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NUMERICAL STUDY OF THE CONTACT ANGLE INFLUENCE ON EQUILIBRIUM FERROFLUID SHAPES

A. V. Gerasimova[1], **O. A. Lavrova**[2], **V. K. Polevikov**[3]
e-mails: [1]alinagerasimovax@gmail.com, [2]lavrovaoa@bsu.by, [3]viktor.polevikov@gmail.com

The problem of forming equilibrium ferrofluid free surfaces, realized under the action of the magnetic field of a vertical line conductor and gravity, is considered. A ferrofluid drop is in contact with the vertical conductor and the horizontal plane on which it is located. The goal is to investigate the influence of contact angles of ferrofluid surface with solid boundaries of the conductor and the plane on equilibrium shapes of the free surface using numerical modelling. The mathematical model contains ordinary differential equations for the parametric representation of an axisymmetric free surface for a known magnetic field, generated by the conductor with a given current. The computational algorithm is constructed on a non-uniform adaptive mesh, based on a finite-difference approximation, in the form of recurrence relations. A comparative analysis of the equilibrium surface shapes is carried out for acute and obtuse contact angles.

Keywords: ferrofluid; current-carrying conductor; free surface; contact angle; finite-difference method; adaptive mesh.

Mathematics Subject Classification (2020): Primary 76M20, Secondary 76W05.

ЧИСЛЕННОЕ ИССЛЕДОВАНИЕ ВЛИЯНИЯ УГЛА КОНТАКТА НА РАВНОВЕСНЫЕ ФОРМЫ МАГНИТНОЙ ЖИДКОСТИ

А.В. Герасимова[1], **О.А. Лаврова**[2], **В.К. Полевиков**[3]
e-mails: [1] alinagerasimovax@gmail.com, [2] lavrovaoa@bsu.by, [3] viktor.polevikov@gmail.com

В работе рассматривается задача о равновесных формах свободной поверхности магнитной жидкости, которые образуются под воздействием магнитного поля вертикального проводника с током и силы тяжести. Капля магнитной жидкости располагается на горизонтальной плоскости вокруг вертикального проводника. Целью работы является исследование влияния углов контакта свободной поверхности жидкости с проводником и горизонтальной плоскостью на равновесные формы свободной поверхности с использованием численного моделирования. Математическая модель задачи содержит обыкновенные дифференциальные уравнения для параметрического описания осесимметричной свободной поверхности для известного магнитного поля, создаваемого проводником с током. Вычислительный алгоритм, основанный на конечно-разностной аппроксимации, построен на неравномерной адаптивной сетке в виде рекуррентных соотношений. В работе представлен сравнительный анализ равновесных форм свободной поверхности для острых и тупых углов контакта.

Ключевые слова: магнитная жидкость; проводник с током; свободная поверхность; угол контакта; конечно-разностный метод; адаптивная сетка.

1 Introduction

Ferrofluid is a synthesized soft material that is sensitive to external magnetic field, which results in special behavior of ferrofluid interfaces with other materials, see e.g. in [1]. The problem of forming equilibrium ferrofluid free surfaces, realized under the action of the magnetic field of a vertical line conductor and gravity, is considered. This is a classical problem of ferrohydrodynamics, see [1]. A ferrofluid drop is in contact with the vertical conductor and the horizontal plane on which it is located. These contacts are characterized by two different contact angles α_1 and α_2, see Fig. 1. The goal of the present study is to investigate the influence of the contact angles of ferrofluid surface with solid boundaries of the conductor and the plane on equilibrium shapes of the free surface using numerical modelling.

A most of experimental, theoretical and numerical studies of equilibrium shapes around a conductor consider a situation with acute contact angles $0 < \alpha_1, \alpha_2 \leq \pi/2$ for any conductor current, see e.g. [2–7]. Numerical studies in [6, 8, 9] investigate the possibility of hysteresis of the free-surface shape with obtuse contact angles $\pi/2 < \alpha_1, \alpha_2 < \pi$ during cyclic quasi-static increase and decrease of current. In the present study, different geometric configurations with $\alpha_1, \alpha_2 \in \{\pi/4, 3\pi/4\}$ are considered, where other ferrofluid parameters are fixed.

The mathematical model of the problem under study is presented in Section 2. The model is constructed, following the ideas in [2, 5, 7], where special attention is paid to taking into account acute and obtuse contact angles of different values. The model contains ordinary differential equations for the parametric representation of an axisymmetric free surface for a known magnetic field, generated by the conductor with a given current, assuming a uniform distribution of particles within the ferrofluid. In contrast, the mathematical models in [5, 7] take into account particle diffusion processes. The computational algorithm of the problem under study is presented in Section 3. It is constructed similarly to [5] with a modification related to a non-uniform mesh. A comparative analysis of the equilibrium surface shapes is carried out in Section 4 for four different geometric configurations with fixed contact angles a) $\alpha_1 = \alpha_2 = \pi/4$, b) $\alpha_1 = \pi/4, \alpha_2 = 3\pi/4$, c) $\alpha_1 = 3\pi/4, \alpha_2 = \pi/4$, d) $\alpha_1 = \alpha_2 = 3\pi/4$, moreover the volume of the ferrofluid is fixed.

Future studies may investigate the situation of varying contact angles with respect to the intensity of the magnetic field, generated by the conductor. This behavior of ferrofluid was experimentally observed in [3, 4] for the problem under study.

2 Mathematical model

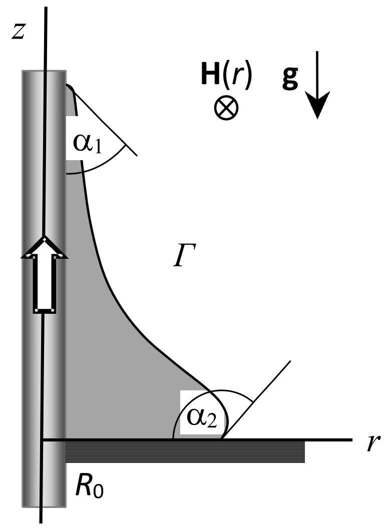

Рис. 1: Problem statement in the meridional plane.

Let us consider the problem under study in the cylindrical coordinate system (r, z), see Fig. 1. The ferrofluid is located on a horizontal plane at $z = 0$ around a vertical conductor. The ferrofluid has a free surface Γ, which touches the conductor at a contact angle α_1 and the horizontal plane at a contact angle α_2. The conductor has the shape of a cylinder with radius R_0, its axis coincides with the Oz axis. The magnetic field, induced by current I in the conductor, is azumuthal. The corresponding magnetic field intensity outside of the conductor is described by the formula $H = I/(2\pi r)$, see e.g. in [1]. The ferrofluid is affected not only by the magnetic force but also by the force of gravity, which is directed opposite to the Oz axis, see Fig. 1. The equilibrium free surface Γ of the ferrofluid is formed due to magnetic, gravitational and capillary forces.

The mathematical model is constructed, following the ideas in [2, 5, 7]. It takes into account the condition of pressure jump on the free surface of ferrofluid, the known magnetic field $H = I/(2\pi r)$ and the property $H_n = 0$ on Γ. The mathematical model has the form of a boundary value problem for the unknown dimensionless functions $r = r(s)$, $z = z(s)$:

$$\begin{aligned} &r'' = -z'F(r, z, z', L), \quad z'' = r'F(r, z, z', L), \quad 0 < s < 1, \\ &r(0) = 1/L,\ r'(0) = \sin\alpha_1,\ z'(0) = -\cos\alpha_1, \\ &r'(1) = \cos\alpha_2,\ z(1) = 0,\ z'(1) = -\sin\alpha_2, \end{aligned} \tag{1}$$

where

$$\begin{aligned} &F(r, z, z', L) = \mathrm{Bo}zL^2 - \mathrm{Bo_m}\varphi(r) - \frac{z'}{r} + C, \quad L = \left(\frac{V}{I_1}\right)^{1/3}, \\ &C = \frac{2}{r^2(1) - r^2(0)}\left(-r(1)\sin\alpha_2 + r(0)\cos\alpha_1 + \mathrm{Bo_m}I_2 - \frac{\mathrm{Bo}V}{2\pi L}\right), \\ &I_1 = 2\pi\int_0^1 zrr'ds, \quad I_2 = \int_0^1 rr'\varphi(r)ds, \quad \varphi(r) = L\ln\left(\frac{rL}{\mathrm{A}^*}\sinh\frac{\mathrm{A}^*}{rL}\right) \end{aligned}$$

for the dimensionless parameters

$$\mathrm{Bo} = \frac{\rho g R_0^2}{\sigma}, \quad \mathrm{Bo_m} = \mu_0\frac{M_S H_* R_0}{\sigma}, \quad \mathrm{A}^* = \frac{I}{2\pi H_* R_0}, \quad V = \frac{U}{R_0^3}.$$

Here ρ is the density of the ferrofluid, g is the gravity acceleration, σ is the surface tension coefficient, μ_0 is the magnetic permeability of vacuum, M_S is the saturation magnetisation of the ferrofluid, $H_* = kT/\mu_0 m$, k is the Boltzmann constant, T is the absolute temperature, m is the magnetic moment of a ferroparticle, U is the volume of the fluid.

Following the approach in [10], we introduce a new unknown function $\beta = \beta(s)$ such that $z' = \sin\beta$, $r' = \cos\beta$. Increasing the number of unknowns in the model from two to three allows us to lower the order of the differential equations in (1) from two to one. We construct the following reformulation of the mathematical model (1), taking into account acute and obtuse contact angles:

$$\begin{aligned} \beta' &= \Phi(\beta, r, z, L), \\ r' &= \cos\beta, \quad r(0) = 1/L, \\ z' &= \sin\beta, \quad z(1) = 0, \end{aligned} \tag{2}$$

with the boundary conditions for β:
$\beta(0) = \alpha_1 - \pi/2$ for $0 < \alpha_1 \le \pi/2$, $\beta(0) = \pi/2 - \alpha_1$ for $\pi/2 < \alpha_1 < \pi$ and $\beta(1) = -\alpha_2$. Here

$$\Phi(\beta, r, z, L) = \mathrm{Bo}zL^2 - \mathrm{Bo_m}\varphi(r) - \frac{\sin\beta}{r} + C,\ I_1 = 2\pi\int_0^1 zr\cos\beta ds,\ I_2 = \int_0^1 r\cos(\beta)\varphi(r)ds.$$

The equilibrium shape of the free surface Γ is defined as the solution $(r(s), z(s))$ of the mathematical model (2) for given values of the dimensionless parameters: α_1 and α_2 (contact angles), Bo (Bond number), $\mathrm{Bo_m}$ (magnetic Bond number), A^* (dimensionless current intensity), V (dimensionless fluid volume).

3 Computational algorithm

To construct a non-uniform adaptive mesh with respect to the variable s, we apply a transformation $s = s(t), t \in [0,1]$ in the following form $s(t) = -a + 2(a+1)/(1+(1+2/a)^{1-t})$ for a given parameter $a > 0$, see [11]. This transformation does not change the endpoints: $s(0) = 0, s(1) = 1$, and concentrates mesh nodes $\{s_i = s(t_i), t_i = ih, h = 1/N, i = \overline{0,N}\}$ closer to the endpoint s_0, when the parameter a tends to 0. To find the value of the parameter a in computations, the condition is used that the product of the curvature at a fixed point of the free surface $(r(0), z(0))$ by s_1 is equal to the product of the curvature of the circle by h, for the solution of which Newton's method is applied. The adaptivity of the mesh in computations is realized by changing the curvature value at point $(r(0), z(0))$. As the height of the ferrofluid drop increases, the curvature at point $(r(0), z(0))$ also increases, causing the value of s_1 to become smaller and the mesh to become more concentrated near $s = 0$. The idea of such mesh adaptivity was suggested and applied in [10,12].

Let us construct a discretization of the mathematical model (2) on the non-uniform mesh $\{s_i, i = \overline{0,N}\}$ for unknown qualities $\beta_i, r_i, z_i, i = \overline{0,N}$, based on a finite-difference approximation. In addition, we linearize the obtained discrete equations by constructing an iterative process with the introduction of a relaxation parameter $\tau > 0$ to control the convergence of the iterations for $n = 0, 1, 2, ...$. The final scheme (3)–(5) is constructed as recurrence relations at each iteration, extending the approach [5] to non-uniform meshes. First, we compute β_i at the $(n+1)$-th iteration, starting from $i = N$ and successively reaching $i = 0$:

$$\begin{aligned} &\beta_N^{n+1} = -\alpha_2, \\ &\beta_i^{n+1} = \beta_{i+1}^{n+1} - hs'(t_{i+1/2})\Phi_{i+1/2}^n + (1-\tau)(\beta_i^n - \beta_{i+1}^n + hs'(t_{i+1/2})\Phi_{i+1/2}^n), \\ &\beta_0^{n+1} = \alpha_1 - \pi/2 \text{ for } 0 < \alpha_1 \le \pi/2 \quad \text{or} \quad \beta_0^{n+1} = \pi/2 - \alpha_1 \text{ for } \pi/2 < \alpha_1 < \pi. \end{aligned} \tag{3}$$

Here $t_{i+1/2} = (i+1/2)h$, $\Phi_{i+1/2}^n = \Phi(\beta_{i+1/2}^n, r_{i+1/2}^n, z_{i+1/2}^n, L^n)$, $\beta_{i+1/2} = (\beta_i + \beta_{i+1})/2$. We then compute r_i^{n+1}, starting from $i = 0$, and z_i^{n+1}, starting from $i = N$, using the values for β_i^{n+1} from (3):

$$r_0^{n+1} = 1/L^n, \quad r_i^{n+1} = r_{i-1}^{n+1} + hs'(t_{i-1/2})\cos\beta_{i-1/2}^{n+1}, \quad i = 1, 2, ..., N; \tag{4}$$

$$z_N^{n+1} = 0, \quad z_i^{n+1} = z_{i+1}^{n+1} - hs'(t_{i+1/2})\sin\beta_{i+1/2}^{n+1}, \quad i = N-1, N-2, ..., 0. \tag{5}$$

At the end of the $(n+1)$-th iteration, we compute L^{n+1} and $\Phi_{i+1/2}^{n+1}$, $\overline{0, N-1}$ for use in the next iteration. To find these values, we approximate the integrals I_1 and I_2 by the trapezoidal quadrature formula, using all necessary data from the $(n+1)$-th iteration.

4 Numerical results

The equilibrium shapes Γ are computed for four geometric configurations with a fixed ferrofluid volume $\mathrm{U} = 400$ and for different contact angles: a) $\alpha_1 = \alpha_2 = \pi/4$, b) $\alpha_1 = \pi/4, \alpha_2 = 3\pi/4$, c) $\alpha_1 = 3\pi/4, \alpha_2 = \pi/4$, d) $\alpha_1 = \alpha_2 = 3\pi/4$. The ferrofluid parameters are taken similarly to computations in [5] and correspond to experimental data in [2]: $\mathrm{Bo} = 1$, $\mathrm{Bo_m} = 6$, where the dimensionless current intensity A^* varies in the range from 0 to 6. Computations are performed on non-uniform adaptive meshes for $N = 1000$. The adaptivity parameter $a \approx 0.006$ corresponds to the most elongated shape of the equilibrium free surface with mesh sizes monotonically changing along the surface from $s_1 - s_0 = 0.04h$ to $s_N - s_{N-1} = 3h$. The iterative scheme (3) was realized for the relaxation parameter $\tau = 0.0001$. The numerically resolved free-surface shapes and their geometric characteristics z_0 and r_N, denoting the height of the ferrofluid drop and the radius of its base, respectively, are shown in Fig. 2 and Fig. 3 for different contact angles.

Fig. 2 presents equilibrium axisymmetric ferrofluid shapes for three values of current intensities $\mathrm{A}^* \in \{0, 3, 6\}$. Fig. 2 shows that ferrofluids with an acute contact angle α_1 with the conductor elongate along the conductor more strongly than ferrofluids with an obtuse contact angle α_1. Namely, the maximum value of z_0/R_0 is greater than 16 for $\alpha_1 = \pi/4$, see Fig. 2a, Fig. 2b in comparison with Fig. 2c, Fig. 2d, where the maximum $z_0/R_0 \approx 12$ for $\alpha_1 = 3\pi/4$. A similar effect is observed for the contact angle α_2 with the horizontal plane. Ferrofluids with an acute contact angle α_2 with the horizontal plane spread along the plane more strongly than ferrofluids with an obtuse contact angle α_2, see r_N/R_0 for $\alpha_2 = \pi/4$ in Fig. 2a, Fig. 2c and for $\alpha_2 = 3\pi/4$ in Fig. 2b, Fig. 2d. We note also that elongated shapes with obtuse contact angle α_1, see solid lines in Fig. 2c and Fig. 2d, require a larger length-scale near the conductor to demonstrate that the contact angle α_1 is obtuse.

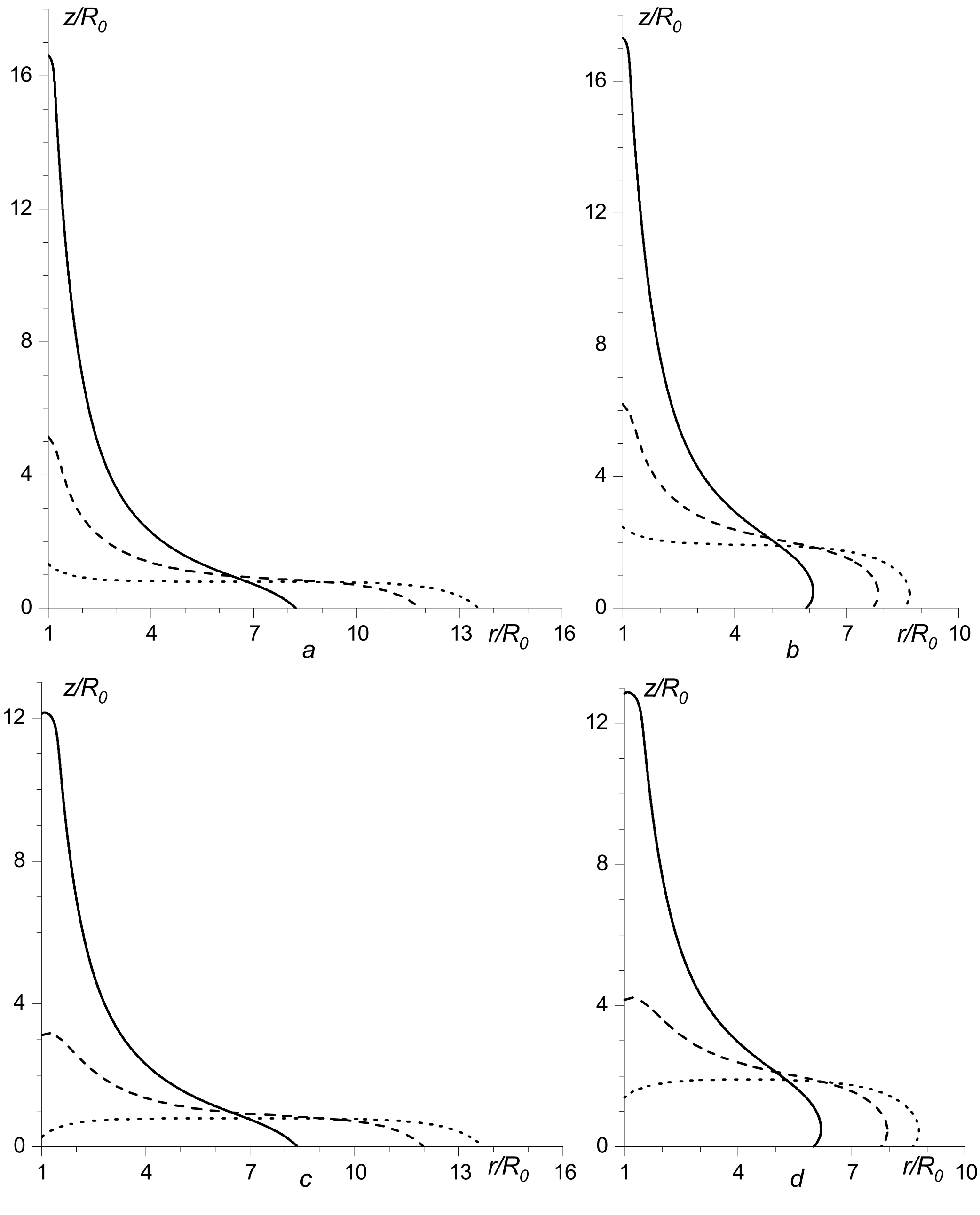

Рис. 2: Computed equilibrium ferrofluid shapes for different contact angles: a) $\alpha_1 = \alpha_2 = \pi/4$, b) $\alpha_1 = \pi/4, \alpha_2 = 3\pi/4$, c) $\alpha_1 = 3\pi/4, \alpha_2 = \pi/4$, d) $\alpha_1 = \alpha_2 = 3\pi/4$, and at different dimensionless current intensities: $A^* = 0$ (dotted lines), $A^* = 3$ (dashed lines), $A^* = 6$ (solid lines).

Fig. 3 demonstrates the position of the contact point z_0/R_0 with the conductor and the contact point r_N/R_0 with the horizontal plane at different current intensities A^*. We observe that with the grows of the values of A*, the values of z_0/R_0 monotonically increase, i.e. the contact line with the conductor rises higher, see Fig. 3 (left), and the values of r_N/R_0 monotonically decrease, i.e. the contact line with the plane moves closer to the conductor, see Fig. 3 (right). Fig. 3 (left) shows that the dynamics of the equilibrium shapes near the conductor mainly depends on the contact angle α_1 at high current intensities, whereas the dynamics near the horizontal plane mainly depends on the contact angle α_2, see Fig. 3 (right). Moreover, Fig. 3 confirms the observations, discussed for Fig. 2, that acute contact angles result in more elongated shapes along solid walls than obtuse contact angles. Namely, Fig. 3 shows that the curves for α_1 and α_2 equal to $\pi/4$ lie above the curves for the contact angles equal to $3\pi/4$. We note that the numerical results, shown in Fig. 3 for the geometric configuration a) $\alpha_1 = \alpha_2 = \pi/4$, are numerically consistent with the results of computations, presented in [5, Fig. 3]. The numerical results in [5] were obtained on a uniform mesh, whereas the current study uses non-uniform adaptive meshes.

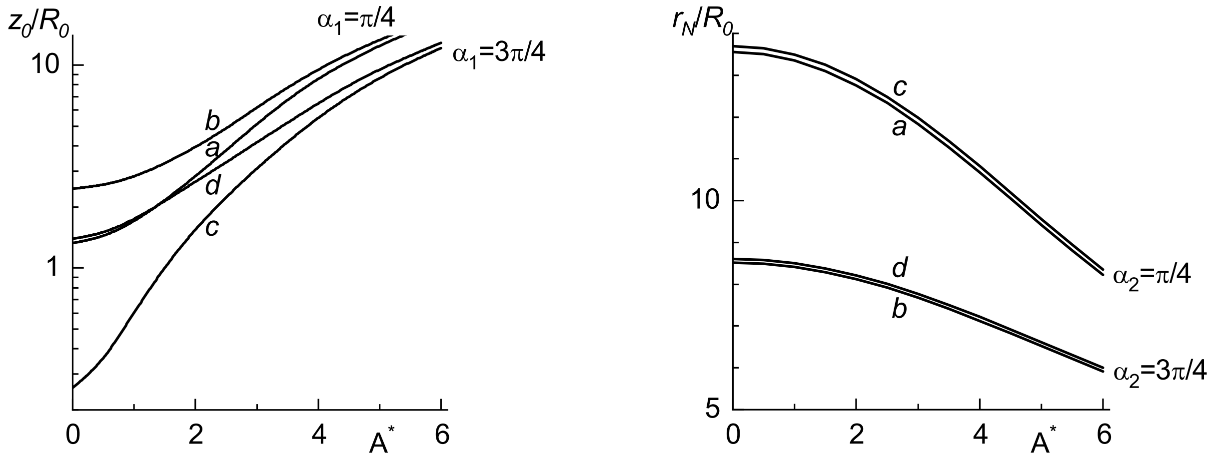

Рис. 3: Dependence of the dimensionless contact points of the free-surface z_0/R_0 (left) and r_N/R_0 (right) on the dimensionless current intensity A* for equilibirum shapes with different contact angles: a) $\alpha_1 = \alpha_2 = \pi/4$, b) $\alpha_1 = \pi/4, \alpha_2 = 3\pi/4$, c) $\alpha_1 = 3\pi/4, \alpha_2 = \pi/4$, d) $\alpha_1 = \alpha_2 = 3\pi/4$.

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