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ВЛИЯНИЕ ЭЛЕКТРОСТАТИЧЕСКИХ И МЕЖМОЛЕКУЛЯРНЫХ СИЛ НА СВОБОДНЫЕ МАЛЫЕ КОЛЕБАНИЯ МИКРОКОНСОЛИ

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Аннотация. Рассматриваются свободные малые колебания микроконсоли вблизи деформированного статического положения с учетом межмолекулярных и электростатических сил, действующих со стороны неподвижного электрода. На первом этапе с применением подхода, основанного на аппроксимации результирующих боковых сил линейными или параболическими функциями аксиальной координаты, определяется начальное статическое отклонение консоли, обусловленное действием внешних сил, при этом начальное отклонение оценивается при значениях напряжения и межмолекулярных сил, меньших критических значений. Для изучения свободных малых колебаний изначально деформированной консоли линеаризуется нелинейное дифференциальное уравнение в окрестности деформированного статического положения. Полученное определяющее уравнение с переменными коэффициентами решается с использованием как асимптотического подхода, так и метода Рунге – Кутты. Проанализировано влияние приложенного напряжения и межмолекулярных сил, включая силы Ван-дер-Ваальса и Казимира.

Ключевые слова: микроконсоль; электростатические силы; межмолекулярные силы; свободные малые колебания.

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EFFECT OF ELECTROSTATIC AND INTERMOLECULAR FORCES ON FREE SMALL VIBRATIONS OF A MICRO-CANTILEVER

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Abstract. The paper deals with free small vibrations of a micro-cantilever near the deformed static position incorporating the electrostatic and intermolecular forces acting from the fixed electrode. First, the initial static deviation of the cantilever due to the external forces is determined using the approach based on the approximation of the resultant lateral forces by the linear or parabolic functions of the axial coordinate, the initial deflection being evaluated under the values of voltage and intermolecular forces less than the critical ones. To study free small vibrations of the initially deformed cantilever, we linearise the nonlinear differential equation in the neighbourhood of the deformed static position. The derived governing equation with variable coefficients is solved using both the asymptotic approach and the Runge – Kutta method. The effect of the applied voltage and the intermolecular forces, including the van der Waals and Casimir ones, is analysed.

Keywords: micro-cantilever; electrostatic forces; intermolecular forces; free small vibrations.

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Introduction

Over the past fifteen years, a large number of papers devoted to studying the phenomenon of pull-in instability and vibrations of electrostatically actuated micro- and nano-beams was published. A detailed analysis of these contributions can be found in the review article [1]. The increased attention to this topic is explained, firstly, by the widespread use of low-dimensional electrically actuated beams as sensing elements in various kinds of micro- and nano-electromechanical systems (MEMS/NEMS), such as micro- and nano-sized sensors and actuators, switches and tweezers and in other nanotube-based devices. The second reason that again and again attracts researchers to this topic is the extreme complexity of the mathematical models governing the mechanical behaviour of low-dimensional beams taking into account the electromechanical and intermolecular forces. The incorporation of the van der Waals (vdW) and Casimir forces into a model leads to a high degree of nonlinearity, which increases when size effects are taken into account. We do not discuss here the problem of capturing the size effects [2] as it was done, for example, in [3; 4], but we focus on free vibrations of a micro-sized cantilever near the deformed static position which is due to the intermolecular and electrostatic forces acting from a fixed electrode (fig. 1).

If a gap between the fixed electrode and electrostatically actuated micro-cantilever is very small, then even if the electrostatic force is absent, the movable electrode is attracted towards the fixed one due to the intermolecular forces. Thus, the initially the micro-cantilever is always pre-strained, and the initial static deformation becoming larger under an applied voltage. If suddenly applied voltage turns out to be less than the critical pull-in voltage, at which the electrical circuit is closed, then the movable cantilever electrode begins vibrating around the initial static deformed position. And vice versa, when the voltage is turned off, the movable electrode, before returning to its original position, makes oscillatory movements around this final position. In other words, any sudden change in the electrical voltage leads to deformation of the micro-cantilever, which may be accompanied by unwanted vibrations and unintended circuit closure. To predict the dynamics of a cantilever as a micro-switch element, it is crucial studying its eigenmodes and corresponding eigenfrequencies, taking into account the intermolecular forces acting on it.

Due to the strong nonlinearity, the dynamic response of electrically actuated micro- or nano-sized beams is as a rule predicted using different semi-analytical approaches and numerical methods, which are usually associated with some computational difficulties. Moreover, in some cases, if the problem is formulated incorrectly, they can lead to solutions that ignore the initial deviation caused by intermolecular forces. For example, in [5–7] zero displacements and velocities for beams were taken as the initial conditions, which in fact lead to a solution that describes oscillations in the vicinity of the undeformed state of a micro- or nano-beam.

In our opinion, an alternative approach taking into account above mentioned issues could be an approach based on splitting the stress-strain state of the beam into a static deformed state and the dynamic state corresponding to the beam vibrations near the deformed static position. This approach has been utilised by K. F. Wang and his colleagues [8] to study large amplitude free vibrations of electrically actuated clamped-clamped nano-beams. In the mentioned paper, the nonlinear dynamic response of the nano-beam under the Casimir forces was analysed considering various complicating factors, such as surface energy and temperature changes. The authors show that the effect of the initial nonlinear static deformation on the fundamental frequency is significant and it increases together with the applied voltage. However, solutions to the nonlinear dynamics equations were found in the form of harmonic functions, which is more consistent with linear oscillations of a mechanical system. It also seems doubtful to ignore the vdW forces when displacements of the beam become very large and comparable to the gap value. A simpler version of this approach is based on the linearisation of the beam dynamic state in the vicinity of the deformed static position. Of course, this simplification does not allow predicting large amplitude vibrations, but it turns to be very effective for analysing small vibrations near the initially deformed pre-buckling position. Such approach has been applied by L. Xu and his colleagues [9; 10] to investigate the effect of the vdW forces on small oscillations of a micro-cantilever, at that the electrostatic and Casimir forces were not considered by the authors, and the static deflection caused by the vdW forces has been determined approximately with a large error (see, for example, equation (13) in [9]).

Taking into account the above critical remarks, we aim in this study to reconsider the approach based on splitting the nonlinear dynamic equation into the static and dynamic ones and give a simple methodology which, in contrast to the mentioned papers, allows more correct predicting the initial static deformed state, which strongly affects the eigenmodes of free vibrations. The novelty of our study lies in the implementation of the approach [3; 4] verified by outcomes of the atomistic simulation [11], which relies on approximations of the lateral forces acting on a beam by linear or parabolic functions and permits to correctly predict the static deviation of the micro-cantilever via correct choosing the type of intermolecular forces. Such approach results in a boundary-value problem for a differential equation with correctly found variable coefficients governing small vibrations of the deformed beam, which is readily integrated by using any numerical method, for example, the Runge – Kutta one. In contrast to many contributions, the results of numerical computations given in the present study are invariant with respect to the geometrical and physical parameters of the micro-mechanical systems, and can be used for the wide range of their variation.

Mathematical model

Consider a micro-switch which consists of a fixed electrode and a micro-cantilever of length L, width b and thickness h separated by a dielectric spacer with an initial gap g, as shown in fig. 1. The beam material is assumed to be elastic with Young's modulus E and density ρ.

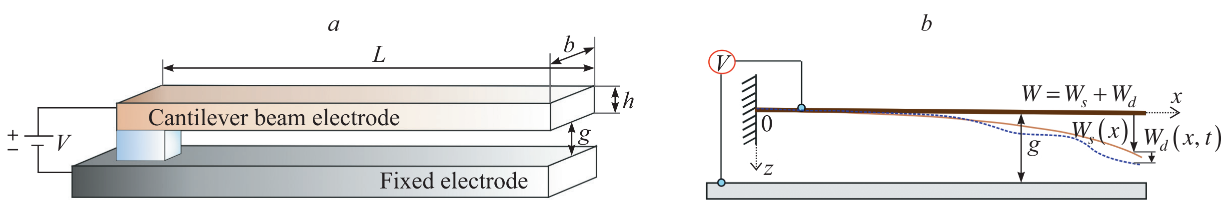

Fig. 1. Schematic configuration of a micro-switch (*a*) and vibration mode W_d of a cantilever near the static deviation W_s (*b*)

Let $W(x, t)$ be a transverse displacement of the cantilever, where x is a coordinate of a point at the midline, and t is time. In general, the displacement W can be forced by the inertia forces and different external forces such as the distributed electrostatic force F_e, generated by a voltage V applied to the fixed electrode, and the intermolecular force F_m, where $m = 3$ and $m = 4$ correspond to the vdW and Casimir forces, respectively. Applied voltage or (and) intermolecular forces result in the deflection of the beam towards the electrode. At a critical voltage value V^*, called the pull-in voltage, or at a very small initial gap g^*, the phenomenon of the pull-in instability of the micro-switch occurs, which consists in the retraction of the cantilever onto the stationary electrode. We assume here that $V < V^*$, and $g > g^*$ so that all forces acting on the micro-cantilever result in a static deviation $W_s(x) < g$ without the pull-in instability effect. The problem is to study small free vibrations of the micro-cantilever with an amplitude $W_s(x, t)$ in the neighbourhood of its initial stationary deflection W_s taking into account both the electrostatic and intermolecular forces.

Free bending vibrations of an elastic beam is governed by the equation

$$EI\frac{\partial^4 W}{\partial x^4}+\rho S\frac{\partial^2 W}{\partial t^2}=q(x), \tag{1}$$

where EI is the bending rigidity of the beam; S is the cross-sectional area; $q(x)=F_\mathrm{e}+F_\mathrm{m}$ is the distributed lateral load per unit length. The electrostatic force, including the fringing one, and the vdW and Casimir forces as well are given by

$$F_\mathrm{e}=\frac{\varepsilon_0 bV^2}{2(g-W)^2}\left(1+0.65\frac{g-W}{b}\right),\ F_3=\frac{Ab}{6\pi(g-W)^3},\ F_4=\frac{\pi^2\bar{h}cb}{240(g-W)^4}. \tag{2}$$

In relations (2), $\varepsilon_0=8.854\cdot 10^{-12}\ \mathrm{C^2\cdot N^{-1}\cdot m^{-2}}$ is the permittivity of vacuum, A is the Hamaker constant, $\bar{h}=$ $=1.055\cdot 10^{-14}$ J · s is Plank's constant divided by 2π, and $c=2.998\cdot 10^8$ m/s is the speed of light. The boundary conditions for the cantilever read

$$W(0,t)=W'_x(0,t)=0,\ W'_{xx}(L,t)=W'_{xxx}(L,t)=0, \tag{3}$$

where the subscript following the hatch denotes differentiation with respect to the corresponding variable.

We introduce dimensionless parameters and variables:

$$s=\frac{x}{L},\ \tau=\omega_c t,\ U=\frac{W}{g},\ \gamma=0.65\frac{g}{b},\ \beta=\frac{\varepsilon_0 bV^2L^4}{2g^3EI},$$
$$\alpha_3=\frac{AbL^4}{6\pi g^4EI},\ \alpha_4=\frac{\pi^2\bar{h}cbL^4}{245g^5EI}, \tag{4}$$

where $\omega_c=\sqrt{\dfrac{EI}{\rho SL^4}}$ is the characteristic frequency. Then, according to (4), equation (1) can be rewritten in the dimensionless form:

$$\frac{\partial^4 U}{\partial s^4}+\frac{\partial^2 U}{\partial \tau^2}=\frac{\gamma\beta}{1-U}+\frac{\beta}{(1-U)^2}+\frac{\alpha_m}{(1-U)^m}, \tag{5}$$

where $m=3$ or $m=4$ for the vdW and Casimir forces, respectively.

Equation (5) possesses a strong nonlinearity and does not admit an explicit solution. We will seek an approximate solution in the form of the superposition of the static and dynamic components:

$$U(x,\tau)=u(s)+w(s,\tau), \tag{6}$$

where $u(s)$ is a static deviation due to the intermolecular and (or) electrostatic forces; $w(s,\tau)$ is an additional small dynamic deflection, which describes free linear vibrations of the micro-cantilever near the deformed static position. We substitute (6) into equation (5) and assume that $|w(s,\tau)|\ll 1$ for any τ and $s\in[0,1]$. Then expanding the right-hand side in equation (5) into a series in powers of w and keeping only linear terms, we arrive at the following equation:

$$\frac{\partial^4 u}{\partial s^4}-\frac{\gamma\beta}{1-u}-\frac{\beta}{(1-u)^2}-\frac{\alpha_m}{(1-u)^m}=$$
$$=-\frac{\partial^4 w}{\partial s^4}-\frac{\partial^2 w}{\partial \tau^2}+\frac{\gamma\beta w}{(1-u)^2}+\frac{2\beta w}{(1-u)^3}+\frac{m\alpha_m w}{(1-u)^{m+1}},\ m=3,4. \tag{7}$$

Because equation (7) should be satisfied for any τ it can be split into the two equations:

$$\frac{\partial^4 u}{\partial s^4}=\frac{\gamma\beta}{1-u}+\frac{\beta}{(1-u)^2}+\frac{\alpha_m}{(1-u)^m} \tag{8}$$

and

$$\frac{\partial^4 w}{\partial s^4} - \left[\frac{\gamma\beta}{(1-u)^2} + \frac{2\beta}{(1-u)^3} + \frac{m\alpha_m}{(1-u)^{m+1}}\right] w + \frac{\partial^2 w}{\partial \tau^2} = 0, \tag{9}$$

from which equation (8) defines the static deviation of the cantilever, and equation (9) governs free small vibrations near this deflected position.

Substituting (6) into (3) leads to the boundary conditions

$$u(0) = u'_s(0) = 0,\ u'_{ss}(1) = u'_{sss}(1) = 0, \tag{10}$$

$$w(0, \tau) = w'_s(0, \tau) = 0,\ w'_{ss}(1, \tau) = w'_{sss}(1, \tau) = 0 \tag{11}$$

for equations (8) and (9), respectively.

Static deviation of micro-cantilever

Consider the static problem (8), (10). Due to the nonlinearity of the external force experienced by the cantilever, this problem does not allow obtaining an exact solution in the explicit form. We use the approach proposed in [12] and later refined in papers [3; 4]. In accordance to this approach, the lateral dimensionless force

$$f[u(s)] = \frac{\gamma\beta}{1-u} + \frac{\beta}{(1-u)^2} + \frac{\alpha_m}{(1-u)^m}, \tag{12}$$

not depending on the its nature, is approximated by a linear or quadratic function of s:

$$f(s) = f_0 + (f_T - f_0)s^n, \tag{13}$$

where $n = 1$ or $n = 2$; $f_0 = \beta + \gamma\beta + \alpha_m$; f_T is the lateral force acting on the cantilever tip. As was proposed in [3], the models, for which $n = 1$ and $n = 2$, are called the linear distributed load (LDL) and quadratic distributed load (QDL) models, respectively.

If we assume the LDL model, then equation (8) with the boundary conditions (10) has the solution

$$u(s) = \frac{f_T - f_0}{120}s^5 + \frac{f_0}{24}s^4 - \frac{f_0 + f_T}{12}s^3 + \frac{f_0 + 2f_T}{12}s^2, \tag{14}$$

and the force f_T introduced by equation (12) will be as follows:

$$f_T = \frac{\gamma\beta}{1-u_T} + \frac{\beta}{(1-u_T)^2} + \frac{\alpha_m}{(1-u_T)^m}, \tag{15}$$

where u_T is the deflection of the cantilever tip calculated by equation (14) and equal to

$$u_T = \frac{11f_T + 4f_0}{120}. \tag{16}$$

In the framework of the QDL model, a solution of the static problem (8), (10) is given by the polynomial

$$u(s) = \frac{f_T - f_0}{360}s^6 + \frac{f_0}{24}s^4 - \frac{f_T + 2f_0}{18}s^3 + \frac{f_T + f_0}{8}s^2 \tag{17}$$

with the force f_T defined from the same equation (15), but with the tip deflection evaluated as

$$u_T = \frac{26f_T + 19f_0}{360}. \tag{18}$$

We note that the accuracy of such models for estimating the static component (17) was verified in [3] by comparing with available data of the atomistic simulations [11].

For the freestanding micro-cantilever ($\beta = 0$), regardless of the model assumed, equation (15) together with (16) or (18) yield the relationship $\alpha_m = \alpha_m(u_T)$. Then, relying on the condition $\frac{d\alpha_m}{du_T} = 0$, one can calculate the critical value α_m^* of a parameter α_m ($m = 3, 4$) and the associated displacement u_T^*. Having known the critical value α_m^*, we can estimate the lower and upper bounds for the gap g and the beam length L beyond which the micro-cantilever may fall onto the base due to the intermolecular forces [3]:

$$\frac{L}{g} < \sqrt[4]{\frac{\pi\alpha_3^* E h^3}{2A}}, \tag{19}$$

and

$$L < \sqrt[4]{\frac{20\alpha_4^* E h^3 g^5}{\pi^2 \overline{h} c}}, \ g > \sqrt[5]{\frac{\pi^2 \overline{h} c L^4}{20\alpha_4^* E h^3}} \tag{20}$$

for the micro-cantilever subjected to the vdW and Casimir forces, respectively.

Under the applied voltage V, $\beta > 0$, and equation (15) together with (16) or (18) give the relationship $\beta = \beta(u_T)$. Then the condition $\frac{d\beta}{du_T} = 0$ allows finding the critical value β^* corresponding to the pull-in voltage V_{PI}, which is defined as

$$V_{\mathrm{PI}} = \sqrt{\frac{E(gh)^3 \beta^*}{6\varepsilon_0 L^4}}. \tag{21}$$

In what follows, we assume that for the freestanding beam, inequalities (19), (20) hold simultaneously, and in the case of applied voltage V, we set the additional inequality $V < V_{\mathrm{PI}}$, where V_{PI} is evaluated by (21). If these conditions are satisfied, then the micro-cantilever just deviates from its initial position toward the fixed electrode by the value $u(s) < u_T^*$ for any $s \in [0, 1]$ and does not fall onto it. Figure 2 displays the dimensionless pre-buckling tip displacement u_T of the micro-cantilever for different values of the vdW parameter α_3 calculated on the base of LDL model. Here, the critical value $\alpha_3^* = 1.004$, while for the QDL model $\alpha_3^* = 1.139$.

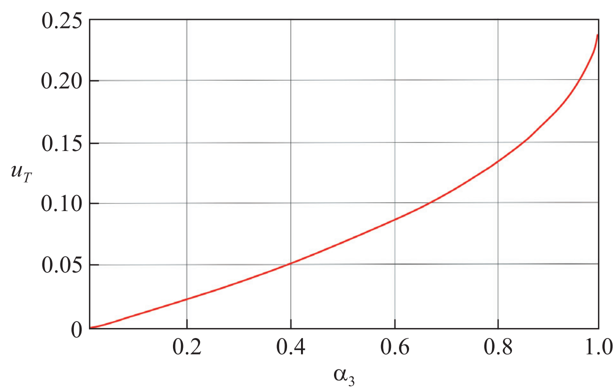

Fig. 2. Tip deflection u_T of a freestanding micro-cantilever versus the parameter α_3 calculated on the base of the LDL model

We study free small vibrations of the micro-cantilever near the deformed static position, which are governed by equation (9) with the boundary conditions (11). Seeking a solution in the form $w(s, t) = y(s)e^{i\omega\tau}$, where $i = \sqrt{-1}$ is the imaginary unit, and ω is a natural frequency, and inserting it into (9), (11), we arrive at the boundary-value problem

$$\frac{d^4 y}{ds^4} - \left[\lambda + \frac{\gamma\beta}{(1-u)^2} + \frac{2\beta}{(1-u)^3} + \frac{m\alpha_m}{(1-u)^{m+1}}\right] y = 0, \tag{22}$$

$$y(0) = y'(0) = 0, \ y''(1) = y'''(1) = 0, \tag{23}$$

where $\lambda = \omega^2$ is a required eigenvalue, and $u(s)$ is the initial static deviation found above.

Free vibrations of freestanding micro-cantilever

First, we consider equation (22) for the case when $\beta = 0$ and $m = 3$:

$$\frac{\partial^4 y}{\partial s^4} - \frac{3\alpha_3 y}{\left[1 - u(s)\right]^4} - \lambda y = 0. \tag{24}$$

It is obvious that equation (24), as well as equation (22) for the general case, do not admit an exact solution due the variable coefficients depending on $u(s)$. However, they can be readily integrated numerically and by using an asymptotic approach for the special case when the initial displacement $u(s)$ is small. The asymptotic solution will be used only to validate further numerical calculations.

Asymptotic approach. Let $u = \mu z(s)$, where $\mu = u(1) \ll 1$. Then the formal asymptotic solution of the boundary-value problem (23), (24) can be sought in the form of series:

$$y(s;\, \mu) = y_0(s) + \mu y_1(s) + \mu^2 y_2(s) + \ldots,$$
$$\lambda = \lambda_0 + \mu\lambda_1 + \mu^2\lambda_2 + \ldots. \tag{25}$$

Substituting (25) into equation (24) and the boundary conditions (23), with the function $(1 - \mu z)^{-4}$ being expanded into the Tailor series, we arrive at the sequence of boundary-value problems which can be considered step-by-step.

In the leading approximation, one has the homogeneous boundary-value problem

$$y_0^{(\mathrm{IV})} - 3\alpha_3 y_0 - \lambda_0 y_0 = 0,$$
$$y_0(0) = 0,\ y_0'(0) = 0,\ y_0''(1) = 0,\ y_0'''(1) = 0, \tag{26}$$

which has the solution

$$y_0 = C\left(F_1(s,\, k_n) - \frac{F_3(s,\, k_n)}{F_2(s,\, k_n)} F_3(s,\, k_n)\right)$$

with

$$k_n^4 = 4\alpha_3 + \lambda_0,\ F_2(s,\, k) = \frac{1}{2}(\sinh ks + \cos ks),$$

$$F_1(s,\, k) = \frac{1}{2}(\cosh ks - \cos ks),\ F_3(s,\, k) = \frac{1}{2}(\sinh ks - \sin ks),$$

$$k_1 = 1.875,\ k_2 = 4.694,\ k_3 = 7.855,\ k_4 = 10.996,\ \ldots.$$

Note that of all available k_n, one needs to consider only those values for which

$$\lambda_0 = \lambda_{0n} = k_n^4 - 3\alpha_3 > 0. \tag{27}$$

In the first-order approximation, we have the inhomogeneous boundary-value problem

$$y_1^{(\mathrm{IV})} - (3\alpha_3 + \lambda_0) y_1 = \lambda_1 y_0 + 12\alpha_3 z(s) y_0,$$
$$y_1(0) = 0,\ y_1'(0) = 0,\ y_1''(1) = 0,\ y_1'''(1) = 0, \tag{28}$$

which is the problem on the spectrum of the homogeneous boundary-value problem (26). With the self-adjointness of problem (26) taken into account, the condition for the existence of a solution to problem (28) results in the relation for a correction:

$$\lambda_1 = \lambda_{1n} = -\frac{12\alpha_3 \int_0^1 z(s) y_0^2(s) ds}{\int_0^1 y_0^2(s) ds}. \tag{29}$$

The procedure for seeking all unknown parameters and functions from (25) can be formally continued. By interrupting this process, we found the parameters λ_{0n}, λ_{1n} as functions of α_3 based on only two approximations. Figure 3, *a*, displays the first four eigenvalues λ_{0n} versus α_3 calculated within the LDL model. It can be seen that inequality (27) holds for any natural n and all values of the vdW parameter $\alpha_3 < 1$ corresponding to the pre-buckling position. It is also seen that the zero approximation of the parameter λ is not strongly influenced by the parameter α_3; an increase in α_3 leads to a decrease only in the first root λ_{01} (it should be noted that curves in fig. 3, *a*, are plotted in the logarithmic scale). As for the correction λ_{1n}, evaluated by (29) and corresponding to λ_{0n}, it reveals a large dependence on the vdW parameter α_3, it increasing in value along with α_3 for any mode number n.

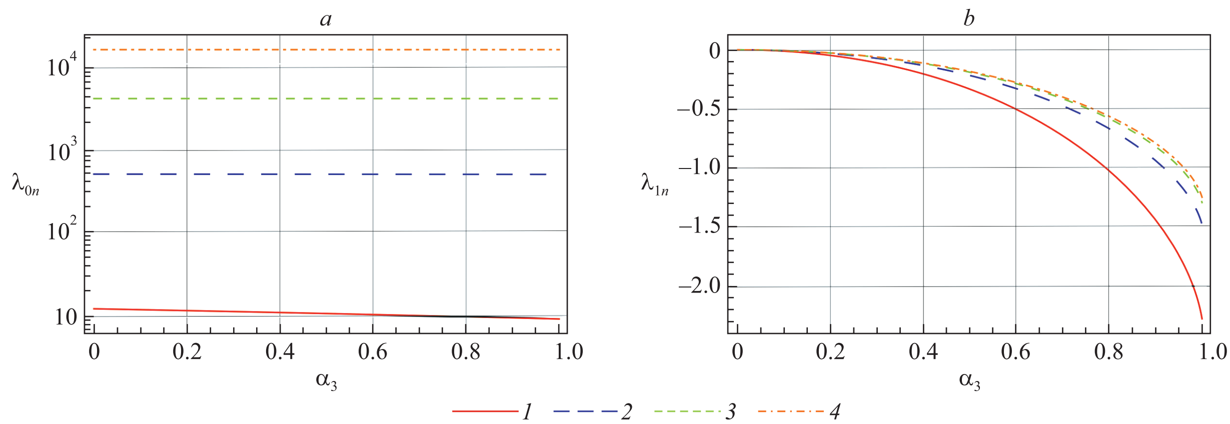

Fig. 3. The first four eigenvalues λ_{0n} (*a*)
and the corresponding corrections λ_{1n} versus the vdW parameter α_3

Numerical solution. Equation (24) with the boundary conditions (23) can be solved numerically. We rewrite it in the form of a system of differential equations

$$\mathbf{Y}' = \mathbf{C}(s;\lambda)\mathbf{Y}^{\mathrm{T}}, \tag{30}$$

where $\mathbf{Y} = (y_1, y_2, y_3, y_4)$ is the four-component vector with $y_1 = y,\ y_2 = y_1',\ y_3 = y_2',\ y_4 = y_3'$ the icon T means transpose, and $\mathbf{C}$ is the (4×4)-matrix introduced as

$$\mathbf{C} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c_{41} & 0 & 0 & 0 \end{pmatrix},\ c_{41} = \lambda + \frac{3\alpha_3}{\left[1 - z(s)\right]^4}.$$

The corresponding boundary conditions read

$$y_1(0) = y_2(0) = 0,\ y_3(1) = y_4(1) = 0. \tag{31}$$

We consider the following independent Cauchy problems for equation (30):

$$\mathbf{Y}\big|_{s=0} = (0, 0, 0, 1) \text{ and } \mathbf{Y}\big|_{s=0} = (0, 0, 1, 0). \tag{32}$$

These problems are to be solved simultaneously using, for example, the Runge – Kutta method. Let $\mathbf{Y}^{(1)} = \left(y_1^{(1)}, y_2^{(1)}, y_3^{(1)}, y_4^{(1)}\right)$ and $\mathbf{Y}^{(2)} = \left(y_1^{(2)}, y_2^{(2)}, y_3^{(2)}, y_4^{(2)}\right)$ be solutions of the problems (30), $(32)_1$ and (30), $(32)_2$, respectively. Composing the function $\mathbf{Y} = c_1\mathbf{Y}^{(1)} + c_2\mathbf{Y}^{(2)}$ and substituting it into conditions $(31)_2$ at point $s = 1$, we arrive at the homogeneous algebraic equations with respect to unknown constants c_1, c_2:

$$\begin{gathered} c_1 y_3^{(1)}(1) + c_2 y_3^{(2)}(1) = 0, \\ c_1 y_4^{(1)}(1) + c_2 y_4^{(2)}(1) = 0. \end{gathered} \tag{33}$$

The condition for the existence of a nontrivial solution to equations (33) leads to the equation

$$y_3^{(1)}(1)\, y_4^{(2)}(1) - y_4^{(1)}(1)\, y_3^{(2)}(1) = 0 \tag{34}$$

with respect to the required parameter λ.

Numerical integration of the above Cauchy problems for equation (30) was performed using the NDSolve-function in the Wolfram code with intermediate vector orthonormalisation. At each integration step, the elements of the vectors $\mathbf{Y}^{(1)}$, $\mathbf{Y}^{(2)}$ were determined with accuracy up to six decimal places, while equation (34) was solved with accuracy up to three decimal places. Figure 4 demonstrates the behaviour of the first positive eigenvalue λ of the boundary-value problem (23), (24) versus the vdW parameter α_3, established both by the numerical integration (dotted line) and using the asymptotic approach (red circles) within the LDL model. Firstly, we note the satisfactory agreement of the results obtained by the two methods, a slight divergence being observed beginning only from $\alpha_3 = 0.5$, which increases and reaches the value not exceeding 2 % for $\alpha_3 = 0.7$. This divergence is due to an error of the asymptotic approach, which increases together with the static tip deviation $\mu = u(1)$. Secondly, fig. 4 shows that the lowest natural frequency of the freestanding cantilever decreases along with the gap g between the electrodes.

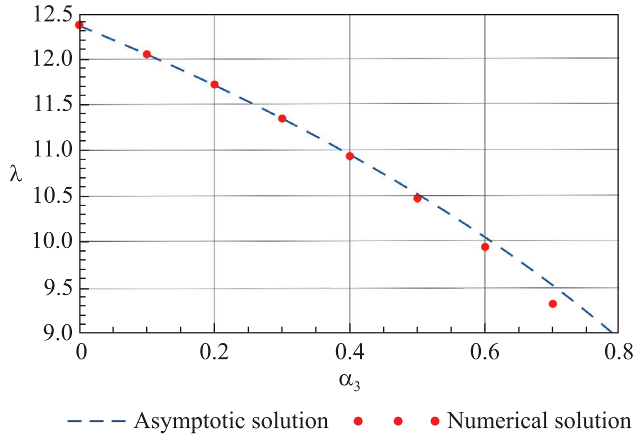

Fig. 4. The first positive eigenvalue λ of the boundary-value problem (23), (24) versus the vdW parameter α_3 defined using the numerical and asymptotic methods

Effect of electrostatic forces on free vibrations of micro-cantilever

Now, returning to equation (22), we will study the influence of the electrostatic forces on the natural frequencies considering the forces of intermolecular interaction. The lateral forces acting on the cantilever from the fixed electrode will be approximated by both the linear and quadratic functions (13) in the framework of the LDL and QDL models. Calculations of the first positive eigenvalue for the boundary-value problem (22), (23) will be done using the numerical procedure developed above with

$$c_{41} = \lambda + \frac{\beta\gamma}{\left[1 - z(s)\right]^2} + \frac{2\beta}{\left[1 - z(s)\right]^3} + \frac{m\alpha_m}{\left[1 - z(s)\right]^4}$$

assumed in the matrix $\mathbf{C}$, where $m = 3$ and $m = 4$ for the vdW and Casimir forces, respectively.

In fig. 5, the first positive dimensionless parameters λ are plotted as functions of the vdW parameter α_3 at $\gamma = 1$ for different values of the voltage parameter β in the framework of the LDL (see fig. 5, *a*) and QDL models (see fig. 5, *b*). Curves marked with numbers *1*, *2*, *3*, *4* correspond to the values $\beta = 0$, 0.15, 0.3, 1.0, respectively. The calculations were carried out only for $\beta < \beta^*$ and at the interval $0 \leq \alpha_3 < \alpha_3^*$, where β^*, α_3^* are the critical values corresponding to pull-in instability of the cantilever, the higher the voltage β being, the shorter the interval of variation of the parameter α_3. When α_3 or β reaches its critical value, the mobile cantilever collapses onto the substrate [10]. It is seen that for fixed values of the parameter α_3, the QDL model gives higher values of the eigenfrequency then the LDL model. The divergence in results is slight for small α_3, however it increases together with α_3. It is also seen that for any values of β the natural frequencies decrease as the vdW parameter α_3 increases, this decrease becoming dramatic as α_3 approaching α_3^*.

Figure 6 shows the results of calculations similar to those given above, but demonstrating λ versus the Casimir parameter α_4. It is seen that behaviour of all curves are the same as in fig. 5.

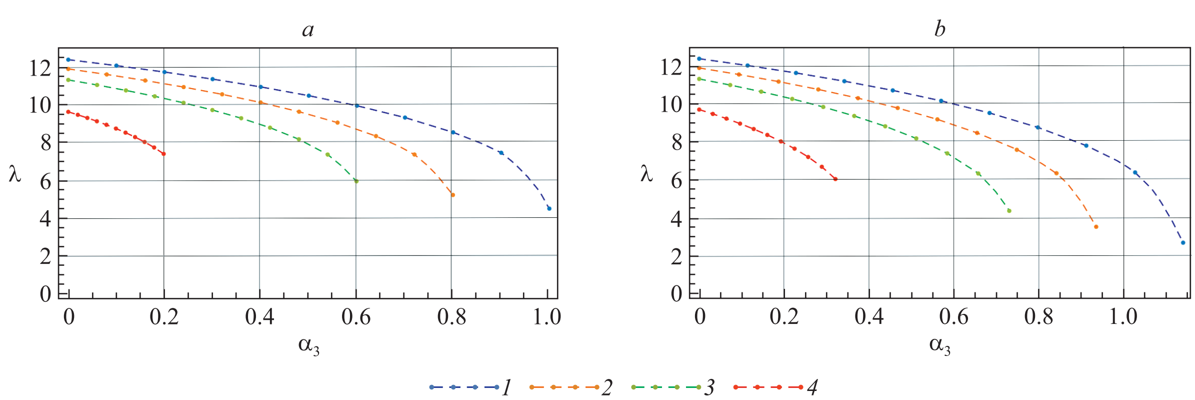

Fig. 5. The first positive eigenvalue λ versus the vdW parameter α_3 for different values of the voltage parameter β calculated in the framework of the LDL (*a*) and QDL (*b*) models: $\beta = 0$ (*1*), $\beta = 0.15$ (*2*), $\beta = 0.3$ (*3*), $\beta = 1.0$ (*4*)

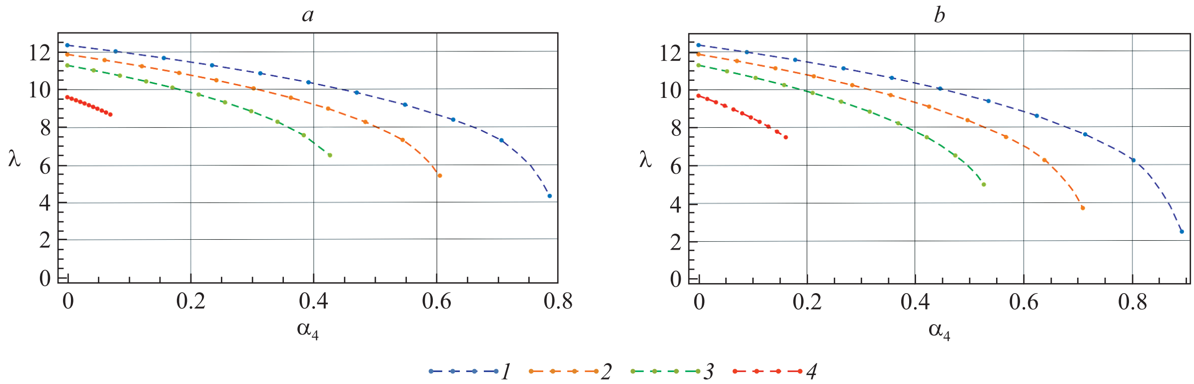

Fig. 6. The first positive eigenvalue λ of the micro-cantilever versus the Casimir parameter α_4 for different values of the voltage parameter β calculated in the framework of the LDL (*a*) and QDL (*b*) models: $\beta = 0$ (*1*), $\beta = 0.15$ (*2*), $\beta = 0.3$ (*3*), $\beta = 1.0$ (*4*)

However, a more detail comparison of the curves in fig. 5 and 6, made for the same parameter β and the adopted model, shows that the incorporation of the Casimir forces results in a weaker reduction in the first natural frequency than accounting for the vdW forces when the voltage becomes close to the critical value β^*.

Conclusions

In this work, free small vibrations of a micro-cantilever as an element of the micro-switch were investigated considering the electrostatic and intermolecular forces, including the vdW and Casimir ones. We revised the approach stated earlier (see, for example, [9; 10]), which is based on splitting the stress-strain state of the micro-cantilever into the static and dynamic states. Assuming a solution of the original nonlinear dynamic equation in the form of the superposition of static and dynamic components, we derived the nonlinear differential equation, governing the static deviation caused by the intermolecular forces, and the linear equation with variable coefficients describing free small oscillations in the vicinity of the static strained state. The novelty of our approach compared to the similar ones realised in [9; 10] is in the effective method, which allows correct estimating the static component strongly influencing the subsequent calculations of the natural frequencies of free vibrations near the deformed state. The static component was first correctly determined within the well-established LDL and QDL models [3; 4], according to which the resulting lateral force acting on the movable cantilever is approximated by either the linear or parabolic function of the axial coordinate. The differential equation governing small vibrations of the micro-cantilever near the static deformed position was derived in the form which is invariant with respect to the geometrical and physical parameters of the micro-electromechanical systems and can be utilised for studying small vibrations with a wide range of variation of these parameters. In the case of a small static deviation of the beam, we determined several first natural frequencies and corresponding modes using the asymptotic approach with a small parameter equal to the tip deviation. For the general case with a finite static tip deviation, we proposed the numerical procedure based on the Runge – Kutta method. All computations were performed for the vdW and Casimir forces not exceeding the critical pull-in instability values. The comparison of results obtained by different methods showed a good agreement for small values of the vdW parameter α_3 corresponding to relatively large clearances between the electrodes. As an expected result, we confirm that increasing the voltage and intermolecular forces leads to a decrease in the natural frequencies, with this effect turning to be strong for the lower natural frequency and becoming weak as the mode number increases. In general, the calculations performed for the adapted models without specifying the parameters of the micro-electromechanical systems also revealed that for the fixed values of the vdW or Casimir parameters, the QDL model gives higher values of the eigenfrequency with respect to the LDL model.

We note that the simple procedure developed in this paper, which relies on the adopted LDL and QDL models [3; 4], may be considered as a benchmark for subsequent investigations to study small and finite vibrations of the electrically actuated nano-beam considering size effects within the nonlocal theory of elasticity [2].

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