УДК 519.9

СИСТЕМЫ УРАВНЕНИЙ В ДИФФЕРЕНЦИАЛАХ С ОБОБЩЕННЫМИ ПРОИЗВОДНЫМИ НЕПРЕРЫВНЫХ ФУНКЦИЙ

А. И. ЖУК[1]***, Е. Н. ЗАЩУК***[1]

[1]*Брестский государственный технический университет,*
ул. Московская, 267, 224023, г. Брест, Беларусь

Аннотация. Исследуются системы неавтономных дифференциальных уравнений с непрерывными обобщенными коэффициентами в алгебре новых обобщенных функций. Система неавтономных дифференциальных уравнений с обобщенными коэффициентами рассматривается как система уравнений в дифференциалах в алгебре новых обобщенных функций. Решением таких систем является новая обобщенная функция. Показано, что различные интерпретации решений данных систем могут быть описаны при помощи единственного подхода, использующего новые обобщенные функции. В настоящей статье в отличие от предшествующих работ описаны ассоциированные решения систем неавтономных дифференциальных уравнений с непрерывными обобщенными коэффициентами в пространстве $L(T)$.

Ключевые слова: алгебра новых обобщенных функций; дифференциальные уравнения с обобщенными коэффициентами; функции ограниченной вариации.

Образец цитирования:
Жук АИ, Защук ЕН. Системы уравнений в дифференциалах с обобщенными производными непрерывных функций. *Журнал Белорусского государственного университета. Математика. Информатика.* 2024;3:22–30 (на англ.).
EDN: DHVDWW

For citation:
Zhuk AI, Zashchuk HN. Systems of equations in differentials with generalised derivatives of continuous functions. *Journal of the Belarusian State University. Mathematics and Informatics.* 2024;3:22–30.
EDN: DHVDWW

Авторы:
Анастасия Игоревна Жук – кандидат физико-математических наук, доцент; доцент кафедры высшей математики факультета электронно-информационных систем.
Елена Николаевна Защук – кандидат физико-математических наук, доцент; доцент кафедры высшей математики факультета электронно-информационных систем.

Authors:
Anastasia I. Zhuk, PhD (physics and mathematics), docent; associate professor at the department of higher mathematics, faculty of electronic informational systems.
aizhuk85@mail.ru
Helena N. Zashchuk, PhD (physics and mathematics), docent; associate professor at the department of higher mathematics, faculty of electronic informational systems.
shvichkina@tut.by

SYSTEMS OF EQUATIONS IN DIFFERENTIALS WITH GENERALISED DERIVATIVES OF CONTINUOUS FUNCTIONS

A. I. ZHUK [a]**,** ***H. N. ZASHCHUK*** [a]

[a] *Brest State Technical University, 267 Maskowskaja Street, Brest 224023, Belarus*
Corresponding author: A. I. Zhuk (aizhuk85@mail.ru)

Abstract. Herein, we investigate systems of nonautonomous differential equations with generalised coefficients using the algebra of new generalised functions. We consider a system of nonautonomous differential equations with generalised coefficients as a system of equations in differentials in the algebra of new generalised functions. The solution of such a system is a new generalised function. It is shown that the different interpretations of the solutions of the given systems can be described by a unique approach of the algebra of new generalised functions. In this paper, for the first time in the literature, we describe associated solutions of the system of nonautonomous differential equations with continuous generalised coefficients in the space $L(T)$.

Keywords: algebra of new generalised functions; differential equations with generalised coefficients; functions of finite variation.

Introduction

In this paper, we will consider the following system of equations with generalised coefficients on $T \in [0, a] \subset R$:

$$\dot{x}^i(t) = \sum_{j=1}^{q} f^{ij}(t, x(t))\dot{L}^j(t),\ i = \overline{1,\ p}, \tag{1}$$

$$x(0) = x_0, \tag{2}$$

where $f^{ij},\ i = \overline{1,\ p},\ j = \overline{1,\ q}$, are some functions; $x(t) = \left[x^1(t),\ x^2(t),\ \ldots,\ x^p(t)\right]$ and $\dot{L}^j(t),\ j = \overline{1,\ q}$, are a continuous function of finite variation on *T*. $\dot{L}^j(t)$ are derivatives in the distributional sense or we can say that $\dot{L}^j(t)$ are derivatives in the Schwartz space. In general, since $\dot{L}^j(t)$ is the distribution and $f^{ij}(t, x(t))$ not smooth functions, the products $f^{ij}(t, x(t))\dot{L}^j(t)$ are not well defined and the solution of system (1) essentially depends on the interpretation. System (1) can describe the model of the rocket flight process or the model of the control problems with impulse actions. Let us recall some approaches to the interpretation of system (1).

The first approach is concerned with considering the system of equations in the framework of the distribution theory. According to this approach, once the product of distributions from some classes is defined, then one tries to find the solution of the system of equations (1) in these classes of distributions. For example, in papers [1; 2] the product of some distributions and discontinuous functions was defined. See also monograph [3] for another definition. Notice that the solutions of system (1) obtained using the products from [1–3] are different.

The second approach is to interpret system (1) as the following system of integral equations:

$$x^i(t) = x_0^i + \sum_{j=1}^{q} \int_0^t f^{ij}(s,\ x(s))\dot{L}^j(s),\ i = \overline{1,\ p},$$

where the integrals are understood in the Lebesgue – Stieltjes, Perron – Stieltjes sense, etc. [4; 5]. But in this approach the solution of the system of integral equations depends on the interpretation of the integral and the definition of the functions $x^i(t)$ in the discontinuity points of $L^j(t)$.

The third approach is based on the idea of the approximation of the solution of system (1) by the solutions of the system of ordinary differential equations, which are constructed using the smooth approximation of the functions $L^j(t)$. In monograph [3], it is shown that in this case the limit of the solutions of the smoothed equations exists.

In this paper, we will consider system of equations (1) using the algebra of new generalised functions from [6]. Thus we will interpret system of equations (1) as a system of equations in the differentials in the algebra of new generalised functions. Such interpretation says that the solution of system (1) is a new generalised function.

We investigate the system of nonlinear differential equations, the coefficients of which are generalised derivatives of the continuous function of finite variation $L^j(t)$. In previous papers [7–11] the general view of system (1) was considered. The coefficients in such systems are generalised derivatives of arbitrary functions of finite variation $L^j(t)$. Using the given sequence of numbers $h_n \to 0$ we construct a sequence of approximating equations, and the generalised solution is defined as the limit of a sequence of the solutions of approximating equations. It is found that generalised solution exists only under some additional conditions for the behaviour of the sequence h_n in the case of discontinuous functions $L^j(t)$ and different generalised solutions exist for different sequences h_n.

The main purpose of this article is to prove that the generalised solution exists for all sequences $h_n \to 0$, it is the solution of system of integral equations (1) and it is independent of the choice of h_n in the case of continuous functions $L^j(t)$.

The algebra of new generalised functions

In this section, we recall the definition of the algebra of new generalised functions from [6]. First, we define an extended real line $\tilde{\mathfrak{R}}$ using a construction typical for nonstandard analysis. Let $\bar{\mathfrak{R}} = \left\{(x_n)_{n=1}^{\infty} : x_n \in R, \forall n \in N\right\}$ be a set of real sequences. We call two sequences $\{x_n\} \in \bar{\mathfrak{R}}$ and $\{y_n\} \in \bar{\mathfrak{R}}$ equivalent if there is a natural number N such that $x_n = y_n$ for all $n > N$. The set $\tilde{\mathfrak{R}}$ of equivalence classes is called the extended real line, and any of the classes is named as a generalised real number.

It follows that $R \subset \tilde{\mathfrak{R}}$ as one may associate a class containing a stationary sequence with $x_n = x$ with any ordinary number $x \in R$. The product $\tilde{x}\tilde{y}$ of two generalised real numbers is defined as the class of sequences equivalent to the sequence $\{x_n y_n\}$, where $\{x_n\}$ and $\{y_n\}$ are the arbitrary representatives of the classes $\tilde{x}$ and $\tilde{y}$, respectively. It is evident that $\tilde{\mathfrak{R}}$ is an algebra. For any segment $T = [0, a] \subset R$ one can construct an extended segment $\tilde{T}$ in a similar way.

Consider the set of sequences of infinity differentiable functions $\{f_n(x)\}$ on R. We name two sequences $\{f_n(x)\}$ and $\{g_n(x)\}$ equivalent if for each compact set $K \subset R$ there is a natural number N such that $f_n(x) = g_n(x)$ for all $n > N$ and $x \in K$. The set of classes of equivalent functions is denoted by $\Im(R)$ and its elements are called new generalised functions. Similarly, one can define the space $\Im(T)$ for any interval $T = [0, a]$. If we endow all these spaces with the natural operations of addition and multiplication they become algebras.

For each distribution f we can construct a sequence $\{f_n\}$ of smooth functions such that f_n converges to f (i. e., one can consider the convolution of f with some δ-sequence). This sequence defines the new generalised function that corresponds to the distribution f. Thus the space of distribution is a subset of the algebra of new generalised functions. However, in this settings, the infinite set of new generalised functions corresponds to one distribution (for example, by taking a different δ-sequence). We will say that the new generalised function $\tilde{f} = [\{f_n\}]$ is associated with a function f from some topological space if f_n converges to f in this space.

Let $\tilde{f} = [\{f_n\}]$ and $\tilde{g} = [\{g_n\}]$ be new generalised functions. Then there is a composition defined by $\tilde{f} \circ \tilde{g} = [\{f_n(g_n(x))\}] \in \Im(R)$. In the same way, one can define the value of the new generalised function $\tilde{f}$ at the generalised real point $\tilde{x} = [\{x_n\}] \in \tilde{\mathfrak{R}}$ as $\tilde{f}(\tilde{x}) = [\{f_n(x_n)\}]$.

Let H denote the subset of $\tilde{\mathfrak{R}}$ of nonnegative «infinitely small numbers»:

$$H = \left\{\tilde{h} \in \tilde{\mathfrak{R}} : \tilde{h} = [\{h_n\}],\ h_n > 0, \forall n \in N,\ \lim_{n \to \infty} h_n = 0\right\}.$$

For each $\tilde{h} = [\{h_n\}] \in H$ and $\tilde{f} = [\{f_n\}] \in \Im(R)$ we define a differential $d_{\tilde{h}}\tilde{f} \in \Im(R)$ by $d_{\tilde{h}}\tilde{f} = [\{f_n(x + h_n) - f_n(x)\}]$. The construction of the differential was proposed by N. V. Lazakovich [6].

Main results

In this section, we will formulate the main results of this paper. The proof of the theorem will be given in the next section.

Using the introduced algebras we can now give an interpretation of system of equations (1). We replace ordinary functions in system (1) by the corresponding new generalised functions and then write the system of equations in differentials in the algebra $\Im(R)$. So we have

$$d_{\tilde{h}}\tilde{x}^i(\tilde{t}) = \sum_{j=1}^{q} \tilde{f}^{ij}(\tilde{t}, \tilde{x}(\tilde{t}))\, d_{\tilde{h}}\tilde{L}^j(\tilde{t}),\ i = \overline{1, p}, \tag{3}$$

with the initial value $\tilde{x}\Big|_{[\tilde{0},\,\tilde{h})} = \tilde{x}_0$, where $\tilde{h} = [\{h_n\}] \in H$, $\tilde{t} = [\{t_n\}] \in T$, $\tilde{x} = [\{x_n\}]$, $\tilde{f} = [\{f_n\}]$, $\tilde{x}_0 = [\{x_{0n}\}]$ and $\tilde{L} = [\{L_n\}]$ are elements of $\Im(R)$. Moreover $\tilde{f}$ and $\tilde{L}$ are associated with f and L, respectively. If $\tilde{x}$ is associated with some (generalised) function x then we say that x is a solution of system (1).

It was shown in [12] that under some minor restrictions on the initial conditions there exists a unique solution of (3). The purpose of the present article is to investigate when the solution $\tilde{x}$ (3) converges to some ordinary function and to describe all possible limits.

Let $L^j(t)$, $j = \overline{1, q}$, $t \in T = [0, a]$, be a continuous function of finite variation. We will assume that $L^j(t) = L(a)$ if $t > a$ and $L^j(t) = L(0)$ if $t < 0$. Denote the total variation of the function $L = [L^1, L^2, \ldots, L^q]$ on the interval T by $\operatorname*{var}_{u \in T} L(u) = \sum_{j=1}^{q} \operatorname*{var}_{u \in T} L^j(u)$. A continuous function f is said to be Lipschitz continuous function with respect to its second variable $x \in R$ if there exists a constant $M > 0$ and for all $x_1, x_2 \in R$ and $t \in T$:

$$|f(t, x_1) - f(t, x_2)| \le M|x_1 - x_2|. \quad (4)$$

In this paper, we consider specific types of representatives of the new generalised functions. We take the following convolutions with δ-sequence as representatives of $\tilde{L}$ from system (3):

$$L_n^j(t) = (L^j * \rho_n)(t) = \int_0^{\frac{1}{n}} L^j(t+s)\rho_n(s)\,ds,$$

where $\rho_n \in C^\infty(R)$, $\rho_n \ge 0$, $\operatorname{sup}\rho_n \subseteq \left[0, \frac{1}{n}\right]$, $\int_0^{\frac{1}{n}} \rho_n(s)\,ds = 1$ and $f_n = f * \tilde{\rho}_n$, $\tilde{\rho} \ge 0$, $\tilde{\rho}_n(x_0, x_1, \ldots, x_p) = n^{p+1}\tilde{\rho} \times (nx_0, nx_1, \ldots, nx_p)$, $\tilde{\rho} \in C^\infty(R^{p+1})$, $\int_{\left[0, \frac{1}{n}\right]^{p+1}} \tilde{\rho}(x_0, \ldots, x_p)\,dx_0 \ldots dx_p = 1$ and $\operatorname{sup}\tilde{\rho} \subseteq \left[0, \frac{1}{n}\right]^{p+1}$.

By using representatives we can rewrite system (3) as follows:

$$x_n^i(t+h_n) - x_n^i(t) = \sum_{j=1}^{q} f_n^{ij}(t, x_n(t))\left[L_n^j(t+h_n) - L_n^j(t)\right], \; i = \overline{1, p}, \quad (5)$$

$$x_n(t)\Big|_{[0,\, h_n)} = x_{0n}(t). \quad (6)$$

The solution of (5) is constructed inductively starting from the interval $[0, h_n)$ where the initial conditions are given. Let t be an arbitrary point of T. There exist $m_t \in N$ and $\tau_t \in [0, h_n)$ such that $t = \tau_t + m_t h_n$. Set $t_k = \tau_t + kh_n$ for $k = 0, 1, \ldots, m_t$. Then the solution of system (5) can be written as

$$x_n^i(t) = x_{0n}^i(\tau_t) + \sum_{j=1}^{q} \sum_{k=0}^{m_t - 1} f_n^{ij}(t_k, x_n(t_k))\left[L_n^j(t_{k+1}) - L_n^j(t_k)\right], \; i = \overline{1, p}. \quad (7)$$

Thus, we will understand the associated solution of (3) as a solution of the system of nonautonomous differential equations (1), (2). Therefore, we have to investigate a limiting behaviour of (5), (6).

In order to describe the limits of the sequence x_n, we consider the following system of integral equations:

$$x^i(t) = x_0^i + \sum_{j=1}^{q} \int_0^t f^{ij}(s, x(s))\,dL^j(s), \; i = \overline{1, p}. \quad (8)$$

Here and below all integrals are understood in the Lebesgue – Stieltjes sense. As it was shown in [13] there exists a unique solution of (8) if f is Lipschitz continuous function.

The following theorem from [12] gives necessary and sufficient conditions for the existence and uniqueness of the solutions of system (3).

Theorem 1. *If the following equality holds for some representatives* $\{f_n^{ij}\} \in \tilde{f}^{ij}$, $\{L_n^j\} \in \tilde{L}^j$, $\{x_n^i\} \in \tilde{x}^i$, $\{x_{0n}^i\} \in \tilde{x}_0^i$, *for all sufficiently large* $n \in N$ *and for all* $l = 0, 1, \ldots$*:*

$$\lim_{t\to 0^+}\left(\frac{d^l}{dt^l}\left[x_{0n}^i(h_n - t) - x_{0n}^i(t)\right] - \sum_{j=1}^{q}\frac{d^l}{dt^l}\left[f_n^{ij}(t,\ x_{0n}(t))\left[L_n^j(h_n + t) - L_n^j(t)\right]\right]\right) = 0$$

then a solution of system (3) exists and it is unique.

Lemma 1. *Let for all* $n = 0, 1, 2, \ldots$ *the following inequality holds:*

$$Z_{n+1} \le A + \sum_{k=1}^{n} A_k + \sum_{k=1}^{n} B_k Z_k, \tag{9}$$

where A, A_k, B_k *are some nonnegative constants and* $Z_k \ge 0$, $k = 1, 2, \ldots, n$. *Then*

$$Z_{n+1} \le \left(A + \sum_{k=1}^{n} A_k\right)\exp\left(\sum_{k=1}^{n} B_k\right). \tag{10}$$

P r o o f. Let us successively apply inequality (9):

$$Z_{n+1} \le A + \sum_{k=1}^{n} A_k + \sum_{k=1}^{n-1} B_k Z_k + B_n Z_n \le$$

$$\le A + \sum_{k=1}^{n} A_k + B_n\left(A + \sum_{k=1}^{n-1} A_k + \sum_{k=1}^{n-1} B_k Z_k\right) + \sum_{k=1}^{n-1} B_k Z_k =$$

$$= A + \sum_{k=1}^{n} A_k + B_n\left(A + \sum_{k=1}^{n-1} A_k\right) + (B_n + 1)\sum_{k=1}^{n-1} B_k Z_k \le \ldots$$

$$\ldots \le A + \sum_{k=1}^{n} A_k + B_n\left(A + \sum_{k=1}^{n-1} A_k\right) + B_{n-1}(B_n + 1)\left(A + \sum_{k=1}^{n-2} A_k\right) + \ldots$$

$$\ldots + B_1\Big(1 + B_2\Big(1 + \ldots\big(1 + B_{n-1}(1 + B_n)\big)\Big)\Big)A \le \left(A + \sum_{k=1}^{n} A_k\right)\prod_{k-1}^{n}(1 + B_n).$$

Let $Z_{n+1} = 0$ then inequality (10) holds, if $A + \sum_{k=1}^{n} A_k = 0$. In other cases we take the logarithm left and right parts of the chain of inequalities

$$\ln Z_{n+1} \le \ln\left(\left(A + \sum_{k=1}^{n} A_k\right)\prod_{k-1}^{n}(1 + B_n)\right) =$$

$$= \ln\left(A + \sum_{k=1}^{n} A_k\right) + \sum_{k=1}^{n}\ln(1 + B_n) \le \ln\left(A + \sum_{k=1}^{n} A_k\right) + \sum_{k=1}^{n} B_n.$$

Then we obtain $Z_{n+1} \le \left(A + \sum_{k=1}^{n} A_k\right)e^{\sum_{k=1}^{n} B_n}$. The proof is completed.

Lemma 2. *Let the function* f *be Lipschitz continuous with constant M satisfying (4). Then for the solutions* x *and* x_n *of (8) and (5), respectively, the following inequalities hold for all* $t, t_1 \in T$, $t > t_1$ *and* $l, n \in N$.

1. $|x(t)| \le C(1 + |x_0|)$, $|x_n(t)| \le C(1 + |x_{n0}(\tau_t)|)$, *where the constant C depends only on M,* $|T|$ *and* $\underset{u\in[a,b]}{\operatorname{var}} L(u)$.

2. $|x(t) - x(t_1)| \le M \underset{u\in[t_1,t]}{\operatorname{var}} L(u)$.

3. $|x_n(t + lh_n) - x_n(t)| \le M \underset{u\in\left[t,\ t + lh_n + \frac{1}{n}\right]}{\operatorname{var}} L(u)$.

P r o o f. Let us prove the first inequality. For this purpose we will consider

$$|x(t)| \le |x_0| + \int_0^t |f(s,\ x(s))|\, d\underset{u\in[0,s]}{\operatorname{var}} L(u) \le |x_0| + M\underset{s\in[0,t]}{\operatorname{var}} L(s) \le C(1 + |x_0|).$$

If we rewrite the solution of system (5) in the vector form, we will get

$$\left|x_n(t)\right| = \left|x_{n0}(\tau_t) + \sum_{k=0}^{m_t-1} f_n\left(t_k, x_n(t_k)\right)\left[L_n(t_{k+1}) - L_n(t_k)\right]\right| \le \left|x_{n0}(\tau_t)\right| +$$

$$+\sum_{k=0}^{m_t-1}\left|f_n\left(t_k, x_n(t_k)\right)\left[L_n(t_{k+1}) - L_n(t_k)\right]\right| \le \left|x_{n0}(\tau_t)\right| + M \operatorname*{var}_{s\in[0,t]} L(s) = C\left(1+\left|x_{n0}(\tau_t)\right|\right),$$

where we used the equality $\operatorname*{var}_{u\in T} L_n(u) = \operatorname*{var}_{u\in T} L(u)$, derived from the definition L_n.

In the same way, we will get the second inequality

$$\left|x(t) - x(t_1)\right| = \left|x_0 + \int_0^t f(s, x(s))\,dL(s) - x_0 - \int_0^{t_1} f(s, x(s))\,dL(s)\right| =$$

$$= \left|\int_{t_1}^t f(s, x(s))\,dL(s)\right| \le M \operatorname*{var}_{u\in[t_1, t]} L(u).$$

It follows the third inequality from the second one

$$\left|x_n(t + lh_n) - x_n(t)\right| \le M \operatorname*{var}_{u\in[t, t+lh_n]} L_n(u) \le M \operatorname*{var}_{u\in\left[t, t+lh_n+\frac{1}{n}\right]} L(u),$$

where we used the inequality $\operatorname*{var}_{u\in[a,b]} L_n(u) \le \operatorname*{var}_{u\in\left[a, b+\frac{1}{n}\right]} L(u)$, derived from the definition L_n. The proof is completed.

Then we will define the module of the $x = \left[x^1, x^2, \ldots, x^p\right]^T$ as $|x| = \sum_{i=1}^p \left|x^i\right|$ and module of $(p\times q)$-matrix as $|f| = \sum_{i=1}^p\sum_{j=1}^q \left|f^{ij}\right|$. We denote variation of the vector-function $L = \left[L^1, L^2, \ldots, L^q\right]$ on A by $\operatorname*{var}_{u\in A} L(u) = \sum_{j=1}^q \operatorname*{var}_{u\in A} L^j(u)$, $\operatorname*{var}_{u\in[a,b)} L(u) = \lim_{\varepsilon\to 0^+} \operatorname*{var}_{u\in[a, b-\varepsilon]} L(u)$.

Theorem 2. *Let f^{ij}, $i=\overline{1, p}$, $j=\overline{1, q}$, be Lipschitz continuous functions satisfying (4) and $L^j(t)$, $j=\overline{1, q}$, be continuous functions of finite variation. Suppose that $\frac{1}{h_n}\int_0^{h_n}\left|x_{n0}(\tau_t) - x_0\right|dt \to 0$ as $n\to\infty$, $h_n\to 0$ for all $t\in T$, then the solution $x_n(t)$ of (5), (6) converges to the solution $x(t)$ from (8) in the Lebesgue space $L_1(T)$.*

Proof of the main theorem

In this section we will prove theorem 2.

P r o o f. Let fix $t\in T$, then using (7) we obtain

$$\left|x_n^1(t) - x^1(t)\right| = \left|x_{n0}^1(\tau_t) - x_0^1 + \sum_{j=1}^q\sum_{k=0}^{m_t-1} f_n^{1j}\left(t_k, x_n(t_k)\right)\left[L_n^j(t_{k+1}) - L_n^j(t_k)\right] -\right.$$

$$\left.-\sum_{j=1}^q\int_0^t f^{1j}(s, x(s))\,dL^j(s)\right| \le \left|x_{n0}^1(\tau_t) - x_0^1\right| + \sum_{j=1}^q\left|\int_0^{\tau_t} f^{1j}(s, x(s))\,dL^j(s)\right| +$$

$$+\sum_{j=1}^q\left|\sum_{k=0}^{m_t-1} f_n^{1j}\left(t_k, x_n(t_k)\right)\left[L_n^j(t_{k+1}) - L_n^j(t_k)\right]\right| +$$

$$+\sum_{j=1}^q\left|\sum_{k=0}^{m_t-1} f^{1j}\left(t_k, x(t_k)\right)\left[L_n^j(t_{k+1}) - L_n^j(t_k)\right] - \sum_{k=0}^{m_t-1} f^{1j}\left(t_k, x(t_k)\right)\left[L^j(t_{k+1}) - L^j(t_k)\right]\right| +$$

$$+\sum_{j=1}^q\left|\sum_{k=0}^{m_t-1} f^{1j}\left(t_k, x(t_k)\right)\left[L^j(t_{k+1}) - L^j(t_k)\right] - \int_{\tau_t}^t f^{1j}(s, x(s))\,dL^j(s)\right| =$$

$$= I_0(t) + \sum_{j=1}^{q}\left(I_1^j(t) + I_2^j(t) + I_3^j(t) + I_4^j(t)\right).$$

We denote the constant C which depends only on M, $|T|$. It does not depend on n, h_n, and $t \in T$ and its value can change from one formula to another. Since f^{ij} are Lipschitz continuous functions, satisfying (4) and bounded for $i = \overline{1, p}$, $j = \overline{1, q}$, functions $L^j(t)$, $j = \overline{1, q}$, are continuous functions of finite variation, then $I_1^j(t) \leq C \operatorname*{var}_{t \in [0, h_n)} L^j(t)$.

Using the view of f_n^{ij} and inequality (4), we obtain

$$I_2^j(t) = \left|\sum_{k=0}^{m_t - 1}\left(f_n^{1j}(t_k, x_n(t_k)) - f^{1j}(t_k, x(t_k))\right)\left[L_n^j(t_{k+1}) - L_n^j(t_k)\right]\right| \leq$$

$$\leq C\sum_{k=0}^{m_t - 1}\left(|x_n(t_k) - x(t_k)| + \frac{1}{n}\right)\left|L_n^j(t_{k+1}) - L_n^j(t_k)\right|.$$

To estimate term $I_3^j(t)$, $j = \overline{1, p}$, we will divide the sum into two parts. Then we will make replacement of indexes of summation in the first part. We will use (4), the view of $x(t)$ and lemma 2:

$$I_3^j(t) = \left|\sum_{k=0}^{m_t - 1} f^{1j}(t_k, x(t_k))\left[L_n^j(t_{k+1}) - L_n^j(t_k)\right] - \sum_{k=0}^{m_t - 1} f^{1j}(t_k, x(t_k))\left[L^j(t_{k+1}) - L^j(t_k)\right]\right| =$$

$$= \left|\sum_{k=0}^{m_t - 1} f^{1j}(t_k, x(t_k))\left[L_n^j(t_{k+1}) - L^j(t_{k+1})\right] - \sum_{k=0}^{m_t - 1} f^{1j}(t_k, x(t_k))\left[L_n^j(t_k) - L^j(t_k)\right]\right| =$$

$$= \left|\sum_{k=1}^{m_t} f^{1j}(t_{k-1}, x(t_{k-1}))\left[L_n^j(t_k) - L^j(t_k)\right] - \sum_{k=0}^{m_t - 1} f^{1j}(t_k, x(t_k))\left[L_n^j(t_k) - L^j(t_k)\right]\right| =$$

$$= \left|\sum_{k=1}^{m_t - 1}\left(f^{1j}(t_{k-1}, x(t_{k-1})) - f^{1j}(t_k, x(t_k))\right)\left[L_n^j(t_k) - L^j(t_k)\right] + f^{1j}(t_{m_t - 1}, x(t_{m_t - 1})) \times\right.$$

$$\left.\times\left[L_n^j(t_{m_t}) - L^j(t_{m_t})\right] - f^{1j}(\tau_t, x(\tau_t))\left[L_n^j(\tau_t) - L^j(\tau_t)\right]\right| \leq$$

$$\leq C\sum_{k=1}^{m_t - 1}\operatorname*{var}_{t \in \left[t_k, t_k + \frac{1}{n}\right]} L^j(t)\left(|x(t_{k-1}) - x(t_k)| + h_n\right) + C\operatorname*{var}_{t \in \left[\tau_t, \tau_t + \frac{1}{n}\right]} L^j(t) + C\operatorname*{var}_{t \in \left[t_{m_t}, t_{m_t} + \frac{1}{n}\right]} L^j(t) \leq$$

$$\leq C\max_{\substack{t_1, t_2 \in T, \\ |t_1 - t_2| \leq \frac{1}{n}}}\operatorname*{var}_{t \in [t_1, t_2]} L^j(t) + Ch_n.$$

Denote $\hat{s}(s) = t_k$, $s \in [t_k, t_{k+1})$, and using the properties of Stieltjes integral we have

$$I_4^j(t) = \left|\sum_{k=0}^{m_t - 1} f^{1j}(t_k, x(t_k))\left[L^j(t_{k+1}) - L^j(t_k)\right] - \int_{\tau_t}^{t} f^{1j}(s, x(s))\,dL^j(s)\right| =$$

$$= \left|\int_{\tau_t}^{t}\left[f^{1j}(\hat{s}(s), x(\hat{s}(s))) - f(s, x(s))\right]dL^j(s)\right| \leq C\sum_{k=1}^{m_t - 1}\left(\left(\operatorname*{var}_{t \in [t_k, t_{k+1}]} x(t) + h_n\right)\operatorname*{var}_{t \in [t_k, t_{k+1}]} L^j(t)\right) \leq$$

$$\leq C\max_{\substack{t_1, t_2 \in T, \\ |t_1 - t_2| \leq \frac{1}{n} + h_n}}\operatorname*{var}_{t \in [t_1, t_2]} L^j(t)\operatorname*{var}_{t \in T} x(t) + Ch_n \leq C\max_{\substack{t_1, t_2 \in T, \\ |t_1 - t_2| \leq \frac{1}{n} + h_n}}\operatorname*{var}_{t \in [t_1, t_2]} L^j(t) + Ch_n.$$

Then

$$\left|x_n^1(t)-x^1(t)\right|\le\left|x_{n0}^1(\tau_t)-x_0^1\right|+C\sum_{j=1}^{q}\sum_{k=1}^{m_t-1}\operatorname*{var}_{t\in[0,h_n)}L^j(t)+C\sum_{j=1}^{q}\sum_{k=0}^{m_t-1}\left(\left|x_n(t_k)-x(t_k)\right|+\frac{1}{n}\right)\times$$

$$\times\left|L_n^j(t_{k+1})-L_n^j(t_k)\right|+C\sum_{j=1}^{q}\max_{\substack{t_1,t_2\in T,\\ |t_1-t_2|\le\frac{1}{n}+h_n}}\operatorname*{var}_{t\in[t_1,t_2]}L^j(t)+Ch_n.$$

We obtain similar inequality for others x_n^i, $i=\overline{1,\ p}$:

$$\left|x_n^i(t)-x^i(t)\right|\le\left|x_{n0}^i(\tau_t)-x_0^i\right|+C\sum_{j=1}^{q}\sum_{k=1}^{m_t-1}\operatorname*{var}_{t\in[0,h_n)}L^j(t)+C\sum_{j=1}^{q}\sum_{k=0}^{m_t-1}\left(\left|x_n(t_k)-x(t_k)\right|+\frac{1}{n}\right)\times$$

$$\times\left|L_n^j(t_{k+1})-L_n^j(t_k)\right|+C\sum_{j=1}^{q}\max_{\substack{t_1,t_2\in T,\\ |t_1-t_2|\le\frac{1}{n}+h_n}}\operatorname*{var}_{t\in[t_1,t_2]}L^j(t)+Ch_n.$$

Combining the last inequality we have

$$\left|x_n(t)-x(t)\right|\le\sum_{i=1}^{p}\left|x_{n0}^i(\tau_t)-x_0^i\right|+C\sum_{j=1}^{q}\sum_{k=1}^{m_t-1}\operatorname*{var}_{t\in[0,h_n)}L^j(t)+C\sum_{j=1}^{q}\sum_{k=0}^{m_t-1}\left(\left|x_n(t_k)-x(t_k)\right|+\frac{1}{n}\right)\times$$

$$\times\left|L_n^j(t_{k+1})-L_n^j(t_k)\right|+C\sum_{j=1}^{q}\max_{\substack{t_1,t_2\in T,\\ |t_1-t_2|\le\frac{1}{n}+h_n}}\operatorname*{var}_{t\in[t_1,t_2]}L^j(t)+Ch_n.$$

Lemma 1 implies the following estimation for the last equation:

$$\left|x_n(t)-x(t)\right|\le\left|x_{n0}(\tau_t)-x_0\right|+C\sum_{j=1}^{q}\operatorname*{var}_{t\in[0,h_n)}L^j(t)+C\sum_{j=1}^{q}\max_{\substack{t_1,t_2\in T,\\ |t_1-t_2|\le\frac{1}{n}+h_n}}\operatorname*{var}_{t\in[t_1,t_2]}L^j(t)+Ch_n.$$

Then

$$\int_T\left|x_n(t)-x(t)\right|dt\le\int_T\left|x_{n0}(\tau_t)-x_0\right|dt+C\int_T\operatorname*{var}_{t\in[0,h_n)}L(t)dt+$$

$$+C\int_T\max_{\substack{t_1,t_2\in T,\\ |t_1-t_2|\le\frac{1}{n}+h_n}}\operatorname*{var}_{t\in[t_1,t_2]}L(t)dt+C\int_T h_n dt$$

and

$$\int_T\left|x_n(t)-x(t)\right|dt\le\int_T\left|x_{n0}(\tau_t)-x_0\right|dt+C\operatorname*{var}_{t\in[0,h_n)}L(t)+C\max_{\substack{t_1,t_2\in T,\\ |t_1-t_2|\le\frac{1}{n}+h_n}}\operatorname*{var}_{t\in[t_1,t_2]}L(t)+Ch_n.$$

Let $n\to\infty$, $h_n\to 0$. Since $L^j(t)$, $j=\overline{1,\ q}$, are continuous on the T and therefore are uniform continuous on it we have $\frac{1}{h_n}\int_0^{h_n}\left|x_{n0}(\tau_t)-x_0\right|dt\to 0$. The proof is completed.

Similar results for the system of autonomous differential equations in other space were obtained in [9; 14].

Theorem 3. *Under the condition of theorem 1, let* f^{ij}, $i=\overline{1,\ p}$, $j=\overline{1,\ q}$, *be Lipschitz continuous functions satisfying (4) and* $L^j(t)$ *are continuous functions of finite variation* ($j=\overline{1,\ q}$). *Suppose that* $\frac{1}{h_n}\int_0^{h_n}\left|x_{n0}(\tau_t)-x_0\right|dt\to 0$ *than the associated solution of (3) is the solution of (8) as* $n\to\infty$, $h_n\to 0$.

P r o o f. This is immediate consequence of the definition of the associated solution of (3) and theorem 1.

Thus, the class of the integral equation to which the associated solutions of systems of the nonautonomous differential equations satisfy, containing the generalised derivatives of continuous functions, is described by system (8).

It should also be noted that from the proof of the theorems in the case of discontinuous functions $L^j(t)$ the proof of the main theorem of this article doesn't follow, because the associated solutions of (3) are obtained in a different topological space.

It was shown in [6; 8; 11] that the solution of system (1) in the sense of the integral and approximate approaches can be obtained from the solution of the system in differentials in the algebra of new generalised functions.

References

1. Antosik P, Legeza J. Products of measures and functions of finite variations. In: *Proceedings of the International conference on generalized functions and operational calculus; 29 September – 6 October 1975; Varna, Bulgaria*. Sophia: Bulgarian Academy of Sciences; 1979. p. 20–26.

2. Pandit SG, Deo SG. *Differential systems involving impulses*. Berlin: Springer; 1982. 106 p. (Lecture notes in mathematics; volume 954). DOI: 10.1007/BFb0067476.

3. Zavalishchin ST, Sesekin AN. *Dynamic impulse systems: theory and applications*. Dordrecht: Kluwer Academic Publishers; 1997. 271 p. (Mathematics and its applications; volume 394).

4. Das PC, Sharma RR. Existence and stability of measure differential equations. *Czechoslovak Mathematical Journal*. 1972; 22(1):145–158. DOI: 10.21136/cmj.1972.101082.

5. Ligeza J. On generalized solutions of some differential nonlinear equations of order *n*. *Annales Polonici Mathematici*. 1975; 31(2):115–120. DOI: 10.4064/ap-31-2-115-120.

6. Лазакович НВ. Стохастические дифференциалы в алгебре обобщенных случайных процессов. *Доклады Академии наук БССР*. 1994;38(5):23–27.

7. Жук АИ, Хмызов АК. Системы квазидифференциальных уравнений в прямом произведении алгебр мнемофункций. Симметрический случай. *Вестник БГУ. Серия 1, Физика. Математика. Информатика.* 2010;2:87–93.

8. Жук АИ, Яблонский ОЛ. Оценки скорости сходимости к ассоциированным решениям дифференциальных уравнений с обобщенными коэффициентами в алгебре мнемофункций. *Доклады Национальной академии наук Беларуси.* 2015;59(2):17–22.

9. Жук АИ, Яблонский ОЛ. Неавтономные системы дифференциальных уравнений в алгебре обобщенных функций. *Труды Института математики.* 2011;19(1):43–51.

10. Жук АИ, Яблонский ОЛ, Спасков СА. Ассоциированные решения системы неавтономных дифференциальных уравнений с обобщенными коэффициентами. Смешанный случай. *Весці БДПУ. Серыя 3, Фізіка. Матэматыка. Інфарматыка. Біялогія. Геаграфія.* 2019;4:16–22.

11. Zhuk AI, Zashchuk HN. On associated solutions of the system of non-autonomous differential equations in the Lebesgue spaces. *Journal of the Belarusian State University. Mathematics and Informatics*. 2022;1:6–13. DOI: 10.33581/2520-6508-2022-1-6-13.

12. Каримова ТИ, Яблонский ОЛ. Об ассоциированных решениях нестационарных систем уравнений в дифференциалах в алгебре обобщенных случайных процессов. *Вестник БГУ. Серия 1, Физика. Математика. Информатика.* 2009;2:81–86.

13. Groh J. A nonlinear Volterra – Stieltjes integral equation and a Gronwall inequality in one dimension. *Illinois Journal of Mathematics*. 1980;24(2):244–263. DOI: 10.1215/ijm/1256047720.

14. Жук АИ, Яблонский ОЛ. Системы дифференциальных уравнений в алгебре обобщенных функций. *Весці Нацыянальнай акадэміі навук Беларусі. Серыя фізіка-матэматычных навук.* 2011;1:12–16.

Received 12.03.2024 / revised 20.10.2024 / accepted 20.10.2024.