for $k + p = 1$. However, we can use the mean value theorem for a continuously differentiable function f:

$$\begin{aligned}\partial_t u^{(1)}\Big(\frac{x}{2a},\frac{x}{2}\Big) - \partial_t u^{(2)}\Big(\frac{x}{2a},\frac{x}{2}\Big) &= aDg^{(1,2)}(0) - aDg^{(1,1)}(0) - \frac{1}{4a}\\ \times \int_0^x \Big\langle \boldsymbol{\alpha}(z) \,\Big|\, \Big(0, 0, (u^{(2)} - u^{(1)})\Big(\frac{z}{2a},\frac{z}{2}\Big)&, \partial_t(u^{(2)} - u^{(1)})\Big(\frac{z}{2a},\frac{z}{2}\Big),\\ \partial_x(u^{(2)} - u^{(1)})\Big(\frac{z}{2a},\frac{z}{2}\Big)\Big)\Big\rangle\, dz,& \quad x \in [0,\infty), \quad (13)\end{aligned}$$

where

$$\begin{aligned}\boldsymbol{\alpha}(z) = \int_0^1 \nabla f\Big(\frac{z}{2a},\frac{z}{2}, (cu^{(2)} + (1-c)u^{(1)})\Big(\frac{z}{2a},\frac{z}{2}\Big),&\\ \partial_t(cu^{(2)} + (1-c)u^{(1)})\Big(\frac{z}{2a},\frac{z}{2}\Big), \partial_x(cu^{(2)} + (1-c)u^{(1)})\Big(\frac{z}{2a},\frac{z}{2}\Big)\Big)\, dc,&\end{aligned}$$

$\langle\cdot|\cdot\rangle$ is the scalar product, and the integral of a vector is understood componentwise.

After some transformations of the expression (13), taking into account the fulfillment of the condition (11) for $k + p = 0$, we obtain

$$\begin{aligned}\partial_t u^{(1)}\Big(\frac{x}{2a},\frac{x}{2}\Big) - \partial_t u^{(2)}\Big(\frac{x}{2a},\frac{x}{2}\Big) &= aDg^{(1,2)}(0) - aDg^{(1,1)}(0)\\ - \frac{1}{4a}\int_0^x (\partial_t u^{(1)} - \partial_t u^{(2)})\Big(\frac{z}{2a},\frac{z}{2}\Big)&\,(\boldsymbol{\alpha}_5(z) - a\boldsymbol{\alpha}_4(z))\, dz, \quad x \in [0,\infty), \quad (14)\end{aligned}$$

where $\boldsymbol{\alpha}_i$ is the i^{th} component of the vector $\boldsymbol{\alpha}$. We can think of the equality (14) as a linear integral equation for the function

$$U_t \colon [0,\infty) \ni z \mapsto U_t(z) = (\partial_t u^{(1)} - \partial_t u^{(2)})\Big(\frac{z}{2a},\frac{z}{2}\Big),$$

i.e.,

$$U_t(x) = aDg^{(1,2)}(0) - aDg^{(1,1)}(0) - \frac{1}{4a}\int_0^x U_t(z)(\boldsymbol{\alpha}_5(z) - a\boldsymbol{\alpha}_4(z))dz, \quad x \in [0,\infty). \quad (15)$$

If the equality (8) holds, then the function $U_t \equiv 0$ satisfies Eq. (15), and due to the theorem of existence and uniqueness of the solution of a linear Volterra integral equation of the second kind, it is unique in the class of continuous functions. Due to the representation (12), this fact implies the fulfillment of the condition (11) for $p+k=1$ if the condition (8) is satisfied.

Let us calculate the second derivatives of the functions $u^{(j)}$, $j=1,2$, on the characteristic $\gamma=\{(t,x): x=at\}$. Taking into account the fulfillment of the comparability conditions (11) for $k+p\le 1$, we have

$$\begin{aligned}\partial_t^2 u^{(1)}\Big(\frac{x}{2a},\frac{x}{2}\Big)-\partial_t^2 u^{(2)}\Big(\frac{x}{2a},\frac{x}{2}\Big)&=a^2\Big(\partial_x^2 u^{(1)}\Big(\frac{x}{2a},\frac{x}{2}\Big)-\partial_x^2 u^{(2)}\Big(\frac{x}{2a},\frac{x}{2}\Big)\Big)\\ &=-a\Big(\partial_t\partial_x u^{(1)}\Big(\frac{x}{2a},\frac{x}{2}\Big)-\partial_t\partial_x u^{(2)}\Big(\frac{x}{2a},\frac{x}{2}\Big)\Big)=a^2\big(D^2g^{(1,1)}(0)-D^2g^{(2)}(0)\big)\\ &+f\big(0,0,g^{(2)}(0)+g^{(1,1)}(0),a^2Dg^{(2)}(0)-a^2Dg^{(1,1)}(0),Dg^{(2)}(0)+Dg^{(1,1)}(0)\big)\\ &+\frac{1}{4a^2}\int\limits_0^x\Big(\partial_t^2 u^{(1)}\Big(\frac{z}{2a},\frac{z}{2}\Big)-\partial_t^2 u^{(2)}\Big(\frac{z}{2a},\frac{z}{2}\Big)\Big)\Big(a\mathcal{P}_4[u^{(1)}]\Big(\frac{z}{2a},\frac{z}{2}\Big)-\mathcal{P}_5[u^{(2)}]\Big(\frac{z}{2a},\frac{z}{2}\Big)\Big)\,dz,\end{aligned}\tag{16}$$

where

$$\mathcal{P}_4[u^{(1)}](t,x)=\partial_{z_4}f\big(t,x,u^{(1)}(t,x),z_4=\partial_t u^{(1)}(t,x),\partial_x u^{(1)}(t,x)\big),$$

and

$$\mathcal{P}_5[u^{(1)}](t,x)=\partial_{z_5}f\big(t,x,u^{(1)}(t,x),\partial_t u^{(1)}(t,x),z_5=\partial_x u^{(1)}(t,x)\big).$$

We can also consider the representation (16) as an integral equation with respect to the function $U_{tt}\colon [0,\infty)\ni z\mapsto U_{tt}(z)=(\partial_t^2u^{(1)}-\partial_t^2u^{(2)})(z/(2a),z/2)$:

$$\begin{aligned}U_{tt}(x)&=f(0,0,g^{(2)}(0)+g^{(1,1)}(0),a^2Dg^{(2)}(0)-a^2Dg^{(1,1)}(0),Dg^{(2)}(0)+Dg^{(1,1)}(0))\\ &+\int\limits_0^x U_{tt}(z)\Big(a\mathcal{P}_4[u^{(1)}]\Big(\frac{z}{2a},\frac{z}{2}\Big)-\mathcal{P}_5[u^{(2)}]\Big(\frac{z}{2a},\frac{z}{2}\Big)\Big)\,dz,\quad x\in[0,\infty).\end{aligned}\tag{17}$$

By the theorem of existence and uniqueness of a solution of a linear Volterra integral equation of the second kind, if the equality (9) is satisfied, then the function $U_{tt}\equiv 0$ is the unique solution of Eq. (17). From the equality $U_{tt}\equiv 0$ and the formulas (16) it follows that the conditions (11) for $k+p=2$ are fulfilled. The proof of the theorem is complete.

Equations (5) contain the functions $g^{(1,1)}$, $g^{(2)}$ and $g^{(1,2)}$, which must be determined before we proceed to the study of the solvability of these equations. These functions should be chosen such that the initial (2) and boundary conditions (3), (4) are satisfied.

Similar to article [1], the functions $g^{(1,1)}$ and $g^{(2)}$ are determined from the Cauchy conditions (2)

$$g^{(1,1)}(x)=\frac{\varphi(x)}{2}-\frac{1}{2a}\int\limits_0^x\psi(z)\,dz-C-\frac{1}{4a^2}\int\limits_0^x dz\int\limits_0^z\mathcal{N}[u^{(1)}]\Big(\frac{z-y}{2a},\frac{z+y}{2}\Big)\,dy,\quad x\ge 0,\tag{18}$$

$$g^{(2)}(x)=\frac{\varphi(x)}{2}+\frac{1}{2a}\int\limits_0^x\psi(z)\,dz+C+\frac{1}{4a^2}\int\limits_0^x dz\int\limits_0^z\mathcal{N}[u^{(1)}]\Big(\frac{z-y}{2a},\frac{z+y}{2}\Big)\,dy,\quad x\ge 0,\tag{19}$$

where C is an arbitrary real constant. Also, according to [1], we determine the function $g^{(1,2)}$ on the segment $[-at_*,0]$ from the boundary condition (3)

$$g^{(1,2)}(x)=\mu_1\Big(-\frac{x}{a}\Big)-\frac{\varphi(-x)}{2}-\frac{1}{2a}\int\limits_0^{-x}\psi(z)\,dz-C-\frac{1}{4a^2}\int\limits_0^{-x}dz\int\limits_0^z\mathcal{N}[u^{(1)}]\Big(\frac{z-y}{2a},\frac{z+y}{2}\Big)\,dy,$$
$$-at_*\le x\le 0.\tag{20}$$

The function $g^{(1,2)}$ on the half-line $(-\infty,-at_*]$ can be defined similarly to [4]. According to the boundary condition (1.4), the function $g^{(1,2)}$ on the interval $(-\infty,-at_*)$ satisfies the following differential equation

$$Dg^{(1,2)}(z)+Dg^{(2)}(-z)-\frac{1}{2a^2}\int\limits_0^z\mathcal{N}[u^{(2)}]\Big(\frac{-y-z}{2a},\frac{y-z}{2}\Big)\,dy=\mu_2\Big(-\frac{z}{a}\Big),z<-at_*\tag{21}$$

According to Theorem 2.1, the function $g^{(1,2)}$ must be continuous, so we have the conjugation condition

$$g^{(1,2)}(-at_* - 0) = g^{(1,2)}(-at_* + 0). \tag{22}$$

We consider Eq. (21) for $g^{(1,2)}$ together with the condition (22) as the Cauchy problem for a first-order differential equation. The solution of this problem is determined by the formula

$$\begin{aligned} g^{(1,2)}(z) = g^{(1,2)}(-at_*) + g^{(2)}(-z) - g^{(2)}(at_*) + \int\limits_{-at_*}^{z} \Big(\mu_2\Big(-\frac{s}{a}\Big) \\ + \frac{1}{2a^2}\int\limits_0^s \mathcal{N}[u^{(2)}]\Big(\frac{-y-s}{2a}, \frac{y-s}{2}\Big)\Big)\, ds = \mu_1(t_*) + \frac{\varphi(-x)}{2} - \varphi(at_*) \\ + \frac{1}{2a}\int\limits_0^{-x} \psi(z)\, dz - \frac{1}{a}\int\limits_0^{at_*} \psi(z) dz - C + \frac{1}{4a^2}\int\limits_0^{-x} dz \int\limits_0^z \mathcal{N}[u^{(1)}]\Big(\frac{z-y}{2a}, \frac{y+z}{2}\Big)\, dy \\ - \frac{1}{2a^2}\int\limits_0^{at_*} dz \int\limits_0^z \mathcal{N}[u^{(1)}]\Big(\frac{z-y}{2a}, \frac{y+z}{2}\Big)\, dy + \int\limits_{-at_*}^{x} \mu_2\Big(-\frac{s}{a}\Big)\, ds \\ + \frac{1}{4a^2}\int\limits_{-at_*}^{x} dz \int\limits_0^z \mathcal{N}[u^{(1)}]\Big(\frac{-z-y}{2a}, \frac{y-z}{2}\Big)\, dy, \quad z \le -at_*. \end{aligned} \tag{23}$$

Substituting the formulas (18) – (20), and (23) into the original integro-differential equations (5), we get the following representations:

$$\begin{aligned} u^{(1)}(t,x) = \frac{\varphi(x-at) + \varphi(x+at)}{2} + \frac{1}{2a}\int\limits_{x-at}^{x+at} \psi(\xi)\, d\xi \\ + \frac{1}{4a^2}\int\limits_{x-at}^{x+at} dz \int\limits_{x-at}^{z} \mathcal{N}[u^{(1)}]\Big(\frac{x-y}{2a}, \frac{x+y}{2}\Big)\, dy, \quad (t,x) \in \overline{Q^{(1)}} \end{aligned} \tag{24}$$

$$\begin{aligned} u^{(2)}(t,x) = \mu_1\Big(t - \frac{x}{a}\Big) - u^{(1)}\Big(\frac{at-x}{2a}, \frac{at-x}{2}\Big) + u^{(1)}\Big(\frac{x+at}{2a}, \frac{x+at}{2}\Big) \\ - \frac{1}{4a^2}\int\limits_{at-x}^{x+at} dz \int\limits_0^{x-at} \mathcal{N}[u^{(2)}]\Big(\frac{z-y}{2a}, \frac{y+z}{2}\Big)\, dy, \quad (t,x) \in \Omega_1, \end{aligned} \tag{25}$$

$$\begin{aligned} u^{(2)}(t,x) = u^{(2)}\Big(\frac{x+at-at_*}{2a}, \frac{x+at-at_*}{2}\Big) - a\int\limits_{t_*}^{t-x/a} \mu_2(z)\, dz \\ + u^{(2)}\Big(\frac{at-x-at_*}{2a}, \frac{at-x-at_*}{2}\Big) - \mu_1(t_*) \\ - \frac{1}{4a^2}\int\limits_{-at_*}^{x-at} dy \int\limits_{at-x}^{x+at} \mathcal{N}[u^{(2)}]\Big(\frac{z-y}{2a}, \frac{y+z}{2}\Big)\, dz \\ - \frac{1}{2a}\int\limits_{t_*}^{t-x/a} dz \int\limits_{-at_*}^{-az} \mathcal{N}[u^{(2)}]\Big(\frac{az-y}{2a}, \frac{y+az}{2}\Big)\, dy, \quad (t,x) \in \Omega_2, \end{aligned} \tag{26}$$

where $\Omega_1 = \overline{Q^{(2)}} \cap \{(t,x) : x - at \ge -at_*\}$ and $\Omega_2 = \overline{Q^{(2)}} \cap \{(t,x) : x - at \le -at_*\}$.

Note that the representation (24) for the function $u^{(1)}$ can be derived from Green's theorem, and the formulas (25) and (26) for the function $u^{(2)}$ can be obtained using the characteristic parallelogram identity [3], similar to what was done in [5–7].

Now, assuming that the nonlinearity f satisfies the Lipschitz condition

$$|f(t,x,u,u_t,u_x) - f(t,x,z,z_t,z_x)| \le L(t,x)(|u-z| + |u_t - z_t| + |u_x - z_x|) \tag{27}$$

with a continuous function $L\colon \overline{Q} \mapsto [0,\infty)$, we solve Eqs. (24) – (26) by the method of successive approximations, i.e., Picard iterations.

For the sake of clarity, let us consider Eq. (24). Let us rewrite it in operator form as

$$u^{(1)}(t,x) = K[u^{(1)}](t,x) = \frac{\varphi(x-at)+\varphi(x+at)}{2} + \frac{1}{2a}\int\limits_{x-at}^{x+at} \psi(\xi)\,d\xi$$
$$+\frac{1}{2a}\int\limits_0^t d\tau \int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} f(\tau,\xi,u^{(1)}(\tau,\xi),\partial_t u^{(1)}(\tau,\xi),\partial_x u^{(1)}(\tau,\xi))\,d\xi, \quad (t,x)\in\overline{Q^{(1)}}. \quad (28)$$

It is natural to solve Eq. (28) in the Fréchet space $C^1(\overline{Q^{(1)}})$, whose topology can be induced by a countable family of seminorms

$$\|\cdot\|_{C^1(\Omega_m)} = \max\Big\{\|\cdot\|_{C(\Omega_m)}, \|\partial_t\cdot\|_{C(\Omega_m)}, \|\partial_x\cdot\|_{C(\Omega_m)}\Big\},$$

where $\Omega_m = \Big\{(t,x) : (t,x)\in\overline{Q^{(1)}} \wedge x+at\le m\Big\}$. Note that the sequence $(w_i)_{i=0}^{\infty}$ converges to the converges in the space $C^1(\overline{Q^{(1)}})$ (with respect to its metric) if and only if it converges with respect to each of the seminorms $\|\cdot\|_{C^1(\Omega_m)}$, $m\in\mathbb{N}$.

It is also necessary that the operator K acts from the space $C^1(\overline{Q^{(1)}})$ into $C^1(\overline{Q^{(1)}})$. It can be achieved by requiring $\varphi\in C^1([0,\infty))$, $\psi\in C([0,\infty))$ and $f\in C(\overline{Q}\times\mathbb{R})$.

Take the initial approximation $u^{(1,0)}\equiv 0$. Then every subsequent approximation is calculated by the formula

$$u^{(1,n)}(t,x) = K[u^{(1,n-1)}](t,x), \quad n\in\mathbb{N}, \quad (t,x)\in\overline{Q^{(1)}} \quad (29)$$

Let $\alpha = \max\big\{1, a^{-1/2}, a^{-1}\big\}$,

$$\mathfrak{L}_m = \max\left\{\|L\|_{C(\Omega_m)}, \left\|\Omega_m \ni (t,x)\mapsto \sqrt{\int\limits_0^t d\tau\int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} L^2(\tau,\xi)\,d\xi}\in\mathbb{R}\right\|_{C(\Omega_m)}\right\}.$$

Then,

$$\max\{|u^{(1,1)}(t,x)-u^{(1,0)}(t,x)|, |\partial_t u^{(1,1)}(t,x)-\partial_t u^{(1,0)}(t,x)|, |\partial_x u^{(1,1)}(t,x)-\partial_x u^{(1,0)}(t,x)|\}$$
$$\le \mathcal{M} := \left\|u^{(1,1)}-u^{(1,0)}\right\|_{C^1(\Omega_m)}, \quad (t,x)\in\Omega_m, \quad m\in\mathbb{N},$$

$$|u^{(1,2)}(t,x)-u^{(1,1)}(t,x)| \le \frac{1}{2a}\left|\int\limits_0^t d\tau\int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} \mathcal{N}[u^{(1,1)}](\tau,\xi)\,d\xi - \int\limits_0^t d\tau\int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} \mathcal{N}[u^{(1,0)}](\tau,\xi)\,d\xi\right|$$

$$\le \frac{1}{2a}\int\limits_0^t d\tau\int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} \big(L(\tau,\xi)(|u^{(1,1)}-u^{(1,0)}|(\tau,\xi) + |\partial_t u^{(1,1)}-\partial_t u^{(1,0)}|(\tau,\xi) +$$

$$+|\partial_x u^{(1,1)}-\partial_x u^{(1,0)}|(\tau,\xi))\big)d\xi \le \frac{1}{2a}\sqrt{\int\limits_0^t d\tau\int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} L^2(\tau,\xi)\,d\xi}\sqrt{\int\limits_0^t d\tau\int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} \mathcal{M}^2 d\xi}$$

$$\le \frac{\mathfrak{L}_m\mathcal{M}t}{2\sqrt{a}} \le \mathfrak{L}_m\mathcal{M}\alpha t, \quad (t,x)\in\Omega_m, \quad m\in\mathbb{N},$$

$$|\partial_t u^{(1,2)}(t,x)-\partial_t u^{(1,1)}(t,x)| \le \frac{1}{2}\left|\int\limits_0^t \big(\mathcal{N}[u^{(1,1)}](\tau,x-a(t-\tau)) + \mathcal{N}[u^{(1,1)}](\tau,x+a(t-\tau))\big)\,d\tau\right.$$

$$\left.-\int\limits_0^t \big(\mathcal{N}[u^{(1,0)}](\tau,x-a(t-\tau)) + \mathcal{N}[u^{(1,0)}](\tau,x+a(t-\tau))\big)\,d\tau\right| \le \int_0^t \mathfrak{L}_m\mathcal{M}d\tau$$

$$\begin{aligned}
&\le \mathfrak{L}_m \mathcal{M} t \le \mathfrak{L}_m \mathcal{M} t\alpha, \quad (t,x) \in \Omega_m, \quad m \in \mathbb{N},\\
&|\partial_x u^{(1,2)}(t,x) - \partial_x u^{(1,1)}(t,x)| \le \frac{\mathfrak{L}_m \mathcal{M} t}{a} \le \mathfrak{L}_m \mathcal{M} t\alpha, \quad (t,x) \in \Omega_m, \quad m \in \mathbb{N},\\
&\max\{|u^{(1,2)}(t,x) - u^{(1,1)}(t,x)|, |\partial_t u^{(1,2)}(t,x) - \partial_t u^{(1,1)}(t,x)|, |\partial_x u^{(1,2)}(t,x) - \partial_x u^{(1,1)}(t,x)|\}\\
&\le \mathfrak{L}_m \mathcal{M} t\alpha, \quad (t,x) \in \Omega_m, \quad m \in \mathbb{N}.
\end{aligned}$$

In what follows, by induction in which the last inequality is chosen as the base case, one can readily prove the estimate

$$\begin{aligned}
\max\{&|u^{(1,j+1)}(t,x) - u^{(1,j)}(t,x)|, |\partial_t u^{(1,j+1)}(t,x) - \partial_t u^{(1,j)}(t,x)|,\\
&|\partial_x u^{(1,j+1)}(t,x) - \partial_x u^{(1,j)}(t,x)|\} \le \frac{\mathfrak{L}_m^j \mathcal{M} t^j \alpha^j}{j!}, \quad (t,x) \in \Omega_m, \quad m \in \mathbb{N},
\end{aligned}$$

which implies

$$\begin{aligned}
\|u^{(1,j+k)}(t,x) - u^{(1,j)}\|_{C^1(\Omega_m)} &\le \sum_{i=j}^{j+k-1} \|u^{(1,i+1)}(t,x) - u^{(1,i)}\|_{C^1(\Omega_m)} = \sum_{i=j}^{j+k-1} \frac{\mathfrak{L}_m^i \mathcal{M} T_m^i \alpha^i}{i!}\\
&\le \sum_{i=j}^{\infty} \frac{\mathfrak{L}_m^i \mathcal{M} T_m^i \alpha^i}{i!} \xrightarrow[j\to\infty]{} 0. \quad (30)
\end{aligned}$$

where $T_m = \max\limits_{(t,x)\in\Omega_m} |t| = a^{-1}m$. The inequality (30) means that the sequence $(u^{(1,i)})_{i=0}^{\infty}$ is Cauchy in the Banach space $C^1(\Omega_m)$. Thus, the successive approximations by the continuously differentiable functions $u^{(1,k)}$, $k = 0, 1, \ldots$, converge to a continuously differentiable function $u^{(1)} \colon \Omega_m \mapsto \mathbb{R}$ with respect to the seminorm $\|\cdot\|_{C^1(\Omega_m)}$, and, by virtue of $\bigcup\limits_{m=1}^{\infty} \Omega_m = \overline{Q^{(1)}}$ to a unique function $u^{(1)} \colon \overline{Q^{(1)}} \mapsto \mathbb{R}$ in the space $C^1(\overline{Q^{(1)}})$. Passing to the limit as $n \to \infty$ in (29), we conclude that the function is a solution of Eq. (28).

Let us prove the uniqueness of the solution of Eq. (28) by contradiction. Let Eq. (28) has two solutions u_1 and u_2. Denote $U = u_1 - u_2$. Then

$$U(t,x) = K[u_1](t,x) - K[u_2](t,x), \quad (t,x) \in \overline{Q^{(1)}}. \tag{31}$$

The function U is continuously differentiable, and hence $\|U\|_{C^1(\Omega_m)} = \mathcal{M}_{U;m}$. It follows from (31) with allowance for the Lipschitz condition and the Cauchy-Bunyakovsky-Schwarz inequality that

$$\max\{|U(t,x)|, |\partial_t U(t,x)|, |\partial_x U(t,x)|\} \le \mathfrak{L}_m \mathcal{M}_{U;m} t\alpha, \quad (t,x) \in \Omega_m, \quad m \in \mathbb{N}.$$

By induction, we arrive at the estimate

$$\max\{|U(t,x)|, |\partial_t U(t,x)|, |\partial_x U(t,x)|\} \le \frac{\mathfrak{L}_m^j \mathcal{M}_{U;m} t^j \alpha^j}{j!}, \quad (t,x) \in \Omega_m, \quad m \in \mathbb{N}, \quad j \in \mathbb{N}.$$

It follows that $U \equiv 0$ on the set Ω_m and, by virtue of $\bigcup\limits_{m=1}^{\infty} \Omega_m = \overline{Q^{(1)}}$, that $U \equiv 0$ on the set $\overline{Q^{(1)}}$.

Thus, we have proved the existence of a unique continuously differentiable solution of Eq. (24). The existence of a unique continuous solution of Eqs. (25) and (26) under the smoothness conditions $f \in C(\overline{Q} \times \mathbb{R})$, $\varphi \in C^1([0,\infty))$, $\psi \in C([0,\infty))$, $\mu_1 \in C^1([0,t_*])$, $\mu_2 \in C([t_*,\infty))$ and the Lipschitz condition (27) with a continuous function L can be proved similarly. We state the result as the following assertion.

Theorem 2.2. *Let the conditions $f \in C(\overline{Q} \times \mathbb{R})$, $\varphi \in C^1([0,\infty))$, $\psi \in C([0,\infty))$, $\mu_1 \in C^1([0,t_*])$, $\mu_2 \in C([t_*,\infty))$ be satisfied, and let the function f satisfy the Lipschitz condition* (27) *with a continuous function $L \colon \overline{Q} \mapsto [0,\infty)$. Then, continuously differentiable solutions of Eqs.* (24) – (26) *exist and are unique.*

Thus, under the smoothness conditions $f \in C(\overline{Q}\times\mathbb{R})$, $\varphi \in C^1([0,\infty))$, $\psi \in C([0,\infty))$, $\mu_1 \in C^1([0,t_*])$, $\mu_2 \in C([t_*,\infty))$ and the Lipschitz condition (27), we have constructed a piecewise smooth solution u of the problem (1) – (4) determined by the formulas (6) and (24) – (26).

Now we can derive compatibility conditions under which the solution u to the problem (1) – (4) is classical. Calculating the quantities appearing in the expressions (7) – (9), we obtain the following compatibility conditions

$$\mu_1(0) = \varphi(0) \tag{32}$$
$$\mu_1'(0) = \psi(0) \tag{33}$$
$$\mu_1''(0) = f(0, 0, \varphi(0), \psi(0), \varphi'(0)) + a^2\varphi''(0). \tag{34}$$

According to Theorem 2.1 for the function u to belong to the class $C^2(\overline{Q})$, it is also necessary that the function $g^{(1,2)}$ belongs to the class $C^2((-\infty, 0])$. By construction we have $g^{(1,2)} \in C^2((-at_*, 0])$ and $g^{(1,2)} \in C^2((-\infty, -at_*))$. So, we have to require the fulfillment of the following equality

$$D^p g^{(1,2)}(-at_* - 0) = D^p g^{(1,2)}(-at_* + 0), \quad p \in \{0, 1, 2\}. \tag{35}$$

Substituting the representations (20) and (23) into (35), we obtain two conditions equivalent to (35):

$$a\mu_2(t_*) - \psi(at_*) = a\partial_x u^{(2)}(t_* - 0, 0) - \partial_t u^{(1)}(0, at_*) \tag{6}$$
$$\mu_2'(t_*) - \psi'(at_*) = \partial_t\partial_x u^{(2)}(t_1 - 0, 0) - \partial_t\partial_x u^{(1)}(0, at_*) \tag{7}$$

which, due to the Cauchy conditions (2), can be simplified to

$$\mu_2(t_*) = \partial_x u^{(2)}(t_* - 0, 0), \quad \mu_2'(t_*) = \partial_t\partial_x u^{(2)}(t_* - 0, 0). \tag{36}$$

In order for the problem to be well posed, in addition to the existence and uniqueness of the solution, it is necessary to prove the continuous dependence of the solution on the initial data. To prove the continuous dependence of the solution on the initial data, we first consider the perturbed equation

$$(u^{(1)} + \Delta u)(t, x) = (G + \Delta G)(t, x) + \frac{1}{2a}\int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\tau, \xi,$$
$$(u^{(1)} + \Delta u)(\tau, \xi), \partial_t(u^{(1)} + \Delta u)(\tau, \xi), \partial_x(u^{(1)} + \Delta u)(\tau, \xi))\, d\xi, \quad (t, x) \in \overline{Q^{(1)}} \tag{37}$$

along with Eq. (24), where

$$G(t, x) = \frac{\varphi(x - at) + \varphi(x + at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\, d\xi$$

and

$$\Delta G(t, x) = \frac{\Delta\varphi(x - at) + \Delta\varphi(x + at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \Delta\psi(\xi)\, d\xi$$

Let us also consider the difference of the perturbed (37) and unperturbed (24) equations

$$\Delta u(t, x) = \Delta G(t, x) + \frac{1}{2a}\int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} \big[f(\tau, \xi, (u^{(1)} + \Delta u)(\tau, \xi), \partial_t(u^{(1)} + \Delta u)(\tau, \xi),$$
$$\partial_x(u^{(1)} + \Delta u)(\tau, \xi)) - f(\tau, \xi, u^{(1)}(\tau, \xi), \partial_t u^{(1)}(\tau, \xi), \partial_x u^{(1)}(\tau, \xi))\big]\, d\xi, \quad (t, x) \in \overline{Q^{(1)}}. \tag{38}$$

Let us introduce for a fixed $m \in \mathbb{N}$ a vector function

$$\begin{aligned}\mathbf{v}^{(m)} : \left[0, \frac{m}{2a}\right] \ni t \mapsto \mathbf{v}^{(m)}(t)& \\ = (x \mapsto (R_m[\Delta u](t, x), R_m[\partial_t\Delta u](t, x), R_m[\partial_x\Delta u](t, x)))& \\ \in PC\left(\left[0, \frac{m}{2a}\right]\right) \times PC\left(\left[0, \frac{m}{2a}\right]\right) \times PC\left(\left[0, \frac{m}{2a}\right]\right)&\end{aligned}$$

where $R_m[h]$ is the extension by zero of the restriction $h|_{\Omega_m}$ of the function h.

Then, we can write the following inequalities

$$\|\mathbf{v}_1^{(m)}(t)\| \le G_0^{(m)} + \frac{1}{2a}\int_0^t L^{(m)} \max_{0\le\xi\le\tau} \|\mathbf{v}^{(m)}(\xi)\|\, d\tau, \tag{39}$$

$$\|\mathbf{v}_2^{(m)}(t)\| \le G_1^{(m)} + \int_0^t L^{(m)} \max_{0\le\xi\le\tau} \|\mathbf{v}^{(m)}(\xi)\|\, d\tau, \tag{40}$$

$$\|\mathbf{v}_2^{(m)}(t)\| \le G_2^{(m)} + \frac{1}{a}\int_0^t L^{(m)} \max_{0\le\xi\le\tau} \|\mathbf{v}^{(m)}(\xi)\|\, d\tau, \tag{41}$$

where

$$G_0^{(m)} = \|\Delta G\|_{C(\Omega_m)},\ G_1^{(m)} = \|\partial_t \Delta G\|_{C(\Omega_m)},\ G_2^{(m)} = \|\partial_x \Delta G\|_{C(\Omega_m)},$$
$$\|\mathbf{v}^{(m)}(t)\| = \max_{i\in\{1,2,3\}} \|\mathbf{v}_i^{(m)}(t)\|,\ L^{(m)} = \|L\|_{C(\Pi_m)},\ \Pi_m = \left[0, \frac{m}{2a}\right] \times [0,m].$$

The estimates (39) – (41) imply an inequality

$$\|\mathbf{v}^{(m)}(t)\| \le \|\Delta G\|_{C^1(\Omega_m)} + \int_0^t \alpha L^{(m)} \max_{0\le\xi\le\tau} \|\mathbf{v}^{(m)}(\xi)\|\, d\tau,$$

Applying [8, p. 39, Lemma 1] to the previous inequality, we obtain

$$\|\mathbf{v}^{(m)}(t)\| \le \|\Delta G\|_{C^1(\Omega_m)} \exp\Big(\alpha L^{(m)} t\Big)$$

So, we have the following estimate of the disturbance modulus:

$$\|\Delta u\|_{C^1(\Omega_m)} \le \|\Delta G\|_{C^1(\Omega_m)} \exp\left(\frac{\alpha m L^{(m)}}{2a}\right)$$

The resulting inequality implies that whatever a small perturbation ΔG,

$$\|\Delta G\|_{C^1(\Omega_m)} \le \beta \|\Delta\varphi\|_{C^1([0,m])} + m\|\Delta\psi\|_{C([0,m])}$$

where $\beta = \max\{1, a\}$, is taken, the perturbation of the solution obeys the inequality

$$\|\Delta u\|_{C^1(\Omega_m)} \le \big(\beta\|\Delta\varphi\|_{C^1([0,m])} + m\|\Delta\psi\|_{C([0,m])}\big) \exp\left(\frac{\alpha m L^{(m)}}{2a}\right),$$

Due to $\bigcup_{m=1}^{\infty} \Omega_m = \overline{Q^{(1)}}$, we conclude that the solution of Eq. (24) depends continuously on the initial data. The continuous dependence of the solution of Eqs. (25) and (26) on the initial data, can be proved in a similar way.

3 Classical solution

Thus, the results of the previous section finally lead to the following theorem.

Theorem 3.1. *Let the conditions $f \in C^2(\overline{Q} \times \mathbb{R})$, $\varphi \in C^2([0,\infty))$, $\psi \in C^1([0,\infty))$, $\mu_1 \in C^2([0,t_*])$, $\mu_2 \in C^1([t_*,\infty))$ be satisfied, and let the function f satisfy the Lipschitz condition (27) with a continuous function $L : \overline{Q} \mapsto [0,\infty)$. The mixed problem (1)–(4) has a unique solution u in the class $C^2(\overline{Q})$ if and only if conditions (32)–(34) and (36) are satisfied. This solution is determined by the formulas (6) and (24)–(26).*

4 Conclusions

In the present paper, we obtain necessary and sufficient conditions under which there exists a unique classical solution of an initial-boundary value problem in a quarter-plane for a mildly quasilinear wave equation. The dependence of the smoothness of the solution on the smoothness of the initial functions is established.

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