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Novel ASEP-inspired solutions of the Yang-Baxter equation

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Abstract
We explore the algebraic structure of a particular ansatz of the Yang-Baxter equation (YBE), which is inspired by the Bethe Ansatz treatment of the asymmetric simple exclusion process spin-model. Various classes of Hamiltonian density arriving from the two types of R-matrices are found, which also appear as solutions of the constant YBE. We identify the idempotent and nilpotent categories of such constant R-matrices and perform a rank-1 numerical search for the lowest dimension. A summary of the final results reveals general non-Hermitian spin-1/2 chain models.

Keywords: Yang-Baxter integrability, non-Hermitian physics, Exactly solvable systems

1. Introduction

In recent decades, significant advances have been made in our understanding of non-equilibrium classical and quantum systems, particularly in one dimension. A large part of this advance is based on the exact results related to integrability. These models are important in studies on integrable probability and interacting particle systems. One of the paradigmatic classes of models in this area is the asymmetric simple exclusion process (ASEP). It is an example of a solvable stochastic interface growth model, which gives rise to the

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Kardar–Parisi–Zhang equation [1–3]; see the surveys by [4]. Other integrable models with similar properties include the stochastic six-vertex model. The particular case of open ASEP is defined as the following: particles occupy sites in a finite chain $\{1,\ldots,N\}$ for some N, and jump to the left at rate q and to the right at rate p. Moreover, particles are inserted into site 1 at rate α and removed at rate γ, whereas at site N insertion occurs at rate δ and removal at rate β. Any moves which violate the rule of one particle per site at a given time are excluded. Such models have been used in various applications. The exactly solvable cases found in the 1990s [1] were generalised and extended further by many authors [5–9]. These models can be mapped to non-Hermitian spin chains and often have hidden algebraic structures such as Temperley-Lieb, Hecke [5], and q-deformed or more general quadratic algebras [6].

In this study, we extend the class of solvable ASEP models related to certain algebraic structures (satisfying hecke relations) and their corresponding spin chains. In our previous work [10], we focus on identifying non-regular Yang-Baxter equation (YBE) solutions. This work however extends the search into certain regular solutions that correspond with constant R-matrices. Some of these constant matrices are low-rank.

The structure of the paper is as follows—Starting from section 2, we derive the Hecke algebra from the YBE with the R-matrix ansatz (2). Necessary relations for symbolic evaluation of the solutions are identified. Section 3 details our computational workflow and results. It includes discussions about the findings. In section 4, we showcase a generalised t-J type Hubbard model satisfying the Hecke relations, which elucidates on the appearance of our results. Finally, we provide some discussions with [10] and draw the conclusions.

2. The R-matrix and an algebra

2.1. Yang Baxter equation

The focus in this paper is to study the YBE

$$\begin{aligned}&\mathcal{R}_{12}\left(f(u_1,u_2)\right)\mathcal{R}_{13}\left(f(u_1,u_3)\right)\mathcal{R}_{23}\left(f(u_2,u_3)\right)\\&\quad=\mathcal{R}_{23}\left(f(u_2,u_3)\right)\mathcal{R}_{13}\left(f(u_1,u_3)\right)\mathcal{R}_{12}\left(f(u_1,u_2)\right)\end{aligned}\tag{1}$$

where we parameterise the rapidities using a general function $f(x,y)$. To examine R-matrices similar to those of the ASEP model, we consider the following R-matrix ansatz:

$$\mathcal{R}\left(f(x,y)\right)=\mathcal{P}\left(I+f(x,y)\,\mathcal{M}\right)\tag{2}$$

where $\mathcal{M}$ is a square matrix with complex coefficients, and $\mathcal{P}$ is the transposition operator. By imposing $\lim_{y\to x}f(x,y)=0$, $\mathcal{R}$ satisfies the regularity condition.

We construct the below transfer matrix of N lattice sites

$$\tau\left(x,y\right)=\mathrm{Tr}_{\mathcal{A}}\left(\mathcal{R}_{0,1}\left(f(x,y)\right)\mathcal{R}_{0,2}\left(f(x,y)\right)\cdots\mathcal{R}_{0,N}\left(f(x,y)\right)\right)\tag{3}$$

where each $\mathcal{R}_{0,n}$ acts on $\mathcal{A}\otimes\mathcal{H}_n$. $\mathcal{H}_n$ is the Hilbert space for local site n. $\mathcal{A}$ is an auxiliary vector space isomorphic to $\mathcal{H}_n$. By considering the following ordering of limits $(y\to x, x\to 0)$, we calculate the first integral of motion:

$$\begin{aligned} T = \tau(0,0) &= \lim_{x\to 0}\lim_{y\to x}\mathrm{Tr}_{\mathcal{A}}\left(\mathcal{R}(f(x,y))_{0,1},\ldots,\mathcal{R}(f(x,y))_{0,N}\right) \\ &= \mathrm{Tr}_{\mathcal{A}}(\mathcal{P}_{0,1},\ldots,\mathcal{P}_{0,N}) \\ &= \mathrm{Tr}_{\mathcal{A}}(\mathcal{P}_{1,0})\mathcal{P}_{1,2},\ldots,\mathcal{P}_{1,N} \\ &= \mathcal{P}_{1,2},\ldots,\mathcal{P}_{1,N} \end{aligned} \tag{4}$$

which is the translation operator that satisfies $T^N = \mathbf{I}$. It generates translations in the periodic lattice. The first derivative with respect to x on τ reveals the second integral of motion:

$$H = \lim_{x\to 0}\lim_{y\to x}\sum_{k=1}^{N}\mathcal{P}_{k,k+1}\left(\frac{d(\mathcal{R}_{k,k+1}(f(x,y)))}{dx}\right), \tag{5}$$

which is the nearest-neighbouring Hamiltonian. By identifying $df(x,y)/dx$ with a constant α after taking the limits, the Hamiltonian simplifies as

$$H = \alpha\sum_{k=1}^{N}\mathcal{M}_{k,k+1}. \tag{6}$$

A key observation is that $\mathcal{M}$ represents the Hamiltonian density. Expanding (1) by substituting (2), and with further simplifications gives the constraint on $\mathcal{M}$

$$\begin{aligned} &(f_{12}+f_{23}-f_{13})(\mathcal{M}_{23}-\mathcal{M}_{12}) + f_{12}f_{23}\left(\mathcal{M}_{23}^2-\mathcal{M}_{12}^2\right) \\ &\quad + f_{12}f_{13}f_{23}(\mathcal{M}_{23}\mathcal{M}_{12}\mathcal{M}_{23}-\mathcal{M}_{12}\mathcal{M}_{23}\mathcal{M}_{12}) = 0 \end{aligned} \tag{7}$$

where $f_{ij} \equiv f(u_i,u_j)$.

2.2. Algebra of the Hamiltonian density

Assuming that the parameterisation $f(x,y)$ does not diverge as $y\to x$ and satisfies $\lim_{y\to x} f(x,y) = 0$, we consider taking all possible pairs of the spectral parameters $u_{1,2,3}$ in (7). The constraint disappears for $u_2\to u_1$ and $u_3\to u_2$. By taking $u_3\to u_1$, we find a nontrivial condition which is

$$\left(\mathcal{M}_{23}^2-\mathcal{M}_{12}^2\right) = \frac{(f_{12}+f_{21})}{f_{12}f_{21}}(\mathcal{M}_{12}-\mathcal{M}_{23}). \tag{8}$$

Since this expression must hold for every $u_{1,2}$, we impose the following:

$$\frac{(f_{12}+f_{21})}{f_{12}f_{21}} = \omega,\ \ \omega\in\mathbf{C}. \tag{9}$$

Using (8) in (7) and rearranging the expression

$$(\mathcal{M}_{23}\mathcal{M}_{12}\mathcal{M}_{23}-\mathcal{M}_{12}\mathcal{M}_{23}\mathcal{M}_{12}) = \frac{(f_{12}+f_{23}-f_{13}-\omega f_{12}f_{23})}{f_{12}f_{13}f_{23}}(\mathcal{M}_{23}-\mathcal{M}_{12}) \tag{10}$$

we find the second constraint that

$$\frac{1}{f_{12}f_{13}f_{23}}(f_{12}+f_{23}-f_{13}-\omega f_{12}f_{23}) = \kappa,\ \ \kappa\in\mathbf{C}. \tag{11}$$

In this paper we will consider the following parameterisation

$$f(x,y) = \frac{x-y}{\sum_{i,j=0}^{N} d_{ij} x^i y^j}, \tag{12}$$

for arbitrary complex coefficients d_{ij}, which automatically satisfies $\lim_{y\to x} f(x,y) = 0$. By using (9) and (11) we obtain

$$f(x,y) = \frac{x-y}{c_0^2 + c_0 c_1 (x+y) + c_1^2 xy + \omega x + \left(\frac{c_1}{c_0}\omega - \frac{\kappa}{c_0^2}\right) xy} \tag{13}$$

where c_0, c_1 are free complex constants. The calculations leading to this expression are given in appendix A.

We arrive at two constraints on $\mathcal{M}$, which are

$$\begin{aligned} &\left(\mathcal{M}_{23}^2 - \mathcal{M}_{12}^2\right) = \omega\left(\mathcal{M}_{12} - \mathcal{M}_{23}\right), \\ &\left(\mathcal{M}_{23}\mathcal{M}_{12}\mathcal{M}_{23} - \mathcal{M}_{12}\mathcal{M}_{23}\mathcal{M}_{12}\right) = \kappa\left(\mathcal{M}_{12} - \mathcal{M}_{23}\right). \end{aligned} \tag{14}$$

2.2.1. Generalising the constraints on $\mathcal{M}$. Extending (14) for arbitrary site indices $(i, i+1, i+2)$ from $(1,2,3)$ correspondingly and using $e_i \equiv \mathcal{M}_{i,i+1}$, we get

$$\begin{aligned} e_i^2 + \omega e_i &= e_{i+1}^2 + \omega e_{i+1}, \\ e_i e_{i+1} e_i + \kappa e_i &= e_{i+1} e_i e_{i+1} + \kappa e_{i+1}. \end{aligned} \tag{15}$$

The Hamiltonian H becomes $\alpha \sum_{i=1}^{N} e_i$ with $\alpha = c_0^{-2}$. From the first constraint, we note that

$$e_1^2 + \omega e_1 = e_2^2 + \omega e_2 \cdots = e_N^2 + \omega e_N \tag{16}$$

which is satisfied if $e_i^2 + \omega e_i = \lambda \mathbf{I}$ for some complex constant λ. Similarly, we impose

$$e_i e_{i+1} e_i + \kappa e_i = e_{i+1} e_i e_{i+1} + \kappa e_{i+1} \equiv t_{i,i+1} \tag{17}$$

where we define $t_{i,i+1}$ as a three site operator acting on $\mathcal{H}_i \otimes \mathcal{H}_{i+1} \otimes \mathcal{H}_{i+2}$ which is invariant under $i \leftrightarrow i+1$ exchange.

Finally the algebraic conditions that e_i must satisfy, starting from (7), are:

$$\begin{aligned} e_i^2 &= \lambda \mathbf{I} - \omega e_i, \\ e_i e_{i+1} e_i &= t_{i,i+1} - \kappa e_i, \\ e_{i+1} e_i e_{i+1} &= t_{i,i+1} - \kappa e_{i+1}. \end{aligned} \tag{18}$$

Up to appropriate identity shifts and multiplicative constants, (15) may be brought into standard Hecke Algebra. For different conditions on κ, λ and $t_{i,i+1}$, the one above indicates various hecke-algebraic structures that the generators need to fulfil.

2.3. Exploring the algebraic conditions

We can gather important aspects of each condition from (15). The first condition

$$e_i^2 + \omega e_i - \lambda \mathbf{I} = 0 \tag{19}$$

is also known as the eigenvalue problem. It can be expressed in the following factorised form

$$(e_i - \nu_+ \mathbf{I})(e_i - \nu_- \mathbf{I}) = \mathbf{0}, \;\; \nu_\pm = \frac{1}{2}(-\omega \pm c_\omega(\lambda)), \tag{20}$$

where we use $c_\omega(x) \equiv \sqrt{\omega^2 + 4x}$ as a shorthand. The second condition in (15) is the intertwining equation, which holds the important constraints arising from the YBE. By rewriting the generator e_i as $q_i + \beta\mathbf{I}$ with $\beta \in \mathbf{C}$, it is rewritten in the braid equation as

$$q_i q_{i+1} q_i = q_{i+1} q_i q_{i+1}, \tag{21}$$

after fixing β such that $\beta^2 + \beta\omega - \kappa = 0$. After selecting the positive branch of the quadratic root, the eigenvalue problem (20) is modified for q_i as follows:

$$\left(q_i + \frac{1}{2}(c_\omega(\kappa) + c_\omega(\lambda))\mathbf{I}\right)\left(q_i + \frac{1}{2}(c_\omega(\kappa) - c_\omega(\lambda))\mathbf{I}\right) = 0. \tag{22}$$

2.3.1. Solution classes. To identify the classes of (22), we consider the constraints on $c_\omega(\kappa)$ and $c_\omega(\lambda)$. For the case where $c_\omega(\kappa) \neq c_\omega(\lambda)$, we arrive at

$$(\tilde{q}_i + \mathbf{I})(\tilde{q}_i - \theta\mathbf{I}) = 0, \;\; \tilde{q}_i = \frac{2q_i}{(c_\omega(\kappa) - c_\omega(\lambda))}, \tag{23}$$

with

$$-\theta = \frac{c_\omega(\kappa) + c_\omega(\lambda)}{c_\omega(\kappa) - c_\omega(\lambda)}, \tag{24}$$

which represents the familiar Iwahori-Hecke algebra [11, 12], with $\theta \neq 0$, while writing (21) with some non-zero C

$$\tilde{q}_i \tilde{q}_{i+1} \tilde{q}_i = \tilde{q}_{i+1} \tilde{q}_i \tilde{q}_{i+1}, \; \tilde{q}_i = C q_i. \tag{25}$$

For the case of $c_\omega(\kappa) = c_\omega(\lambda)$, with $\kappa = \lambda \neq -\omega^2/4$ we obtain $\tilde{q}_i^2 = \tilde{q}_i$, $\tilde{q}_i = -q_i/c_\omega(\lambda)$, which corresponds to the idempotent generators of the braid equation. Finally, for $\kappa = \lambda = -\omega^2/4$, we have $\tilde{q}_i^2 = 0$, $\tilde{q}_i = q_i$, which represent nilpotent generators of degree 2.

The rescaled braid equation in (25) is equivalent to the constant Yang-Baxter equation (cYBE) where the constant R-matrix $Q_{i,i+1}$ and $\tilde{q}_i$ are related by $\tilde{q}_i = \mathcal{P}_{i,i+1} Q_{i,i+1}$. The Hamiltonian then becomes

$$H = \frac{\alpha}{C}\sum_{i=1}^{N} \tilde{q}_i + N\alpha\beta\mathbf{I}, \;\; \alpha = c_0^{-2}, \tag{26}$$

and the R-matrix (2) as

$$R_{ij}(f(x,y)) = (1 + \beta f(x,y))\mathcal{P}_{ij} + \frac{f(x,y)}{C} Q_{ij}. \tag{27}$$

Table 1. Forms of possible Hamiltonian density. The $\mathbf{R}$ column refers the relations which satisfies $\mathbf{R} = \mathbf{0}$.

Type	Case	C	β	$\mathbf{R}$
A	$\kappa \neq \lambda$	$2(c_\omega(\kappa) - c_\omega(\lambda))^{-1}$	$\frac{1}{2}(-\omega + c_\omega(\kappa))$	$(\tilde{q}_i + \mathbf{I})(\tilde{q}_i - \theta\mathbf{I})$
B	$\kappa = \lambda \neq -\frac{1}{4}\omega^2$	$-c_\omega(\lambda)^{-1}$	$\frac{1}{2}(-\omega + c_\omega(\lambda))$	$\tilde{q}^2 - \tilde{q}$
C	$\kappa = \lambda = -\frac{1}{4}\omega^2$	1	$-\frac{1}{2}\omega$	$\tilde{q}^2$

A summary of different eigenvalue problems with corresponding forms of C and β is given in table 1. In this manner, we have transformed the problem into a solution for $\tilde{q}_i$ satisfying (25) with any one of the three eigenvalue problems depending on which constraints λ, κ and ω fulfil.

2.3.2. Representation of the Hamiltonian density. We focus on (20), where e_i is an $N^2 \times N^2$ matrix for $\dim(\mathcal{A}) = N$ and provide the necessary matrix representation. By identifying $p(x) = (x - \nu_+)(x - \nu_-)$ as the minimal polynomial of e_i, we use the Primary Decomposition Theorem [13] to obtain

$$\ker(e_i - \nu_+\mathbf{I}) \oplus \ker(e_i - \nu_-\mathbf{I}) = \mathbf{C}^{N^2}. \tag{28}$$

After applying $\dim(A \oplus B) = \dim(A) + \dim(B)$ and rank-nullity theorem [13, 14], we find

$$R_k(e_i - \nu_+\mathbf{I}) + R_k(e_i - \nu_-\mathbf{I}) = N^2, \tag{29}$$

where $R_k(M)$ is the rank of square matrix M. If we identify $\Lambda_i = e_i - \nu_+\mathbf{I}$ as a rank r matrix, then $e_i - \nu_-\mathbf{I} = \Lambda_i + c_\omega(\lambda)\mathbf{I}$ is a rank $N^2 - r$ matrix. Then, we rewrite (20) as

$$\Lambda_i(\Lambda_i + c_\omega(\lambda)\mathbf{I}) = \mathbf{0} \tag{30}$$

to construct the matrix representation of $e_i = \Lambda_i + \nu_+\mathbf{I}$. The essential property to note is that for $\omega^2 = -4\lambda$, we have Λ_i as a nilpotent matrix of degree 2. For the case where $\omega^2 \neq -4\lambda$, we rewrite (30) with $\Phi_i = -c_\omega(\lambda)^{-1}\Lambda_i$ as $\Phi_i^2 = \Phi_i$ which reveals the idempotent nature of Λ_i. Thus, we require only nilpotent and idempotent matrices to construct two possible solutions for e_i.

2.3.3. General form of e_i and intertwining relations. With $D = N^2$, we have the desired forms of e_i as below

$$e_i = \begin{cases} \mathbf{N}_r - \frac{\omega}{2}\mathbf{I} & \omega^2 = -4\lambda, \\ -c_\omega(\lambda)\mathbf{B}_r + \nu_+\mathbf{I} & \omega^2 \neq -4\lambda, \end{cases} \tag{31}$$

where $\mathbf{N}_r$ and $\mathbf{B}_r$ acting on $\mathcal{H}_i \otimes \mathcal{H}_{i+1}$ are the (degree 2) Nilpotent and Idempotent matrices of rank r, respectively. Using (17), we find the intertwining constraints that both of them must satisfy:

$$\mathbf{Z}\otimes\mathbf{I}\cdot\mathbf{I}\otimes\mathbf{Z}\cdot\mathbf{Z}\otimes\mathbf{I} - \mathbf{I}\otimes\mathbf{Z}\cdot\mathbf{Z}\otimes\mathbf{I}\cdot\mathbf{I}\otimes\mathbf{Z} = f_{\lambda,\kappa}[\mathbf{I}\otimes\mathbf{Z} - \mathbf{Z}\otimes\mathbf{I}], \tag{32}$$

where

$$\mathbf{Z} = \begin{cases} \mathbf{N}_r & f_{\lambda,\kappa} = \frac{1}{4}\left(\omega^2 + 4\kappa\right), \\ \mathbf{B}_r & f_{\lambda,\kappa} = (\kappa - \lambda)\left(\omega^2 + 4\lambda\right)^{-1}. \end{cases} \tag{33}$$

In the end we categorically write all possible forms of $\tilde{q}_i$ using table 1 as

$$\tilde{q}_i = \begin{cases} \mathbf{N}_r & \omega^2 = -4\lambda = -4\kappa\,(BN), \\ \mathbf{B}_r & \omega^2 \neq -4\lambda = -4\kappa\,(BI), \\ \frac{2}{c_\omega(\kappa)}\mathbf{N}_r - \mathbf{I} & \omega^2 = -4\lambda \neq -4\kappa\,(HN), \\ \frac{2c_\omega(\lambda)}{c_\omega(\lambda) - c_\omega(\kappa)}\mathbf{B}_r - \mathbf{I} & \omega^2 \neq -4\lambda \neq -4\kappa\,(HI); \end{cases} \tag{34}$$

with which we use the braid equation (25) to solve for $\mathbf{N}_r$ and $\mathbf{B}_r$ and identify the Hamiltonian (26) and the R-matrix (27). Any solution of types BI and BN from (34) is a low-rank matrix. For an idempotent matrix, the rank r is less than the dimension N. If $r = N$ then it corresponds to the identity matrix. For a nilpotent matrix (of degree 2), we have $r \leqslant N/2$. Hence, we have shown two classes of low-rank solutions of cYBE that are related physically through (27). Solutions from classes HI and HN may not necessarily have $r < N$.

2.4. The three site operator

From (18), the three site operator $t_{i,i+1}$ can be rewritten in the following symmetrised form with respect to q_i as

$$t_{i,i+1} = q_i q_{i+1} q_i + \beta\left\{q_i, q_{i+1}\right\} + \beta^2\left(q_i + q_{i+1}\right) + \beta\left(\beta^2 + \lambda\right)\mathbf{I} \tag{35}$$

after setting the value of β which is considered in (22). Multiplication of the three-site operator with $q_i - q_{i+1}$ simplifies to the following property:

$$t_{i,i+1}\left(q_i - q_{i+1}\right) = \lambda\left[q_i, q_{i+1}\right] \tag{36}$$

and similarly

$$t_{i,i+1}\left(q_i + q_{i+1} + 2(\beta + \omega)\mathbf{I}\right) = \lambda\left(2\kappa\mathbf{I} + \left\{q_i + \beta\mathbf{I},\ q_{i+1} + \beta\mathbf{I}\right\}\right). \tag{37}$$

2.4.1. Deducing the Temperley Lieb algebra.

Now we can use the properties of the operator to show that if $t_{i,i+1} = \mathbf{0}$, then $\lambda = 0$ for non-trivial q_i. Let $t_{i,i+1} = \mathbf{0}$. If $\lambda \neq 0$, then

$$\left[q_i, q_{i+1}\right] = 0,\ \left\{e_i, e_{i+1}\right\} = -2\kappa\mathbf{I},\ \forall i. \tag{38}$$

From the commutator condition, q_i must be a single lattice site term. From the anti-commutator relation, we write

$$q_i q_{i+1} + \beta\left(q_i + q_{i+1}\right) + \left(\kappa + \beta^2\right)\mathbf{I} = 0,\ \forall i \tag{39}$$

which is only possible if each q_i is proportional to identity $\mathbf{I}$. Hence, for the non-trivial generator satisfying that the three site operator is zero, we need $\lambda = 0$.

Then, it follows that a non-trivial representation of e_i satisfies the Temperley Lieb algebra if $t_{i,i+1} = \mathbf{0}$ in (18). We have used it to simplify our numerical computations in the below sections, where we focus on providing a list of lowest-dimensional solutions ($N = 2$) of the R-matrix (2).

3. Numerical analysis

We have computed relevant rank-1 idempotent and (degree 2) nilpotent square matrices of dimension $N^2 = 4$ for the possible forms of e_i through the classification given in (34), which is sufficient to reconstruct the R-matrix (27). The choice of working with rank-1 models lies in avoiding set of equations that are difficult to tackle computationally. The numerical construction of these matrices is described in appendix B.

Another reason we chose to look into rank 1 solutions is that $\mathbf{A}_r$ in (B.1) represents the sum of various rank-1 matrices that are linearly independent. When working with a rank $r \geqslant 1$, it involves $2N^2 r$ variables with r^2 additional constraints either from (B.5) or (B.9). One may think of rank-m models as an added generalisation to a rank-n case for $n < m$. Therefore, solving for the lowest rank is a key step.

In this section, we describe the computational workflow involved in the analysis and methods to simplify the results.

3.1. Computational workflow

Our numerical methodology is divided into three phases.

3.1.1. Main computation. In the specific case of $N = 2$, we can directly calculate the solutions of (34) from the braid equation (25) with $r = 1$. Fortunately, the use of the Gröbner basis for decomposing a maximal $N^6 + r^2 = 65$ overdeterministic equation for $2N^2 r = 8$ unknowns is tenable with Mathematica packages. Hence, for every solution pool (BI, BN, HI and HN), we gathered results using the `Solve[]` and `Reduce[]` modules available in the package.

3.1.2. Removing redundant results. The next step is to remove redundant solutions from the gathered results. Using the symmetries of the R-matrix mentioned in appendix C, we created routines to identify repeating solutions. Pseudocodes of the routines used are provided in appendix D.

For the proceeding step, we introduce the structure matrix

$$S(M) = [s_{ij}], \;\; s_{ij} = \begin{cases} 1 & if m_{ij} \neq 0 \\ 0 & if m_{ij} = 0 \end{cases} \tag{40}$$

to identify the non-zero elements of matrix M. Any solution whose structure matrix does not have zero elements is called a *full-case* matrix.

We segment full-case solutions into various subcases with zero elements. For this purpose, we made valid substitution of their free constant parameters with zero. All of them are then combined with the rest of the results and repeated solutions are removed.

The symmetries of the R-matrix act equivalently on the Nilpotent/Idempotent matrices owing to their form in (27). Hence, we can remove the repetitions within each solution category in (34) while ensuring the distinction among them.

3.1.3. Post-simplifications. In the final step, we nullify the three-site operator in (35) for every solution to simplify our results, which then fulfils the Temperley-Lieb algebra after writing them in the form of e_i. External parameters, such as λ, ω, κ are reduced case-wise if they do not contribute to the solution.

3.2. *Results of rank-1 models*

Here, we present all rank-1 models with zero matrix terms. For simplicity, we also mention the subcases of (18) that each result class fulfils. We name each class according to which of the eigenvalue problems and intertwining relations they satisfy. Parameters c_1, c_2 and c_3 are free in these solutions.

3.2.1. Braid-Nilpotent. The solutions are of form $e_i = \mathbf{N}_1$, satisfying

$$e_i^2 = \mathbf{0}, \;\; e_i e_{i+1} e_i = \mathbf{0}, \;\; e_{i+1} e_i e_{i+1} = \mathbf{0}. \tag{41}$$

The list of these $\mathbf{N}_1$ is in (42).

$$M_{BN}(a) = \begin{pmatrix} 0 & -c_2 & 0 & -c_2^2 c_1^{-1} \\ 0 & 0 & 0 & 0 \\ 0 & c_1 & 0 & c_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \; M_{BN}(b) = \begin{pmatrix} 0 & 0 & 0 & c_3 \\ 0 & 0 & 0 & c_2 \\ 0 & 0 & 0 & c_1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{42}$$

3.2.2. Braid-Idempotent. The solutions are of form $e_i = \mathbf{B}_1$, satisfying

$$e_i^2 = e_i, \;\; e_i e_{i+1} e_i = \mathbf{0}, \;\; e_{i+1} e_i e_{i+1} = \mathbf{0}. \tag{43}$$

The list of these $\mathbf{B}_1$ is in (44).

$$\begin{gathered} M_{BI}(a) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -c_2 & 1-c_1 & -c_1 & c_1(1-c_1)c_2^{-1} \\ c_2 & -1+c_1 & c_1 & c_1(c_1-1)c_2^{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ M_{BI}(b) = \begin{pmatrix} 0 & c_2 & 0 & -c_2^2(c_1+1)^{-1} \\ 0 & 1 & 0 & -c_2(c_1+1)^{-1} \\ 0 & c_1 & 0 & -c_1 c_2 (c_1+1)^{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix} M_{BI}(c) = \begin{pmatrix} 0 & 0 & c_2 c_1^{-1} & c_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ M_{BI}(d) = \begin{pmatrix} 0 & 0 & -c_2 & c_2 c_1^{-1} \\ 0 & 0 & -(c_1c_2+1) & (c_1c_2+1)c_1^{-1} \\ 0 & 0 & -c_1c_2 & c_2 \\ 0 & 0 & -c_1(c_1c_2+1) & c_1c_2+1 \end{pmatrix}. \end{gathered} \tag{44}$$

There is a model of form $e_i = \mathbf{B}_1$

$$M_{BI}(e) = \begin{pmatrix} 0 & 0 & 0 & c_1^2 \\ 0 & 0 & 0 & c_1 \\ 0 & 0 & 0 & c_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{45}$$

which satisfies $e_i^2 = e_i, \;\; e_i e_{i+1} e_i = e_{i+1} e_i e_{i+1} \neq \mathbf{0}$.

3.2.3. Hecke-Nilpotent. The solutions are of form $e_i = \mathbf{N}_1$, satisfying

$$e_i^2 = 0, \; e_i e_{i+1} e_i = -\kappa e_i, \; e_{i+1} e_i e_{i+1} = -\kappa e_{i+1}. \tag{46}$$

The list of these $\mathbf{N}_1$ is in (47).

$$\begin{aligned}
M_{HN}(a) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -c_2 & -c_1 & -c_1 & \left(\kappa - c_1^2\right) c_2^{-1} \\ c_2 & c_1 & c_1 & \left(-\kappa + c_1^2\right) c_2^{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
M_{HN}(b) &= \begin{pmatrix} 0 & \left(c_2 \kappa^{1/2}\right) c_1^{-1} & c_2 & \left(q_\kappa c_2^2\right) c_1^{-1} \\ 0 & \kappa^{1/2} & c_1 & c_2 q_\kappa \\ 0 & -\kappa c_1^{-1} & -\kappa^{1/2} & -\left(\kappa^{1/2} q_\kappa c_2\right) c_1^{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
M_{HN}(c) &= \begin{pmatrix} -2c_1 & 2c_1^2 c_2 \left(\kappa - c_1^2\right)^{-1} & 2c_1^2 c_2 \left(\kappa - c_1^2\right)^{-1} & 2c_1 c_2^2 \left(\kappa - c_1^2\right)^{-1} \\ \left(c_1^2 - \kappa\right) c_2^{-1} & c_1 & c_1 & c_2 \\ \left(c_1^2 - \kappa\right) c_2^{-1} & c_1 & c_1 & c_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{47}$$

where $q_\kappa = \left(c_1 + \kappa^{1/2}\right)\left(\kappa^{1/2} - c_1\right)^{-1}$.

3.2.4. Hecke-Idempotent. We found two types of solutions for this type. The first type is of form $e_i = \mathbf{B}_1$, satisfying

$$e_i^2 = e_i, \; e_i e_{i+1} e_i = \frac{1}{4} e_i, \; e_{i+1} e_i e_{i+1} = \frac{1}{4} e_{i+1}. \tag{48}$$

The list of these $\mathbf{B}_1$ is in (49)

$$\begin{aligned}
M_{HIa}(a) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ (1 - 2c_2)^2 (4c_1)^{-1} & c_2 & c_2 - 1 & c_1 \\ -(1 - 2c_2)^2 (4c_1)^{-1} & -c_2 & 1 - c_2 & -c_1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
M_{HIa}(b) &= \frac{1}{2} \begin{pmatrix} 1 & 0 & -c_1 & c_2 \\ c_1 c_2^{-1} & 0 & -c_1^2 c_2^{-1} & c_1 \\ 0 & 0 & 0 & 0 \\ c_2^{-1} & 0 & -c_1 c_2^{-1} & 1 \end{pmatrix} \\
M_{HIa}(c) &= \frac{1}{2(c_1 - c_2)} \begin{pmatrix} c_1 + c_2 & 0 & -4 & 4(c_1 + c_2)^{-1} \\ c_1 (c_1 + c_2) & 0 & -4c_1 & 4c_1 (c_1 + c_2)^{-1} \\ c_2 (c_1 + c_2) & 0 & -4c_2 & 4c_2 (c_1 + c_2)^{-1} \\ \frac{1}{4}(c_1 + c_2)^3 & 0 & -(c_1 + c_2)^2 & c_1 + c_2 \end{pmatrix}.
\end{aligned} \tag{49}$$

The second type is of form $e_i = \mathbf{B}_1$, satisfying

$$e_i^2 = e_i, \; e_i e_{i+1} e_i = -\kappa e_i, \; e_{i+1} e_i e_{i+1} = -\kappa e_{i+1}. \tag{50}$$

The list of these $\mathbf{B}_1$ is in (51)

$$
\begin{aligned}
M_{HIb}(a) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\left(\kappa - c_2^2 + c_2\right)c_1^{-1} & c_2 & c_2 - 1 & c_1 \\ \left(\kappa - c_2^2 + c_2\right)c_1^{-1} & -c_2 & 1 - c_2 & -c_1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
M_{HIb}(b) &= \frac{1}{2}\begin{pmatrix} 0 & c_1(1-\gamma) & c_1(1-\gamma) & -c_1^2(1-\gamma)^2 \\ 0 & (1-\gamma) & (1-\gamma) & -c_1(1-\gamma)^2 \\ 0 & (1+\gamma) & (1+\gamma) & 4c_1\kappa \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
M_{HIb}(c) &= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (1-\gamma) & (1-\gamma)^2(1+\gamma)^{-1} & 0 \\ 0 & (1+\gamma)^2(1-\gamma)^{-1} & (\gamma+1) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{51}
$$

where $\gamma = \sqrt{4\kappa + 1}$.

3.3. Extensions of known models

In this section, we examine asymmetric spin-hopping models from our numerical results. They are grouped together according to the similarities in their nearest neighbouring dynamics. We decompose each two-site e_i in terms of $SU(2)$ spin matrices $S_i^q = \sigma_i^q/2$. For brevity, we use the notation in (52).

$$
\begin{aligned}
S_{i,xy}^{\pm} &= S_i^x \pm i S_i^y \\
S_{i,xz}^{\pm} &= S_i^x \pm i S_i^z \\
\mathcal{P}_{i,i+1} &= \frac{1}{2}\left(I + \sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \sigma_i^z\sigma_{i+1}^z\right).
\end{aligned} \tag{52}
$$

We reiterate that our results are derived from the general treatment of the solutions of the YBE. One may consider mapping them into spinless fermions to interpret them as Markov processes, and is not solely limited by it. For instance, they may present themselves after vectorising the Lindblad equation [15]. Hence, our results are given as non-Hermitian spin chains for a broader consideration.

3.3.1. The asymmetric hopping models.

$M_{BI}(a)$ from (44), $M_{HIa}(a)$ from (49), and $M_{HIb}(c)$ from (51) have dynamics similar to $M_{HIb}(a)$ from (51), which we write as follows

$$
\begin{aligned}
\mathcal{M}_{i,i+1} &= -\mathcal{K}_{i,i+1}(c_2) + \mathcal{Q}_{i,i+1} \\
\mathcal{K}_{i,i+1}(c_2) &= c_2 S_{i,xy}^- S_{i+1,xy}^+ + (1-c_2) S_{i,xy}^+ S_{i+1,xy}^- + S_i^z S_{i+1}^z + \left(c_2 - \frac{1}{2}\right)\left(S_{i+1}^z - S_i^z\right) - \frac{1}{4} \\
\mathcal{Q}_{i,i+1} &= S_i^z A_{i+1}^- - A_i^- S_{i+1}^z + \frac{1}{2}\left(A_i^+ - A_{i+1}^+\right) \\
A_i^{\pm} &= c_1 S_{i,xy}^+ \pm \left(\frac{\kappa + c_2(1-c_2)}{c_1}\right) S_{i,xy}^-
\end{aligned} \tag{53}
$$

where $\mathcal{K}_{i,i+1}$ resembles a spin-ASEP model. In addition, $\mathcal{Q}_{i,i+1}$ appears in the model and is additional in various subparts. To understand this extra term, we simplify the model by substituting $c_1 = (c_2(c_2-1)-\kappa)^{1/2}$. Then,

$$\mathcal{Q}_{i,i+1} = \left(\sqrt{c_2(c_2-1)-\kappa}\right)\left[\frac{1}{2i}(S^+_{i,xz}S^-_{i+1,xz} - S^-_{i,xz}S^+_{i+1,xz}) + i(S^y_i - S^y_{i+1}))\right] \tag{54}$$

which represents an XZ-aligned spin-chain with a hermiticity-breaking subterm. With $\kappa = c_2(c_2-1)$, the extra term vanishes and we recover the ASEP model. In this manner, we identify *extensions* of previously studied models.

Another extension is provided by the models of $M_{BN}(a)$ from (42), $M_{HN}(b), M_{HN}(c)$ from (47) and $M_{HIb}(b)$ from (51). We write $M_{HIb}(b)$ as follows

$$\begin{aligned}
\mathcal{M}_{i,i+1} &= \mathcal{K}_{i,i+1}\left(\frac{1+\gamma}{2}\right) + \mathcal{Q}_{i,i+1} \\
\mathcal{Q}_{i,i+1} &= -c_1^2\frac{(1-\gamma)^2}{2}S^+_{xy,i}S^+_{xy,i+1} - \frac{1}{4}c_1\gamma(1-\gamma)\left(S^+_{xy,i+1} - S^+_{xy,i}\right) - 2S^z_iS^z_{i+1} \\
&\quad - \frac{1}{2}c_1(\gamma-1)(\gamma+2)S^z_iS^+_{xy,i+1} + \frac{1}{2}c_1(\gamma-1)(\gamma-2)S^+_{xy,i}S^z_{i+1} + \frac{1}{2}.
\end{aligned} \tag{55}$$

The additional term represents the dynamics of the spin-creation operations. Imposing $c_1 = 0$ recovers the model.

Finally, we conclude that there are similar extensions to the TASEP (Totally ASEP) model, where spin hopping is allowed in one direction in the periodic chain. These are the $M_{BI}(b), M_{BI}(d)$ from (44), $M_{HIa}(b), M_{HIa}(c)$ from (49).

3.3.2. Anti-Hermitian model. We find a model $M_{HN}(a)$ from (47) which extends an anti-Hermitian spin chain. $M_{HN}(a)$ is written as follows

$$\begin{aligned}
\mathcal{M}_{i,i+1} &= \left[c_1S^-_{i,xy}S^+_{i+1,xy} - c_1S^+_{i,xy}S^-_{i+1,xy} + c_1\left(S^z_{i+1} - S^z_i\right)\right] + \mathcal{Q}_{i,i+1} \\
\mathcal{Q}_{i,i+1} &= S^z_iA^-_{i+1} - A^-_iS^z_{i+1} + \frac{1}{2}\left(A^+_i - A^+_{i+1}\right) \\
A^\pm_i &= \left(\left(\frac{\kappa - c_1^2}{c_2}\right)S^+_{i,xy} \pm c_2S^-_{i,xy}\right).
\end{aligned} \tag{56}$$

For relevant parameter substitutions, we can rewrite $\mathcal{Q}_{i,i+1}$ in (56), similar to that in (54). Both systems can be further combined into the following rank-1 model

$$\begin{aligned}
e_i = &-\frac{p}{4} + \frac{2q-p}{2}\left(S^z_i - S^z_{i+1}\right) + \left(qS^+_{i,xy}S^-_{i+1,xy} + (p-q)S^-_{i,xy}S^+_{i+1,xy}\right) \\
&+ pS^z_iS^z_{i+1}2s\left(S^z_iS^x_{i+1} - S^x_iS^z_{i+1}\right) + is\left(S^y_i - S^y_{i+1}\right)
\end{aligned} \tag{57}$$

satisfying the below TL algebra,

$$\begin{aligned}
e_ie_{i+1}e_i &= \left(s^2 - q(q-p)\right)e_i \\
e_{i+1}e_ie_{i+1} &= \left(s^2 - q(q-p)\right)e_{i+1} \\
e_i^2 &= -pe_i
\end{aligned} \tag{58}$$

which is also a solution of YBE with the R-matrix given by (2).

Table 2. Some rank-2 Hecke-Idempotent models.

Model	e_i	ω	λ
$B1$	$-\frac{s}{2}+\left((p+s)S_i^z+pS_{i+1}^z\right)+qS_{i,xy}^-S_{i+1,xy}^- + \left(pS_{i,xy}^+S_{i+1,xy}^- + (p+s)S_{i,xy}^-S_{i+1,xy}^+\right)$	s	$p(p+s)$
$B2$	$-\left(S_{i+1}^y+S_i^y\right)+rS_{i,xz}^-S_{i+1,xz}^- + \left(S_{i,xz}^-S_{i+1,xz}^+ + S_{i,xz}^+S_{i+1,xz}^-\right)$	0	1
B3	$k\left(S_i^z+S_{i+1}^z\right)+\left(k^2S_{i,xy}^+S_{i+1,xy}^- + S_{i,xy}^-S_{i+1,xy}^+\right)$	0	k^2

3.3.3. Spin-creation models. Finally, we have $M_{BN}(b)$ from (42) and $M_{BI}(c), M_{BI}(e)$ from (44), which represents the model with sole spin-creation operations. For example, we write $M_{BI}(c)$ as follows:

$$\begin{aligned}\mathcal{M}_{i,i+1} &= c_2S_{i,xy}^+S_{i+1,xy}^+ - S_i^zS_{i+1}^z + \frac{1}{2}\left(S_{i+1}^z - S_i^z\right) + \frac{1}{4} + \mathcal{Q}_{i,i+1}\\ \mathcal{Q}_{i,i+1} &= \frac{c_2}{c_1}S_{i,xy}^+S_{i+1}^z - c_1S_i^zS_{i+1,xy}^+ + \frac{1}{2}\left(\frac{c_2}{c_1}S_{i,xy}^+ + c_1S_{i+1,xy}^+\right).\end{aligned} \tag{59}$$

This completes the summary of all rank-1 models obtained through our numerical analysis. We note that the free parameters in our models are not reduced further by similarity transformations. Such reductions might reveal models that we already know are integrable. Thus, we do not claim to have found any new models. Nevertheless the problem of identifying higher-rank models is yet to be investigated.

3.4. Higher rank models

For completeness, we also present some of the models e_i constructed using the nilpotent and idempotent matrices of rank $r > 1$ which we could identify from our numerical analysis. For brevity, λ, κ and ω are expressed with respect to the model parameters.

3.4.1. Some rank 2 cases. We have some Hecke-Idempotent models in table 2 where $e_i = -c_\omega(\lambda)\mathbf{B}_2 + \nu_+\mathbf{I}$. They all satisfy the below subcase of (18)

$$\begin{aligned}e_ie_{i+1}e_i &= e_{i+1}e_ie_{i+1} \neq 0,\\ e_i^2 &= \lambda - \omega e_i.\end{aligned} \tag{60}$$

It is interesting to note that model $B1$ becomes a Hecke-Nilpotent model of rank 2 when $s = -2p$.

3.4.2. A rank 3 case. In the end, we provide one model of the form $e_i = -c_\omega(\lambda)\mathbf{B}_3 + \nu_+\mathbf{I}$ which follows the algebra below

$$\begin{aligned}e_ie_{i+1}e_i &= t_{i,i+1} - \kappa e_i,\\ e_{i+1}e_ie_{i+1} &= t_{i,i+1} - \kappa e_{i+1},\\ e_i^2 &= \lambda;\end{aligned} \tag{61}$$

and is given as follows

$$\begin{aligned} e_i &= -r\mathcal{P}_{i,i+1} + 2\sqrt{p-r^2}\left(S_i^z S_{i+1}^x - S_i^x S_{i+1}^z\right) + i\sqrt{p-r^2}\left(S_{i+1}^y - S_i^y\right), \\ t_{i,i+1} &= \left(pr - 2r^3\right) - r^2\left(e_i + e_{i+1}\right) - r\left\{e_i, e_{i+1}\right\}, \\ \kappa &= r^2 - p, \\ \lambda &= r^2. \end{aligned} \tag{62}$$

This model is also discussed within the setting of [2] and is presented in our classification as follows: a rank-3 HI model when $p \neq 0$, a rank-3 BI model when $p = 0$ and a rank-1 HN model when $r = 0$. The case where $p = r^2$ represents the well-known Heisenberg XXX model.

3.5. As stochastic operators

In order to consider the Hamiltonian densities e_i as stochastic operators $M = [m_{i,j}]$, they need to satisfy these criteria: (i) $m_{i,j} \in \mathbf{R}$, (ii) $\sum_i m_{i,c} = 0$ for all c, (iii) $m_{i,i} \geqslant 0$ for all i, and (iv) $m_{i,j} \leqslant 0$ for all $i \neq j$. The sign of the elements of M can be interchanged by an overall multiplication of -1.

If we consider M to be a *rank-1* matrix of dimension 4, pertaining to the discussions before, it leads in restricted choices. Notably, there is no valid M of form $\mathbf{N}_1$. Turning to M satisfying the form $\mathbf{B}_1$, we have

$$M_{s,1}(a) = \begin{pmatrix} \cdot & \cdot & \cdot & \alpha \\ \cdot & \cdot & \cdot & \beta \\ \cdot & \cdot & \cdot & \gamma \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \; M_{s,1}(b) = \begin{pmatrix} \cdot & \cdot & \alpha & \cdot \\ \cdot & \cdot & \beta & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \gamma & \cdot \end{pmatrix} \; M_{s,1}(c) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & 1-p & -p & \cdot \\ \cdot & p-1 & p & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$
$$M_{s,1}(d) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1-p & -p \\ \cdot & \cdot & p-1 & p \end{pmatrix} \; M_{s,1}(e) = \begin{pmatrix} 1-p & \cdot & \cdot & -p \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ p-1 & \cdot & \cdot & p \end{pmatrix} \tag{63}$$

where $\alpha + \beta + \gamma = -1$ and $0 \leqslant p \leqslant 1$. $M_{s,1}(c)$ is the familiar ASEP model. $M_{s,1}(a)$ also satisfies the integrable conditions for $\{\alpha = 1,\ \beta, \gamma = -1\}$ as it becomes $M_{BI}(e)$ with $c_1 = -1$. $M_{s,1}(e)$ for $p = 1/2$ also overlaps with $M_{HIa}(b)$ with $\{c_1 = 0, c_2 = -1\}$.

The question of higher-rank stochastic operators additionally satisfying Yang-Baxter integrability is an open problem.

4. Hubbard-type representation of (degenerate) Hecke algebraic models

In this section we are interested to bring note of a (degenerate) Hecke algebraic model within the representation of Hubbard X-operators which illuminates on the appearance of non-trivial spin-chain models from our numerical analysis.

First we introduce the Hubbard operators as

$$\begin{aligned} X_i^{\alpha\beta} &= \left(|\alpha\rangle\langle\beta|\right)_i, \quad X_i^{\alpha\beta} X_i^{\lambda\gamma} = \delta^{\beta\lambda} X_i^{\alpha\gamma} \\ \sum_\alpha X_i^{\alpha\alpha} &= 1, \quad \left[X_i^{\alpha\beta}, X_j^{\delta\gamma}\right]_\pm = \left(\delta^{\beta\delta} X_i^{\alpha\gamma} \pm \delta^{\alpha\gamma} X_i^{\beta\gamma}\right)\delta_{ij}, \end{aligned} \tag{64}$$

where $[A,B]_{\pm} = AB - (-1)^{p(A)p(B)}BA$ denotes a graded commutator, and $p(A)$ is the fermionic parity of operator A. We consider a particular choice of basis for the operators

$$|0\rangle = |\uparrow\downarrow\rangle, \quad |1\rangle = |\downarrow\rangle, \quad |2\rangle = |\uparrow\rangle, \quad |3\rangle = |\circ\rangle, \tag{65}$$

where states $|0\rangle$ and $|3\rangle$ are bosonic whereas states $|1\rangle$ and $|2\rangle$ are fermionic. With the Greek indices running through integers 0 to 3, a particular representation of the X-operators is given by

$$\left[X_i^{\alpha\beta}\right] = \begin{pmatrix} n_\downarrow n_\uparrow & n_\downarrow c_\uparrow^\dagger & -c_\downarrow^\dagger n_\uparrow & c_\uparrow^\dagger c_\downarrow^\dagger \\ n_\downarrow c_\uparrow & n_\downarrow(1-n_\uparrow) & c_\downarrow^\dagger c_\uparrow & c_\downarrow^\dagger(1-n_\uparrow) \\ -c_\downarrow n_\uparrow & c_\uparrow^\dagger c_\downarrow & (1-n_\downarrow)n_\uparrow & c_\uparrow^\dagger(1-n_\downarrow \\ c_\downarrow c_\uparrow & c_\downarrow(1-n_\uparrow) & c_\uparrow(1-n_\downarrow) & (1-n_\uparrow)(1-n_\downarrow) \end{pmatrix}. \tag{66}$$

Next by using the bond notation $O_{i,i+1} \equiv O_i$, we introduce the following set of operators

$$a_i^\dagger = X_i^{30}X_{i+1}^{30} - X_i^{10}X_{i+1}^{20} + X_i^{20}X_{i+1}^{10}, \qquad a_i = X_i^{03}X_{i+1}^{03} + X_i^{01}X_{i+1}^{02} - X_i^{02}X_{i+1}^{01}, \tag{67a}$$

$$b_i = \sum_a (-1)^{p(a)} X_i^{0a}X_{i+1}^{a0}, \quad b_i^\dagger = \sum_a X_i^{a0}X_{i+1}^{0a}, \tag{67b}$$

$$p_i^0 = \left(1 - X_i^{00}\right)\left(1 - X_{i+1}^{00}\right), \; p_i^1 = X_i^{00}\left(1 - X_{i+1}^{00}\right), \qquad p_i^2 = \left(1 - X_i^{00}\right)X_{i+1}^{00}, \; p_i^3 = X_i^{00}X_{i+1}^{00}, \tag{67c}$$

$$B_i = \sum_{a,b} (-1)^{p(b)} X_i^{ab}X_{i+1}^{ba}, \quad r_i = a_i^\dagger a_i, \tag{67d}$$

where the Latin indices run from integers 1 to 3. These operators satisfy the following quasi-local algebra [16]

$$r_i r_{i+1} r_i = r_i p_{i+1}^0, \quad r_{i+1} r_i r_{i+1} = p_i^0 r_{i+1}, \tag{68a}$$

$$B_i B_{i+1} B_i = B_{i+1} B_i B_{i+1}, \tag{68b}$$

$$b_i^\dagger r_{i+1} b_i = b_{i+1} r_i b_{i+1}^\dagger, \quad b_i^\dagger B_{i+1} b_i = b_{i+1} B_i b_{i+1}^\dagger, \tag{68c}$$

$$b_i b_{i+1} a_i^\dagger = a_{i+1}^\dagger p_i^3, \quad b_i a_{i+1} a_i^\dagger = p_i^1 b_{i+1}^\dagger, \tag{68d}$$

$$b_i a_{i+1} r_i = b_{i+1}^\dagger a_i, \quad b_i a_{i+1} B_i = b_{i+1}^\dagger a_i B_{i+1}, \tag{68e}$$

$$b_i b_{i+1} B_i = B_{i+1} b_i b_{i+1}, \tag{68f}$$

$$b_i^\dagger a_{i+1}^\dagger = b_{i+1} a_i^\dagger \tag{68g}$$

among other relations. For a single bond, we have $b_i^\dagger b_i = p_i^2$, $b_i b_i^\dagger = p_i^1$, $B_1^2 = p_i^0$ and all operators commute for $|i-j| > 1$. One *particular* baxterization of this algebra is provided by the operator

$$e_i = -\left[b_i + b_i^\dagger + \left(\alpha + \frac{1}{\alpha}\right)\left(p_i^0 + p_i^3\right) + \alpha p_i^1 + \frac{1}{\alpha} p_i^2 - \left(\alpha + \frac{1}{\alpha}\right)\right], \tag{69}$$

which satisfies the following Hecke algebra relations (18)

$$\begin{aligned} e_i e_{i+1} e_i - e_i &= e_{i+1} e_i e_{i+1} - e_{i+1}, \\ e_i e_j &= e_j e_i, \qquad |i-j| > 1, \\ e_i^2 &= \left(\alpha + \frac{1}{\alpha}\right) e_i. \end{aligned} \tag{70}$$

In the limit of $\alpha = -1/\alpha$ the relations are reduced to the Temperley-Lieb algebra. For the limit of $\alpha = 1$, one can write the generator $h_i = 1 - q_i$, where $q_i = b_i + b_i^\dagger + p_i^0 + p_i^3$ further satisfies the braid equation (21).

These algebraic structures correspond to a variation of an integrable $t - J$-type models

$$\begin{aligned} H = -\sum_{i,\sigma} \Big(& n_{i,\bar{\sigma}} c_{i,\sigma}^\dagger c_{i+1,\sigma} n_{i,\bar{\sigma}} + n_{i+1,\bar{\sigma}} c_{i+1,\sigma}^\dagger c_{i,\sigma} n_{i,\bar{\sigma}} \\ & + \eta_i^+ \eta_{i+1}^- + \eta_{i+1}^+ \eta_i^- + V n_{i,\uparrow} n_{i,\downarrow} n_{i+1,\uparrow} n_{i+1,\downarrow} + U n_{i,\uparrow} n_{i,\downarrow} \Big) \end{aligned} \tag{71}$$

with arbitrary V and U. Here $\eta_i^+ = c_{i,\uparrow}^\dagger c_{i,\downarrow}^\dagger$, $\eta_i^- = c_{i,\downarrow} c_{i,\uparrow}$ are generators of the pairing $su(2)$ algebra

$$\left[\eta_i^+, \eta_i^-\right] = 2\eta_i^z, \quad \left[\eta_i^z, \eta_i^\pm\right] = 2\eta_i^z, \quad 2\eta_i^z = n_{i,\uparrow} + n_{i,\downarrow} - 1. \tag{72}$$

Furthermore, the Hamiltonian (71) commutes with the generators of the supersymmetric $su(2|1)$ algebra. In practice, all of our models in section 3.3 may be written into fermionic operations through a pseudospin representation, and we can tie our spin-chain models as a manifestation of integrable variations of the Hubbard model.

5. Discussions and conclusion

We have identified certain forms of the constant solutions to the YBE (given in (34) upto a left multiplication of $\mathcal{P}_{i,i+1}$) which also solves the main relation in (2) via the R-matrix (27). In particular, many of them are low-rank matrices which are often overlooked. In our previous work [10], we devised an algorithm for identifying non-trivial solutions to the YBE of the difference form and enlisted many novel results.

To elucidate the relationship with this work, the algorithm is applicable for identifying those constant solutions. Hence we extend the scope of finding non-trivial models solely by studying the cYBE[3] (constant YBE), which is a much easier task than dealing with spectral-dependent relations. Additionally, we further draw attention to the parameterisation (13) which has correspondences with the one employed in [17].

To conclude, motivated by earlier studies on exactly solvable exclusion processes, we present a number of (possibly) new solutions of the YBE related to low-rank matrices and degenerate versions of Hecke-related algebraic structures. We also wrote spin-1/2 versions of the corresponding exclusion processes and demonstrated their connection to the integrable hubbard-type models. In future, we plan to examine their critical and dynamical properties.

[3] It is equation (25) after substituting $\tilde{q}_i \to \mathcal{P}_{i,i+1} Q_{i,i+1}$.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Acknowledgments

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Appendix A. Calculation towards finding $f(x,y)$

We first note the constraints on the function $f_{ij} \equiv f(u_i, u_j)$

$$\begin{aligned} \lim_{y\to x} f_{x,y} &= 0 \\ \frac{(f_{12}+f_{21})}{f_{12}f_{21}} &= \omega \\ \frac{1}{f_{12}f_{13}f_{23}}\left(f_{12}+f_{23}-f_{13}-\omega f_{12}f_{23}\right) &= \kappa \end{aligned} \tag{A.1}$$

where f_{ij} is the following ansatz

$$f(x,y) = \frac{x-y}{S(x,y)},\;\; S(x,y) = \sum_{i,j=0}^{N} d_{ij}x^i y^j. \tag{A.2}$$

It immediately satisfies $\lim_{y\to x} f(x,y) = 0$. Substituting the ansatz to the second constraint reveals

$$\sum_{i,j=0,i\neq j}^{N} (d_{ij}-d_{ji})\, x^i y^j = \omega\,(x-y) \tag{A.3}$$

which resolves by identifying

$$\begin{aligned} &d_{10} = d_{01} + \omega \\ &d_{ij} = d_{ji},\;\; i\neq j,\;\; (i,j)\notin\{(1,0),(0,1)\} \end{aligned} \tag{A.4}$$

Expanding the third constraint becomes tedious if all the variables $u_{1,2,3}$ are considered. Hence we consider $u_2 = 0$ and expand it as follows

$$\begin{aligned} f(u_1,0) + f(0,u_3) - f(u_1,u_3) - \omega f(u_1,0)f(0,u_3) \\ -\kappa f(u_1,0)f(u_1,u_3)f(0,u_3) = 0 \end{aligned} \tag{A.5}$$

which after substituting the form of $f(x,y)$, becomes

$$\begin{aligned} &S(u_1,u_3)(u_1S(0,u_3)-u_3S(u_1,0)+\omega u_1u_3)\\ &-(u_1-u_3)(S(u_1,0)S(0,u_3)+\kappa u_1u_3)=0 \end{aligned} \tag{A.6}$$

A.1. Expansions

The numerator terms of (A.6) are expanded term-wise as follows

$$\begin{aligned} u_1S(0,u_3)S(u_1,u_3) &= u_1\left(\sum_{i=0}^{N} d_{0,i}u_3^i\right)\left(\sum_{j,k=0}^{N} d_{j,k}u_1^ju_3^k\right)\\ &= \sum_{i,j,k=0}^{N} d_{0,i}d_{j,k}u_1^{j+1}u_3^{i+k}\\ &= \sum_{j=0}^{N}\sum_{n=0}^{2N}\left(\sum_{n=i+k}^{0\leqslant i,k\leqslant N} d_{0,i}d_{j,k}u_3^n\right)u_1^{j+1}\\ &= \sum_{i=0}^{N}\sum_{j=0}^{2N}\left(\sum_{j=\alpha+\beta}^{0\leqslant\alpha,\beta\leqslant N} d_{0,\alpha}d_{i,\beta}\right)u_3^ju_1^{i+1} \end{aligned} \tag{A.7}$$

$$\begin{aligned} u_3S(u_1,0)S(u_1,u_3) &= u_3\left(\sum_{i=0}^{N} d_{i,0}u_1^i\right)\left(\sum_{j,k=0}^{N} d_{j,k}u_1^ju_3^k\right)\\ &= \sum_{i,j,k=0}^{N} d_{i,0}d_{j,k}u_1^{i+j}u_3^{k+1}\\ &= \sum_{k=0}^{N}\sum_{n=0}^{2N}\left(\sum_{n=i+j}^{0\leqslant i,j\leqslant N} d_{i,0}d_{j,k}u_1^n\right)u_3^{k+1}\\ &= \sum_{j=0}^{N}\sum_{i=0}^{2N}\left(\sum_{i=\alpha+\beta}^{0\leqslant\alpha,\beta\leqslant N} d_{\alpha,0}d_{\beta,j}\right)u_1^iu_3^{j+1} \end{aligned} \tag{A.8}$$

$$\omega u_1u_3S(u_1,u_3) = \omega\sum_{i,j=0}^{N} d_{i,j}u_1^{i+1}u_3^{j+1} \tag{A.9}$$

$$\begin{aligned} (u_1-u_3)S(u_1,0)S(0,u_3) &= (u_1-u_3)\sum_{i,j=0}^{N}\left(d_{i,0}d_{0,j}u_1^iu_3^j\right)\\ &= \sum_{i,j=0}^{N} d_{i,0}d_{0,j}\left(u_1^{i+1}u_3^j-u_1^iu_3^{j+1}\right) \end{aligned} \tag{A.10}$$

$$\kappa u_1u_3(u_1-u_3) = \kappa\left(u_1^2u_3-u_1u_3^2\right) \tag{A.11}$$

A.2. Cases

We now consider the pre-factors of each monomial terms of the numerator of (A.6) and equate it to zero.

(i) $u_1^i u_3^{j+1}$, $i \geqslant N+2,\ 0 \leqslant j \leqslant N$:

$$\sum_{i=\alpha+\beta}^{0\leqslant\alpha,\beta\leqslant N} (-d_{\alpha,0}d_{\beta,j}) = 0 \tag{A.12}$$

We start by taking $i = 2N$, the maximal power possible in this case and find $d_{N,0}d_{N,j} = 0$. By setting $d_{N,0} = 0$ for all values of j, we can proceed with $i = 2N-1$ and impose $d_{N-1,0} = 0$ similarly. In this way, we put $d_{i,0} = 0,\ 2 \leqslant i \leqslant N$.

(ii) $u_1^{i+1} u_3^j$, $j \geqslant N+2,\ 0 \leqslant i \leqslant N$:

$$\sum_{j=\alpha+\beta}^{0\leqslant\alpha,\beta\leqslant N} d_{0,\alpha}d_{i,\beta} = 0. \tag{A.13}$$

We impose $d_{i,j} = d_{j,i}$ for all $i \neq j$ except for $(i,j) \in \{(1,0),(0,1)\}$ to use the previous remark and keep it zero.

(iii) $u_1^{N+1} u_3^{N+1}$:

$$(d_{0,1}d_{N,N} - d_{1,0}d_{N,N} + \omega d_{N,N}) = 0 \tag{A.14}$$

which is satisfied by the earlier result that $d_{1,0} = d_{0,1} + \omega$.

(iv) $u_1^{N+1} u_3^j$, $1 \leqslant j \leqslant N$:

$$\begin{aligned} &\sum_{j=\alpha+\beta}^{0\leqslant\alpha,\beta\leqslant N} (d_{0,\alpha}d_{N,\beta}) - \sum_{N+1=\alpha+\beta}^{0\leqslant\alpha,\beta\leqslant N} (d_{\alpha,0}d_{\beta,j-1}) - d_{N,0}d_{0,j} + \omega d_{N,j-1} \\ &= d_{0,0}d_{N,j} + d_{0,1}d_{N,j-1} - d_{1,0}d_{N,j-1} + \omega d_{N,j-1} \\ &= d_{0,0}d_{N,j} = 0 \end{aligned} \tag{A.15}$$

If $d_{0,0} = 0$, then $f(u,u)$ is indeterminate. It is null for $d_{N,j} = 0$ for all j.
For the u_1^{N+1} term, the prefactor is $d_{0,0}d_{N,0} - d_{N,0}d_{0,0} = 0$.
For $u_1^i u_3^{N+1}$, $1 \leqslant i \leqslant N$,

$$\begin{aligned} &\sum_{N+1=\alpha+\beta}^{0\leqslant\alpha,\beta\leqslant N} (d_{0,\alpha}d_{i-1,\beta}) - \sum_{i=\alpha+\beta}^{0\leqslant\alpha,\beta\leqslant N} (d_{\alpha,0}d_{\beta,N}) + d_{i,0}d_{0,N} + \omega d_{N,j-1} \\ &= d_{0,1}d_{i-1,N} - d_{0,0}d_{i,N} - d_{1,0}d_{i-1,N} + \omega d_{N,j-1} \\ &= -d_{0,0}d_{i,N} = 0 \end{aligned} \tag{A.16}$$

For u_3^{N+1}, the prefactor is $-d_{0,0}d_{0,N} + d_{0,0}d_{0,N} = 0$.

(v) $u_1^p u_3^q,\ 2 \leqslant p,q \leqslant N:$

$$\sum_{p=\alpha+\beta}^{0\leqslant\alpha,\beta\leqslant N}(d_{0,\alpha}d_{p-1,\beta})-\sum_{q=\alpha+\beta}^{0\leqslant\alpha,\beta\leqslant N}(d_{\alpha,0}d_{\beta,q-1})-(d_{p-1,0}d_{0,q}-d_{p,0}d_{0,q-1})+\omega d_{p-1,q-1}$$
$$=d_{0,0}\left(d_{p-1,q}-d_{p,q-1}\right)+d_{p-1,q-1}\left(d_{0,1}+\omega-d_{1,0}\right)$$
$$=d_{0,0}\left(d_{p-1,q}-d_{p,q-1}\right)=0 \tag{A.17}$$

which is satisfied by putting $d_{p-1,q}=d_{p,q-1}$.

(vi) $u_1 u_3^q,\ 3\leqslant q\leqslant N:$

$$\sum_{q=\alpha+\beta}^{0\leqslant\alpha,\beta\leqslant N}(d_{0,\alpha}d_{0,\beta})-\sum_{1=\alpha+\beta}^{0\leqslant\alpha,\beta\leqslant N}(d_{\alpha,0}d_{\beta,q-1})-(d_{0,0}d_{0,q}-d_{1,0}d_{0,q-1})+\omega d_{0,q-1}$$
$$=d_{0,0}\left(d_{0,q}-d_{1,q-1}\right)+d_{0,q-1}\left(d_{0,1}+\omega-d_{1,0}\right) \tag{A.18}$$
$$=-d_{0,0}\left(d_{1,q-1}\right)=0$$

which is satisfied by putting $d_{1,q-1}=0$ for all $q=3,\ldots,N$.
Similarly for $u_1^p u_3,\ 3\leqslant p\leqslant N$, the prefactor is

$$\sum_{1=\alpha+\beta}^{0\leqslant\alpha,\beta\leqslant N}(d_{0,\alpha}d_{p-1,\beta})-\sum_{p=\alpha+\beta}^{0\leqslant\alpha,\beta\leqslant N}(d_{\alpha,0}d_{\beta,0})-(d_{p-1,0}d_{0,1}-d_{p,0}d_{0,0})+\omega d_{p-1,0}$$
$$=d_{0,0}\left(d_{p-1,1}-d_{p,0}\right)+d_{p-1,0}\left(d_{0,1}+\omega-d_{1,0}\right) \tag{A.19}$$
$$=d_{0,0}d_{p-1,1}=0$$

(vii) $u_1^2 u_3:$

$$\sum_{1=\alpha+\beta}^{0\leqslant\alpha,\beta\leqslant N}(d_{0,\alpha}d_{1,\beta})-\sum_{2=\alpha+\beta}^{0\leqslant\alpha,\beta\leqslant N}(d_{\alpha,0}d_{\beta,0})-(d_{1,0}d_{0,1}-d_{2,0}d_{0,0})+\omega d_{1,0}+\kappa$$
$$=d_{0,0}\left(d_{1,1}-d_{2,0}\right)+d_{1,0}\left(d_{0,1}+\omega-d_{1,0}\right)-d_{1,0}d_{0,1}+\kappa \tag{A.20}$$
$$=d_{0,0}d_{1,1}-d_{1,0}d_{0,1}+\kappa=0.$$

Similarly for $u_1u_3^2$, the prefactor is

$$\sum_{2=\alpha+\beta}^{0\leqslant\alpha,\beta\leqslant N}d_{0,\alpha}d_{0,\beta}-\sum_{1=\alpha+\beta}^{0\leqslant\alpha,\beta\leqslant N}d_{\alpha,0}d_{\beta,1}-(d_{0,0}d_{0,2}-d_{1,0}d_{0,1})+\omega d_{0,1}-\kappa$$
$$=d_{0,0}\left(d_{0,2}-d_{1,1}\right)+d_{0,1}\left(d_{0,1}+\omega-d_{1,0}\right)+d_{1,0}d_{0,1}-\kappa \tag{A.21}$$
$$=-\left(d_{0,0}d_{1,1}-d_{1,0}d_{0,1}+\kappa\right)=0.$$

For both cases, it requires

$$d_{1,1}=\frac{d_{1,0}d_{0,1}-\kappa}{d_{0,0}} \tag{A.22}$$

(vii) u_1u_3 :

$$\begin{aligned}&\sum_{1=\alpha+\beta}^{0\leqslant\alpha,\beta\leqslant N}(d_{0,\alpha}d_{0,\beta})-\sum_{1=\alpha+\beta}^{0\leqslant\alpha,\beta\leqslant N}(d_{\alpha,0}d_{\beta,0})-(d_{0,0}d_{0,1}-d_{1,0}d_{0,0})+\omega d_{0,0}\\&=d_{0,0}(d_{0,1}-d_{1,0})+d_{0,0}(d_{0,1}+\omega-d_{1,0})-d_{0,0}(d_{0,1}-d_{1,0})\\&=0.\end{aligned} \tag{A.23}$$

For u_1^p, $1\leqslant p\leqslant N$, the prefactor is $d_{0,0}d_{p-1,0}-d_{p-1,0}d_{0,0}=0$.
For u_3^q, $1\leqslant q\leqslant N$, the prefactor is $-d_{0,0}d_{0,q-1}+d_{0,0}d_{0,q-1}=0$.

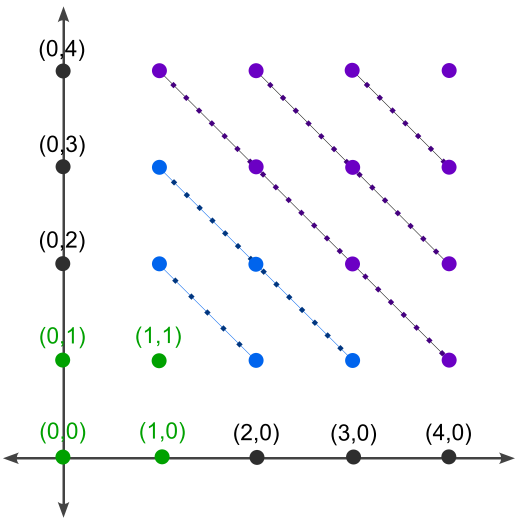
Consolidating the conditions obtained for nullifying each sub-terms, we have

1. $d_{1,0}=d_{0,1}+\omega$
2. $d_{1,1}=(d_{1,0}d_{0,1}-\kappa)/d_{0,0}$
3. $d_{i,0}=d_{0,i}=0,\ 2\leqslant i\leqslant N$
4. $d_{j,N}=d_{N,j}=0,\ 1\leqslant j\leqslant N$
5. $d_{1,p}=d_{p,1}=0,\ 2\leqslant p\leqslant N-1$
6. $d_{p-1,q}=d_{p,q-1},\ 2\leqslant p,q\leqslant N.$

We use condition 4, 5 and 6 in the list to show that, except for $d_{0,0},d_{1,1},d_{0,1}$ and $d_{1,0}$, all the values of $d_{i,j}$ are 0. We demonstrate this using $d_{p-1,q}=d_{p,q-1}$ as follows:

$$\begin{aligned}&0=d_{j,N}=d_{j+1,N-1}=d_{j+2,N-2}=\cdots=d_{N,j},\ \ 1\leqslant j\leqslant N\\&0=d_{1,j}=d_{2,j-1}=d_{3,j-2}=\cdots=d_{j,1},\ \ 2\leqslant j\leqslant N-1\end{aligned} \tag{A.24}$$

We represent (A.24) diagrammatically by representing $d_{i,j}$ as (i,j), and indicate the nullification as follows (for $N=4$ as an example):



The green points represents the coefficients which are non-zero.

The ansatz (A.2) then becomes

$$f(x,y)=\frac{x-y}{d_{0,0}+d_{0,1}\,(x+y)+\omega x+d_{1,1}xy},\;\; d_{1,1}=\frac{(d_{0,1}+\omega)\,d_{0,1}-\kappa}{d_{0,0}} \tag{A.25}$$

and also satisfies

$$\begin{aligned} f(u_1,u_2)+f(u_2,u_3)-f(u_1,u_3)-\omega f(u_1,u_2)f(u_2,u_3)\\ -\kappa f(u_1,u_2)f(u_1,u_3)f(u_2,u_3)=0. \end{aligned} \tag{A.26}$$

With the given form of $f(x,y)$, we substitute $d_{0,0}=c_0^2$ and $d_{0,1}=c_0c_1$ to rewrite it as

$$f(x,y)=\frac{x-y}{c_0^2+c_0c_1\,(x+y)+c_1^2xy+\omega x+\left(\frac{c_1}{c_0}\omega-\frac{\kappa}{c_0^2}\right)xy}. \tag{A.27}$$

Appendix B. Idempotent and degree-2 Nilpotent matrices of rank *r*

A rank r square matrix of dimension D is

$$\mathbf{A}_r=\sum_{i=1}^{r}C_iX_i^T \tag{B.1}$$

where $\{C_i|\ 1\leqslant i\leqslant r\}$ and $\{X_i|\ 1\leqslant i\leqslant r\}$ are sets of D-dimensional linearly independent column vectors.

To construct an idempotent matrix $\mathbf{B}_r$ of rank r, we require its Jordan canonical form to be a diagonal matrix, with r entries of 1 and 0 for the remaining entries. Hence, it is expressed in the below form

$$\mathbf{B}_r=\mathbf{Q}\ \mathrm{diag}\left[\underbrace{1,\ldots,1}_{r},0,\ldots,0\right]\ \mathbf{Q}^{-1} \tag{B.2}$$

where Q is any general invertible $D\times D$ matrix. By using

$$E_i=\left(0,\ldots,\underbrace{1}_{i},0,\ldots,0\right)^T \tag{B.3}$$

where 1 is in the i-th position of the column vector E_i, $\mathbf{B}_r$ is written as

$$\mathbf{B}_r=\sum_{i=1}^{r}\mathbf{Q}E_iE_i^T\mathbf{Q}^{-1}=\sum_{i=1}^{r}C_iX_i^T \tag{B.4}$$

with $C_i = \mathbf{Q}E_i$ and $X_i^T = E_i^T\mathbf{Q}^{-1}$. Then we need

$$X_i^T C_j = \delta_{ij} \tag{B.5}$$

for any $\mathbf{A}_r$ to be idempotent.

Similarly, to construct a Nilpotent matrix $\mathbf{N}_r$ of rank r of degree 2, that is, $\mathbf{N}_r^2 = \mathbf{0}$, we identify the following Jordan normal form (upto similarity):

$$\mathbf{Q}^{-1}\mathbf{N}_r\mathbf{Q} = \begin{bmatrix} S_1 & 0 & \dots & 0 & \dots & 0 \\ 0 & S_2 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \dots & 0 \\ 0 & 0 & \dots & S_r & \dots & 0 \\ 0 & 0 & \dots & 0 & \ddots & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 \end{bmatrix} \tag{B.6}$$

where

$$S_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{B.7}$$

and the remaining diagonal blocks are null. Note that the rank of the nilpotent matrix satisfies $2r \leqslant D$. Writing Jordan matrix in terms of E_i, we have

$$\mathbf{N}_r = \sum_{i=1}^{r} \mathbf{Q}E_{2i-1}E_{2i}^T\mathbf{Q}^{-1} = \sum_{i=1}^{r} C_i X_i^T \tag{B.8}$$

with $C_i = \mathbf{Q}E_{2i-1}$ and $X_i^T = E_{2i}^T\mathbf{Q}^{-1}$. Then we have

$$X_i^T C_j = 0,\ \forall i,j \tag{B.9}$$

for any $\mathbf{A}_r$ to be a nilpotent matrix of degree 2.

Appendix C. Symmetries of the R-matrix

The YBE is also an over-determined system of atmost cubic polynomials for solving the matrix elements of $\mathcal{R}(f(u,v))$, which resides in $\mathcal{A}\otimes\mathcal{A}$. We define an algebra homomorphism ϕ_{ij} : $\mathcal{A}\otimes\mathcal{A} \to \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}$ where

$$\begin{aligned} \phi_{12}(x\otimes y) &= a\otimes b\otimes 1 \\ \phi_{23}(x\otimes y) &= 1\otimes a\otimes b \\ \phi_{13}(x\otimes y) &= a\otimes 1\otimes b \end{aligned} \tag{C.1}$$

such that $\mathcal{R}_{ij} = \phi_{ij}(\mathcal{R})$. In this manner (1) is constructed.

By considering $\mathcal{A} \equiv \mathbf{C}^2$, the R-matrix becomes a 4×4 matrix. Subsequently, using (1), we construct a maximal set of 64 equations with a total unknown of 16 variables. Using the notation followed in [18] for the YBE equations of an $N^2 \times N^2$ R-matrix ($N = \dim(\mathcal{A})$) :

$$\sum R_{ij}^{kl} E_{jl} \otimes E_{ik}, \quad E_{ij} = [(\delta_{ai}\delta_{bj})], \; a, b \in \left\{1, 2, \ldots, N^2\right\}, \tag{C.2}$$

which we call the Hietarinta notation, the YBE is written in the index notation

$$\mathcal{R}_{j_1 j_2}^{k_1 k_2}(u_1, u_2)\, \mathcal{R}_{k_1 j_3}^{l_1 k_3}(u_1, u_3)\, \mathcal{R}_{k_2 k_3}^{l_2 l_3}(u_2, u_3) = \mathcal{R}_{j_2 j_3}^{k_2 k_3}(u_2, u_3)\, \mathcal{R}_{j_1 k_3}^{k_1 l_3}(u_1, u_3)\, \mathcal{R}_{k_1 k_2}^{l_1 l_2}(u_1, u_2) \tag{C.3}$$

where the repeated indices imply summation. By using the above notation, the following symmetries of the R-matrix are revealed:

(i) $\mathcal{R}_{ij}^{kl} \to \mathcal{R}_{kl}^{ij}$ [Transposition]
(ii) $\mathcal{R}_{ij}^{kl} \to \mathcal{R}_{(i+n) \bmod N,\ (j+n) \bmod N}^{(k+n) \bmod N,\ (l+n) \bmod N}$ [Index incremention]
(iii) $\mathcal{R}_{ij}^{kl} \to \mathcal{R}_{ji}^{lk}$[Inversions]

along with the local basis transformation and multiplicity freedom of the R-matrix

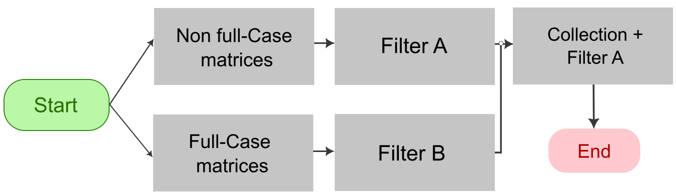
$$\mathcal{R} \to g\,(K \otimes K)\,\mathcal{R}\,(K \otimes K)^{-1} \tag{C.4}$$

for some non-singular $K \in \mathcal{A}$ and complex function g. These invariances allow for identification of the repeated solutions.

Appendix D. Pseudocodes towards removing repeated CYBE solutions

D.1. Algorithm workflow

To remove repeating R-matrix solutions, we utilise their symmetries. We refer to Transposition, Inversions and Index incrementions as *point transformations* and employ similarity transformations separately. The following workflow demonstrates this algorithm.



D.1.1. Filter A workflow. We consider all the matrix results Rl that have some zero elements and generate equivalence class `caseunion[i]` $\equiv [i]$ based on point transformations. Correspondingly, we construct subcase graph `subgraphs[i]` $\equiv g[i]$ for every $[i]$.

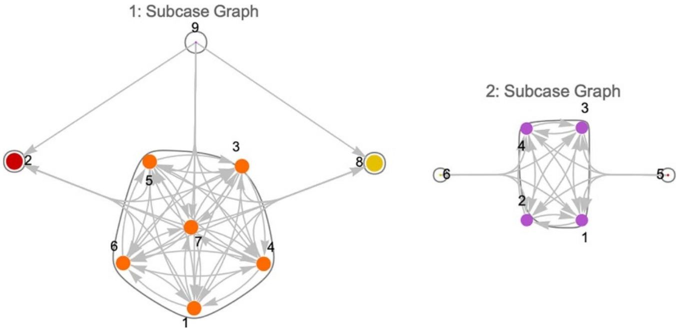


Figure D1. Examples of subcase graphs found in our numerical analysis. The solution index $2, 8$ and any of $1, 2, 3, 4$ are considered from the $[i]$s represented from left figure respectively.

The subcase graph is defined as $g[i] = \{a \to b \text{if } a \text{is transformable from } b\ \forall\, a, b\ \in [i]\}$. To check whether b should be brought into a, we first transform both matrices closer to a triangular matrix via point transformations using algorithm 6. Then, we solve for the re-substitution of parameters in b towards a using algorithm 4.

The pseudocode (algorithm 2) is the main routine for generating classifier objects. Algorithm 3 produces all possible permutations of matrices invariant under transposition, inversions and index incrementions.

For every corresponding $[i]$ we use the generated $g[i]$ to manually choose results that are not subcases of other solutions. Some of the generated $g[i]$s are shown in figure D1. In general, we circumvent the highly parameterised results from our computation to consider simpler results for the similarity transformations.

The union U of all the selected results from every $[i]$ is then used to generate the following subcase graph $gs = \{b \to a \text{if } f(a, b, t)\, \forall\, a, b \in U\}$, where $f(a, b, t) \equiv$ `IsSimilar[a,b,t]` from algorithm 5. The parameter t is used to limit the computation time (in seconds). We manually remove the subcases and finalise the list of solutions.

D.1.2. Filter B workflow. All full-case matrices Rf have a structure matrix that has no zero elements. These correspond to heavily coupled models untenable for our study. Hence, divide them into many non-full matrices to identify new results for further simplification.

First, we generate a subcase graph $g[i] = \{a \to b \text{if } a \text{is transformable from } b\ \forall\, a, b\ \in Rf\}$. Results that are not subcases of other solutions are considered. Then, we use algorithm 1 to break them into valid matrix solutions with zero elements and finalise the list.

D.1.3. Collection + filter A workflow. From the Filter A and Filter B workflows, all results are collated and run through the Filter A again which then provides a set of unique solutions.

D.2. Pseudocodes for the algorithm/routines used

Algorithm 1. Full model simplification routine.

Description: Breaks list of full-matrix results into non-full matrix results
Input: List of matrices M
Output: List of non-full matrices `Ml`
Require: Each matrix from M are square matrices

```
procedure FullCaseSimplify(Ml)
    Local Ml ← {}
    foreach m ∈ Ml do
        Local vars ← all variables of m
        Local varsubs ← Variable replacement of vars to unique c_i
        m ← m after applying varsubs
        Local temp ← {m with c_i = 0 ∀c_i}
        temp ← temp after removing Indeterminate cases
        Ml ← Ml ∪ temp
    Return Ml
```

Algorithm 2. Non-full model classifier.

Description: Classifies all the non-full matrix results
Input: List of matrix results Rl
Output: List `caseunion`, List `subgraphs`
Require: Elements N of Rl are square matrices and $S(N)$ has some zero elements

```
procedure (FullCaseSimplify(Rl))
    Local struct ← {S(x) ∀ x ∈ Rl} (after removing duplicates)
    Local matrsolgraph[i] ← {x ∈ Rl s.t S(x) = S(y)} where y is ith element of struct
    Local grpcases ← {1,2,...,Length(struct)}/ ∼ where i ∼ j if S(i)
        ∈ RmatrixInvariances(S(j),20)
    Local caseunion[i] ← ⋃_{j∈[i]} matrsolgraph[j] ∀ [i] ∈ grpcases
    forall the caseunion[i] ≡ x do
        x ← {TriFormat(i,120), i ∈ x}
        Local subgraphs[i] ← {If IsTransformable(x_n, x_m, 120) then m → n, ∀ 1 ⩽ n,m ⩽ #x}
        subgraphs[i] ← subgraphs[i] with all cliques identified
    caseunion ← {caseunion[i], 1 ⩽ i ⩽ Length(grpcases)}
    subgraphs ← {subgraphs[i], 1 ⩽ i ⩽ Length(grpcases)}
    Return caseunion, subgraphs
```

Algorithm 3. Routine for making a table of R-matrices invariant under their symmetries.

Description: Produce a set of matrices invariant to R-matrix symmetries (except for similarity transformation)
Input: Matrix R, Integer N
Output: List `RMatrices`
Require: $N > 0$, R is a square matrix

```
procedure RMatrixInvariances(R, N)
    Local RMatrices ← {R}
    Local NewCases ← {}
        for i = 1, i < N+1, i++ do
        NewCases ← {}
        NewCases ← NewCases ∪ (RMatrices after transposition)
        NewCases ← NewCases ∪ (RMatrices after index incremention)
        NewCases ← NewCases ∪ (RMatrices after inversion)
        NewCases ← DeleteDuplicates(NewCases)
        foreach m ∈ NewCases do
            if m ∉ RMatrices then
                RMatrices ← RMatrices ∪{m}
    RMatrices ← DeleteDuplicates(RMatrices)
    Return RMatrices
```

Algorithm 4. Routine to check if one R-matrix is transformed into another.

Description: Checks if constant matrix M_1 be transformed to M_2 by variable substitution
Input: Matrix M_1, M_2, Integer t (time, seconds)
Output: Boolean `isvalid`
Require: M_1, M_2 are of same dimensions

```
procedure IsTransformable(M1, M2, t)
    Local isvalid ← False
    Local vars1 ← all variables from M1
    Local vars2 ← all variables from M2
    Local varssubs[1] ← Variable replacement of vars1to unique a_i
    Local varssubs[2] ← Variable replacement of vars2to unique b_i
    if Length(vars1)<Length(vars2) then
        Returnisvalid ← False
    Local solset ← (M1 − M2) after applying varsubs[1], varsubs[2]
    Local varset ← {all a_i}
    Local sols ← TimeConstrained(Solve(solset = 0,varset),t,{})
    foreach s ∈ sols do
        if M1 = M2 after applying varsubs[1], varsubs[2] and s then
            Local isvalid ← True
    Return isvalid
```

Algorithm 5. Routine to check if one R-matrix is similar to another.

Description: Checks if constant matrix M_1 is similar to M_2 upto variable substitution
Input: Matrix M_1, M_2, Integer t (time, seconds)
Output: Boolean `isvalid`
Require: M_1, M_2 are of dimension 4×4

procedure (IsSimilar(M_1, M_2, t))
- Local `isvalid` ← False
- Local $Q = \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix} \otimes \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}$
- Local `vars1` ← all variables from M_1
- Local `vars2` ← all variables from M_2
- Local `varssubs[1]` ← Variable replacement of `vars1` to unique a_i
- Local `varssubs[2]` ← Variable replacement of `vars2` to unique b_i
- **if** Length(*vars*1)+4 <Length(*vars*2) **then**
  - Return `isvalid` ← False
- Local `solset` ← $(Q \cdot M_1 \cdot Q^{-1} - M_2)$ after applying `varsubs[1]`, `varsubs[2]`
- Local `varset` ← $\{$all $a_i$and $q_1, q_2, q_3, q_4\}$
- Local `sols` ← TimeConstrained(Solve(`solset` $= \mathbf{0}$, `varset`), t, {})
- **foreach** $s \in$ *sols* **do**
  - **if** $Q.M_1.Q^{-1} = M_2$ *after applying* *varsubs[1]*, *varsubs[2]* and s **then**
    - Local `isvalid` ← True
- Return `isvalid`

Algorithm 6. Routine to transform R-matrix closer to a triangular matrix.

Description: Transform the matrix into an upper-triangular matrix structure upto R-matrix symmetries
Input: Matrix M_1, Integer N
Output: Matrix `M`
Require: M_1 is a square matrix

procedure TriFormat(M_1, N)
- Local `d` ← Dimension of M_1
- Local `rlist` ← RMatrixInvariances(M_1, N)
- Local `weight` ← {TriWeight(x) $\forall\, x \in$ `rlist`}
- Local `weight` ← $\{x[1] + 2^d * ($Sum$(x[2] + x[3]))\ \forall\, x \in$ `weight`$\}$
- Local `index` ← Position of highest value in `weight`
- Local `mout` ← *rlist*[*index*]
- Return `mout`

Algorithm 7. Helper function to Triformat routine.

Description: Assign a weight to a matrix to indicate its proximity to an upper triangular matrix.
Input: Matrix M_1
Output: Number `n`
Require: M_1 is a square matrix

```
procedure (TRIWEIGHT(M_1))
    Local d ← Dimension of M_1
    Local w1 ← (Sum of upper triangular elements of S(M_1)+1) / (Sum of lower triangular elements of S(M_1)+1)
    MASK(s) = [(1+s)d + s + (−1)^s j − |i − j|]_ij ,  1 ⩽ i,j ⩽ d
    Local w2 ← table of sum of every upper diagonal terms of MASK(1) ∘ S(M_1)
    Local w3 ← table of sum of every lower diagonal terms of MASK(0) ∘ S(M_1)
    Local n ← {w1, w2, w3}
    Return n
```

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