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On unconditionality of fractional Rademacher chaos in symmetric spaces

S. V. Astashkin and K. V. Lykov

Abstract. We study density estimates of an index set $\mathcal{A}$ under which the unconditionality (or even the weaker property of random unconditional divergence) of the corresponding Rademacher fractional chaos $\{r_{j_1}(t)\times r_{j_2}(t)\cdots r_{j_d}(t)\}_{(j_1,j_2,\dots,j_d)\in\mathcal{A}}$ in a symmetric space X implies its equivalence in X to the canonical basis in ℓ_2. In the special case of Orlicz spaces L_M, unconditionality of this system is also shown to be equivalent to the fact that a certain exponential Orlicz space embeds into L_M.

Keywords: Rademacher functions, Rademacher chaos, symmetric space, combinatorial dimension, unconditional convergence.

§ 1. Introduction

As usual, the Rademacher functions are defined as follows: if $0\leqslant t\leqslant 1$, then

$$r_j(t):=(-1)^{[2^j t]},\qquad j=1,2,\dots,$$

where $[x]$ denotes the integer part of a real number x (that is, the greatest integer not exceeding x). According to the classical Khintchine inequality (see [1], and also [2]), for any $p\geqslant 1$, there exists a constant C_p such that, for arbitrary $a_j\in\mathbb{R}$, $j=1,2,\dots$,

$$\biggl\|\sum_{j=1}^\infty a_j r_j\biggr\|_{L_p[0,1]}\leqslant C_p\biggl(\sum_{j=1}^\infty a_j^2\biggr)^{1/2}. \tag{1}$$

It is well known that $C_p\leqslant\sqrt{p}$ (the sharp values of the constants in this inequality were given by Haagerup [3]). In the opposite direction, Szarek [4] proved that, for all $p\geqslant 1$ and $a_k\in\mathbb{R}$, $k=1,2,\dots$,

$$\frac{1}{\sqrt{2}}\biggl(\sum_{j=1}^\infty a_j^2\biggr)^{1/2}\leqslant\biggl\|\sum_{j=1}^\infty a_j r_j\biggr\|_{L_p[0,1]}. \tag{2}$$

These inequalities, which gave an impetus for an enormous number of investigations and generalizations, have found numerous applications in various fields of analysis. Recall that Khintchine proved inequality (1) "by pursuing the goal of finding

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the 'right' rate of convergence in the strong law of large numbers of Borel" [5]. At the same time, from the point of view of the geometry of Banach spaces, inequalities (1) and (2) indicate that the spaces $L_p[0,1]$, $1 \leqslant p < \infty$, which are not Hilbert spaces for $p \neq 2$, still contain subspaces isomorphic to ℓ_2. A characterization of the symmetric spaces X in which the sequence $\{r_j\}_{j=1}^\infty$ is equivalent to the canonical basis in ℓ_2 was given by Rodin and Semenov in [6], who proved that this equivalence holds if and only if X contains the separable part of the Orlicz space $\operatorname{Exp} L^2$ generated by the function $N_2(u) = e^{u^2} - 1$. In [7], a similar question was studied for the system $\{r_{j_1}(t) \cdot r_{j_2}(t)\}_{j_1 > j_2}$ of products of Rademacher functions, which is usually called the second-order Rademacher chaos. Specifically, it was shown that this system is equivalent in X to the canonical basis in ℓ_2 if and only if X contains the separable part of the Orlicz space $\operatorname{Exp} L$ generated by the function $N_1(u) = e^u - 1$. Moreover, both these properties were found to be equivalent to the formally weaker (than the equivalence to the canonical basis in ℓ_2) property of unconditionality of the basic sequence $\{r_{j_1}(t) \cdot r_{j_2}(t)\}_{j_1 > j_2}$ in X (see [8]). Note that the Rademacher system itself is an unconditional (and even symmetric with constant 1) basic sequence in any symmetric space (see for example, Proposition 2.2 in [2]). The next step in the study of the behaviour of the Rademacher chaos in symmetric spaces was made by the authors of the present paper by employing the important concept of combinatorial dimension developed earlier by Blei (see [10]–[14]). Namely, in [9] it was shown that the above results in [7] and [8] can be extended to a non-complete chaos $\{r_{j_1}(t) \cdot r_{j_2}(t) \cdots r_{j_d}(t)\}_{(j_1, j_2, \dots, j_d) \in \mathcal{A}}$ if the combinatorial dimension of the corresponding index set $\mathcal{A} \subset \mathbb{N}^d$ is d.

The main purpose of this paper is to find conditions on an index set $\mathcal{A}$ under which the unconditionality of the system $\{r_{j_1}(t) \cdot r_{j_2}(t) \cdots r_{j_d}(t)\}_{(j_1, j_2, \dots, j_d) \in \mathcal{A}}$ in a symmetric space X guaranties its equivalence in X to the canonical basis in ℓ_2. In particular, bearing in mind the aforementioned specifics in the behaviour of the chaos in comparison with the Rademacher system itself, we investigate a quantitative dependence of the behaviour of such a subsystem on the combinatorial dimension of the corresponding index set. To achieve this goal, we slightly modify the notion of combinatorial dimension using one-sided density estimates for an index set $\mathcal{A}$, which allows us to substantially extend the scope of estimates of the form (1).

A new effect appearing in the present paper is worth pointing out. According to Theorem 1 below, certain density estimates of an index set guarantee that "remoteness" of a symmetric space X from the "extreme" space L_∞ is a consequence of the so-called random unconditional divergence (RUD) property of the system $\{r_{j_1}(t) \cdot r_{j_2}(t) \cdots r_{j_d}(t)\}_{(j_1, j_2, \dots, j_d) \in \mathcal{A}}$ in X, which is weaker than its unconditionality. Thus, in this case, such a system possesses the RUD property in a symmetric space X if and only if it is equivalent in X to the canonical basis in ℓ_2 (see Theorem 2). In the special case of the Orlicz spaces L_M, basic properties of the system $\{r_{j_1}(t) \cdot r_{j_2}(t) \cdots r_{j_d}(t)\}_{(j_1, j_2, \dots, j_d) \in \mathcal{A}}$ can also be characterized in terms of continuous embeddings of certain exponential Orlicz spaces into L_M (see Theorem 3). Note that related results for Orlicz spaces were obtained earlier by Blei and Ge [15] and [16], who, instead of dealing with unconditionality properties of the system, provide a more detailed analysis of the combinatorial dimension of the corresponding index set.

In the concluding part of the present paper, we show that every uniformly bounded Bessel system (in particular, any Rademacher chaos) in a symmetric space X such that $\operatorname{Exp} L^2 \subset X$ possesses the random unconditional convergence (RUC) property, which is in a certain sense opposite to the RUD property. In addition, we give a concrete example illustrating the interesting fact of “divergence” of the moment estimates of a Rademacher fractional chaos and its asymptotic behaviour (see also [14]).

§ 2. Preliminaries

In what follows, any embedding of a given Banach space into another one is assumed to be continuous, that is, $X_1 \subset X_0$ means that if $x \in X_1$, then $x \in X_0$ and $\|x\|_{X_0} \leqslant C\|x\|_{X_1}$ for some $C > 0$. If the value of the embedding constant C is important for our analysis, we will additionally write $X_1 \overset{C}{\subset} X_0$. The notation of $F_1 \asymp F_2$ means that $cF_1 \leqslant F_2 \leqslant CF_1$ for some constants $c > 0$ and $C > 0$, and these constants are independent of all or a part of the arguments of F_1 and F_2; it should be clear from the context which arguments are involved.

By $|\cdot|$ we denote either the absolute value of a number (or a function) or the cardinality of a set, depending on the context.

2.1. Symmetric spaces. A detailed exposition of the theory of symmetric spaces can be found in the books [17]–[19].

Let $\mathcal{S}$ be the set of (equivalence classes of) measurable almost everywhere finite real-valued functions on $[0,1]$ with the usual Lebesgue measure μ.

The *distribution function* of a function $x = x(t) \in \mathcal{S}$ is defined as follows:

$$n_x(\tau) = \mu\{t \colon x(t) > \tau\}, \qquad \tau \in \mathbb{R}.$$

Two functions x and y are called *equidistributed* if they have the same distribution functions; they are *equimeasurable* if the functions $|x|$ and $|y|$ are equidistributed.

For any function $x = x(t) \in \mathcal{S}$, there exists a unique decreasing left-continuous non-negative function $x^* = x^*(t)$ on $[0,1]$ equimeasurable with $x(t)$; this function, which is referred to as the *rearrangement* of x, is given by the formula (see [17], § 2.2)

$$x^*(t) = \inf\{\tau : \ n_{|x|}(\tau) < t\}.$$

Definition 1. A Banach space X, $X \subset \mathcal{S}$, is said to be *ideal* if the conditions $x \in X$, $y \in \mathcal{S}$ and $|y| \leqslant |x|$ imply that $y \in X$ and $\|y\|_X \leqslant \|x\|_X$. A Banach ideal space X is said to be *symmetric* if the conditions $x \in X$, $y \in \mathcal{S}$ and $y^* = x^*$ imply that $y \in X$ and $\|y\|_X = \|x\|_X$.

By definition, if x lies in a symmetric space, then this space also contains all the functions equimeasurable with x.

Let us give some examples of symmetric spaces on $[0,1]$. As usual, the space $L_p = L_p[0,1]$, $1 \leqslant p < \infty$, consists of all functions $x \in \mathcal{S}$ with

$$\|x\|_p := \left(\int_0^1 |x(t)|^p \, dt\right)^{1/p} < \infty.$$

For $p > q$, we have $L_p \overset{1}{\subset} L_q$. In the limit case $p \to \infty$, we have the space L_∞ with the norm

$$\|x\|_\infty := \operatorname*{ess\,sup}_{t\in[0,1]} |x(t)| = \inf\bigl\{C\colon \mu\{t \in [0,1]\colon |x(t)| > C\} = 0\bigr\}.$$

Orlicz spaces appear as natural generalizations of L_p-spaces. Let $M = M(u)$ be an *Orlicz function*, that is, a convex non-negative function on $[0,\infty)$ which is not identically zero and $M(0) = 0$. The Orlicz space L_M consists of all functions $x = x(t)$ such that

$$\int_0^1 M\left(\frac{|x(t)|}{\lambda}\right) dt < \infty$$

for some $\lambda > 0$. The norm in L_M is defined by

$$\|x\|_{L_M} := \inf\left\{\lambda > 0\colon \int_0^1 M\left(\frac{|x(t)|}{\lambda}\right) dt \leqslant 1\right\}.$$

In particular, $L_{M_p} = L_p$ isometrically if $M_p(u) = u^p$. By $\operatorname{Exp} L^r$, $r > 0$, we will denote the exponential Orlicz space generated by an Orlicz function $N_r(u)$ such that, for some $u_0 > 0$, $\log N_r(u) \asymp u^r$ if $u > u_0$.

We will repeatedly use the following extrapolation description of the exponential Orlicz spaces $\operatorname{Exp} L^r$ (see [20], formulas (2)–(4), [21], § 2, or [13], Ch. X, Lemma 18):

$$\|x\|_{\operatorname{Exp} L^r} \asymp \sup_{p\geqslant 1} \frac{\|x\|_p}{p^{1/r}}. \tag{3}$$

For a more detailed account of Orlicz spaces, see, for instance, the book [22].

Let φ be a continuous increasing concave function on $[0,1]$, $\varphi(0) = 0$. The *Lorentz space* $\Lambda(\varphi)$ consists of all functions $x \in \mathcal{S}$ such that

$$\|x\|_{\Lambda(\varphi)} := \int_0^1 x^*(t)\, d\varphi(t),$$

and the *Marcinkiewicz space* $\mathcal{M}(\varphi)$ consists of all functions $x \in \mathcal{S}$ such that

$$\|x\|_{\mathcal{M}(\varphi)} := \sup_{t\in(0,1]} \frac{\varphi(t)}{t} \int_0^t x^*(s)\, ds.$$

For any symmetric space X on $[0,1]$, we have $L_\infty \subset X \subset L_1$ (see Theorem II.4.1 in [17]). The closure of L_∞ in a symmetric space X is referred as the *separable part* of X, and is denoted by X°. If $X \neq L_\infty$, then X° is a separable symmetric space.

An important characteristic of a symmetric space X is its fundamental function ϕ_X defined by

$$\phi_X(t) := \|\chi_{(0,t)}\|_X, \qquad t \in [0,1].$$

Throughout the paper, χ_A is the characteristic function (indicator) of a set $A \subset [0,1]$. The fundamental function of a symmetric space is quasiconcave (that is, $\phi_X(t)$ is increasing, $\phi_X(t)/t$ is decreasing, and $\phi_X(0) = 0$). Recall also that any quasiconcave function is equivalent to its smallest concave majorant (in the sense of the relation $\asymp$ defined above; see [17], the corollary after Theorem II.1.1). In particular,

$$\phi_{\mathcal{M}(\varphi)}(t) = \phi_{\Lambda(\varphi)}(t) = \varphi(t), \qquad \phi_{L_M}(t) = \frac{1}{M^{-1}(1/t)}.$$

Note that Orlicz and Marcinkiewicz spaces are equal under certain conditions. Namely (see [23], [24]), $L_M = \mathcal{M}(\varphi)$ if and only if

$$\varphi(t) \asymp \frac{1}{M^{-1}(1/t)} \tag{4}$$

and

$$\int_0^1 M\left(\frac{\varepsilon}{\varphi(t)}\right) dt < \infty \quad \text{for some } \varepsilon > 0. \tag{5}$$

The Lorentz space $\Lambda(\varphi)$ has the following extremal property in the class of symmetric spaces: if $\phi_X(t) \leqslant C\varphi(t)$ for some $C > 0$ and all $t \in [0,1]$, then $\Lambda(\varphi) \subset X$ (see [17], Theorem II.5.5). In particular, the Lorentz space $\Lambda(\varphi)$ is the smallest space among all symmetric spaces with the fundamental function $\varphi(t)$. The Marcinkiewicz space $\mathcal{M}(\varphi)$ is the biggest space in the same class (see [17], Theorem II.5.7). So, if a symmetric space X is such that $\phi_X = \varphi$, then the following continuous embeddings holds:

$$\Lambda(\varphi) \subset X \subset \mathcal{M}(\varphi). \tag{6}$$

2.2. Combinatorial dimension and (α, β)-sets. Based on the notion of the fractional Cartesian product (see [10]), Blei put forward the following definition of the combinatorial dimension of a set (see [11] and Ch. XIII in [13], which is a good source of many interesting applications of this notion). Let $d \in \mathbb{N}$ and $\mathbb{N}^d := \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}$ (d factors), where $\mathbb{N}$ is the set of positive integers.

Definition 2. A set $\mathcal{A} \subset \mathbb{N}^d$ is said to have *combinatorial dimension* α if

1) for an arbitrary $\beta > \alpha$, there exists $C_\beta > 0$ such that, for any $n \in \mathbb{N}$ and every collection of sets $B_1, B_2, \dots, B_d \subset \mathbb{N}$, $|B_1| = |B_2| = \cdots = |B_d| = n$,

$$|\mathcal{A} \cap (B_1 \times B_2 \times \cdots \times B_d)| < C_\beta n^\beta;$$

2) for an arbitrary $\gamma < \alpha$ and $k \in \mathbb{N}$, there exist $n > k$ and sets $B_1, B_2, \dots, B_d \subset \mathbb{N}$, $|B_1| = |B_2| = \cdots = |B_d| = n$, such that

$$|\mathcal{A} \cap (B_1 \times B_2 \times \cdots \times B_d)| > n^\gamma.$$

It is known that, for each real $\alpha \in [1, d]$, there exists a set of combinatorial dimension α (see [12] or Ch. XIII in [13]).

Note that in Definition 2 there is a certain asymmetry between the lower and upper density estimates for a set $\mathcal{A}$. We will use the following modification of this definition in which these estimates are considered separately.

Definition 3. Let $\mathcal{A} \subset \mathbb{N}^d$, $\alpha \geqslant 1$. We will say that a set $\mathcal{A}$ is a *super-α-set* if, for some $c_{\mathcal{A}} > 0$ and each $n \in \mathbb{N}$, there exist sets $B_1, B_2, \dots, B_d$ such that $|B_j| = n$, $j = 1, 2, \dots, d$, and

$$|\mathcal{A} \cap (B_1 \times B_2 \times \cdots \times B_d)| \geqslant c_{\mathcal{A}} n^\alpha.$$

Let us emphasize that, in contrast to the second condition of Definition 2, in Definition 3, for *each* positive integer n, there exist sets $B_1, B_2, \dots, B_d$ for which the lower density estimate holds.

Definition 4. Let $\mathcal{A} \subset \mathbb{N}^d$, $\beta \leqslant d$. We will say that $\mathcal{A}$ is a *sub-β-set* if, for some $C_{\mathcal{A}} > 0$, each $n \in \mathbb{N}$, and all sets $B_1, B_2, \dots, B_d$, $|B_j| = n$, $j = 1, 2, \dots, d$,

$$|\mathcal{A} \cap (B_1 \times B_2 \times \cdots \times B_d)| \leqslant C_{\mathcal{A}} n^{\beta}.$$

Definition 5. A set $\mathcal{A} \subset \mathbb{N}^d$ which is both a super-α-set and a sub-β-set will be called an (α, β)-*set*.

Let us mention some immediate consequences of the above definitions. If $\mathcal{A}$ is an (α, β)-set, then $\alpha \leqslant \beta$. Each super-α-set is an (α, d)-set. Each (α, α)-set $\mathcal{A}$ has combinatorial dimension α; we will say that such a set has *exact* combinatorial dimension α. Note also that, for any $1 \leqslant \alpha < \beta \leqslant d$, there exists an (α, β)-set that is not a (α', β')-set if at least one of the inequalities $\alpha < \alpha'$ or $\beta > \beta'$ holds (see Ch. XIII, Theorem 19 in [13]).

2.3. Systems of random unconditional convergence and divergence in Banach spaces. Recall that a sequence $\{x_k\}_{k=1}^{\infty}$ of elements of a Banach space X is called *basic* if it is a basis in its closed linear span. A sequence $\{x_{\pi(k)}\}_{k=1}^{\infty}$ which is a basic sequence for any bijection $\pi\colon \mathbb{N} \to \mathbb{N}$ is said to be an *unconditional basic sequence*. It is well known that a basic sequence $\{x_k\}_{k=1}^{\infty}$ in a Banach space X is unconditional in X if and only if there exists $D > 0$ such that, for any $n \in \mathbb{N}$, any collection of signs $\{\theta_k\}_{k=1}^{n}$, $\theta_k = \pm 1$, and all $a_k \in \mathbb{R}$,

$$\left\| \sum_{k=1}^{n} \theta_k a_k x_k \right\|_X \leqslant D \left\| \sum_{k=1}^{n} a_k x_k \right\|_X.$$

A detailed account of basic and unconditional basic sequences can be found, for instance, in the books [25]–[27].

Each of the next notions is a natural relaxation of that of an unconditional basic sequence.

Definition 6. A basic sequence $\{x_k\}_{k=1}^{\infty}$ in a Banach space X is called a *system of random unconditional convergence with constant* D (a D-RUC system, for short), where $D > 0$, if, for any $n \in \mathbb{N}$ and $a_k \in \mathbb{R}$, $k = 1, 2, \dots, n$,

$$\int_0^1 \left\| \sum_{k=1}^{n} r_k(u) a_k x_k \right\|_X du \leqslant D \left\| \sum_{k=1}^{n} a_k x_k \right\|_X.$$

A basic sequence $\{x_k\}_{k=1}^{\infty}$ in a Banach space X is called a *system of random unconditional divergence with constant* D (a D-RUD system, for short), where $D > 0$, if, for any $n \in \mathbb{N}$ and $a_k \in \mathbb{R}$, $k = 1, 2, \dots, n$,

$$\left\| \sum_{k=1}^{n} a_k x_k \right\|_X \leqslant D \int_0^1 \left\| \sum_{k=1}^{n} r_k(u) a_k x_k \right\|_X du.$$

If the exact value of the constant D of a D-RUC (a D-RUD, respectively) system is immaterial for us, such a system will simply be called an RUC (respectively, an RUD) *system*.

The abbreviation RUC (respectively, RUD) stands for “Random Unconditional Convergence” (respectively, “Random Unconditional Divergence”). The concept of an RUC system was introduced in [28], where many important properties of such systems were also established. Subsequently, the behaviour of RUC and RUD systems in various function spaces was intensively studied by many authors (see for example, [29]–[34]).

It is clear that a basic sequence is unconditional in a Banach space if and only if it is both an RUC and an RUD sequence in this space (see also Proposition 2.3 in [32]). Moreover, it easily follows from the definitions that a basic sequence is a 1-RUC system (respectively, a 1-RUD system) if and only if it is 1-unconditional (see Propositions 2.7 and 2.8 in [32]).

Let $d \in \mathbb{N}$. By Δ^d we will denote the “lower triangular” subset of the set $\mathbb{N}^d$, that is,

$$\Delta^d := \{(j_1, j_2, \dots, j_d) \in \mathbb{N}^d \colon j_1 > j_2 > \dots > j_d\}.$$

Throughout, by $\jmath$ we denote multi-indices $(j_1, j_2, \dots, j_d) \in \Delta^d$, $d \in \mathbb{N}$. Next, $\{r_\jmath\}_{\jmath \in \Delta^d}$ is the usual sequence of Rademacher functions (see § 1) numbered in some (fixed) order by multi-indices $\jmath \in \Delta^d$. We also set $\mathbf{r}_\jmath(t) := r_{j_1}(t) \cdot r_{j_2}(t) \cdots r_{j_d}(t)$, $\jmath = (j_1, j_2, \dots, j_d) \in \Delta^d$. It is known that the system $\{\mathbf{r}_\jmath\}_{\jmath \in \Delta^d}$ (considered in the lexicographic order of $\jmath \in \Delta^d$) is basic in any symmetric space X (see Theorem 2 in [9]). However, in this paper, the numbering order of the system $\{\mathbf{r}_\jmath\}_{\jmath \in \Delta^d}$ is immaterial.

§ 3. Main results

Our first result, which plays a key role in this paper, shows that, under certain non-restrictive conditions on the density characteristics of an index set, the RUD property of the corresponding subsystem of the Rademacher chaos in a symmetric space X ensures that X is located sufficiently “far” from the space L_∞.

Theorem 1. *Let X be a symmetric space, $d \in \mathbb{N}$, $\alpha, \beta, b \in \mathbb{R}$, $1 \leqslant \alpha, \beta, b \leqslant d$, $\alpha + b/\beta > b + 1$. Let also $\mathcal{A} \subset \Delta^d$ be an (α, β)-set such that, for some $D > 0$ and any finite set $\mathcal{A}' \subset \mathcal{A}$,*

$$\Big\| \sum_{\jmath \in \mathcal{A}'} \mathbf{r}_\jmath \Big\|_X \leqslant D \int_0^1 \Big\| \sum_{\jmath \in \mathcal{A}'} r_\jmath(u) \mathbf{r}_\jmath \Big\|_X \, du. \tag{7}$$

Then $X \supset \operatorname{Exp} L^{2/b}$.

In particular, this embedding holds if $\{\mathbf{r}_\jmath\}_{\jmath \in \mathcal{A}}$ is an RUD sequence in X for some $(\alpha - \varepsilon, \alpha + \varepsilon)$-set $\mathcal{A}$ whenever $\alpha > b$ and $\varepsilon > 0$ is sufficiently small.

Proof. Note first that the functions $\varphi(t) = \log^{-b/2}(e/t)$ and $M(u) = \exp(u^{2/b}) - 1$ satisfy conditions (4) and (5). Therefore, $\operatorname{Exp} L^{2/b} = \mathcal{M}(\log^{-b/2}(e/t))$. Since, for each $\gamma > b/2$, the space $\mathcal{M}(\log^{-b/2}(e/t))$ is continuously embedded into the Lorentz space $\Lambda(\log^{-\gamma}(e/t))$ (see Corollary 1 in [9]), the theorem will be proved once we show that $\Lambda(\log^{-\gamma}(e/t)) \subset X$ for some $\gamma > b/2$.

It follows from the conditions of the theorem that $\alpha > 1$. We choose $\alpha_0 \in (1, \alpha)$ so that $\alpha_0 + b/\beta > b + 1$. By the assumption, for each sufficiently large $n \in \mathbb{N}$, there exist sets $B_1, B_2, \ldots, B_d$ such that $|B_j| = n$, $j = 1, 2, \ldots, d$, and

$$|\mathcal{A} \cap \mathcal{B}_n| \geqslant n^{\alpha_0},$$

where $\mathcal{B}_n := B_1 \times B_2 \times \cdots \times B_d$. Let us fix an n and a set $\mathcal{B}_n$ satisfying the above conditions. Since $|\mathcal{A} \cap \mathcal{B}_n| \leqslant n^d$, there exists $\delta \in [\alpha_0, d]$ depending on n and $\mathcal{B}_n$ such that

$$|\mathcal{A} \cap \mathcal{B}_n| = n^{\delta}. \tag{8}$$

We claim that there exists a set $U_n \subset [0,1]$ such that $\mu(U_n) > 1 - 2(e/2)^{-dn}$ and, for all $u \in U_n$,

$$\Big\| \sum_{\jmath \in \mathcal{A} \cap \mathcal{B}_n} r_\jmath(u) \mathbf{r}_\jmath \Big\|_\infty \leqslant \sqrt{2d}\, n^{(\delta+1)/2}. \tag{9}$$

Indeed, by (8) and in view of Bernstein's inequality (see for example, [35], Ch. 1, § 6, formula (42), or [2], Proposition 1.2), we have, for any $t \in [0,1]$ and $\lambda > 0$,

$$\mu\Big\{ u \in [0,1] \colon \Big| \sum_{\jmath \in \mathcal{A} \cap \mathcal{B}_n} r_\jmath(u) \mathbf{r}_\jmath(t) \Big| > \lambda \Big\} < 2e^{-\lambda^2/(2n^\delta)},$$

which implies

$$\mu\Big\{ u \in [0,1] \colon \Big| \sum_{\jmath \in \mathcal{A} \cap \mathcal{B}_n} r_\jmath(u) \mathbf{r}_\jmath(t) \Big| > \sqrt{2d}\, n^{(\delta+1)/2} \Big\} < 2e^{-dn}.$$

Note that $\{\mathbf{r}_\jmath\}_{\jmath \in \mathcal{A} \cap \mathcal{B}_n}$ contains at most dn distinct Rademacher functions. Therefore, there are at most 2^{dn} variants of the values of the sequence $\{\mathbf{r}_\jmath(t)\}_{\jmath \in \mathcal{A} \cap \mathcal{B}_n}$ where t runs over $[0,1]$. Therefore, from the preceding estimate we have

$$\mu\Big\{ u \colon \Big| \sum_{\jmath \in \mathcal{A} \cap \mathcal{B}_n} r_\jmath(u) \mathbf{r}_\jmath(t) \Big| > \sqrt{2d}\, n^{(\delta+1)/2} \text{ for some } t \in [0,1] \Big\} < 2^{dn} \cdot 2e^{-dn}.$$

If now U_n is the complement of the set from the last estimate, then $\mu(U_n) > 1 - 2(e/2)^{-dn}$ and, for all $u \in U_n$, we have (9). This proves the claim.

For all $u \in [0,1]$, we have

$$\Big\| \sum_{\jmath \in \mathcal{A} \cap \mathcal{B}_n} r_\jmath(u) \mathbf{r}_\jmath \Big\|_\infty \leqslant n^{\delta}$$

(see (8)), and hence, by (9)

$$\begin{aligned}
&\int_0^1 \Big\| \sum_{\jmath \in \mathcal{A} \cap \mathcal{B}_n} r_\jmath(u) \mathbf{r}_\jmath \Big\|_\infty du \\
&\quad \leqslant \int_{[0,1] \setminus U_n} \Big\| \sum_{\jmath \in \mathcal{A} \cap \mathcal{B}_n} r_\jmath(u) \mathbf{r}_\jmath \Big\|_\infty du + \int_{U_n} \Big\| \sum_{\jmath \in \mathcal{A} \cap \mathcal{B}_n} r_\jmath(u) \mathbf{r}_\jmath \Big\|_\infty du \\
&\quad \leqslant n^{\delta} \cdot 2 \Big(\frac{2}{e} \Big)^{dn} + \sqrt{2d}\, n^{(\delta+1)/2}.
\end{aligned}$$

Therefore,

$$\int_0^1 \Big\| \sum_{\jmath \in \mathcal{A} \cap \mathcal{B}_n} r_\jmath(u) \mathbf{r}_\jmath \Big\|_\infty du \leqslant C n^{(\delta+1)/2}, \tag{10}$$

where the constant C depends only on d.

On the other hand, for some set of points $t \in [0,1]$ of measure 2^{-dn}, each Rademacher function involved in the sum assumes the value 1, and hence

$$\Big\| \sum_{\jmath \in \mathcal{A} \cap \mathcal{B}_n} \mathbf{r}_\jmath \Big\|_X \geqslant \|n^\delta \chi_{(0,2^{-dn})}\|_X \geqslant n^\delta \phi_X(2^{-dn}), \tag{11}$$

where ϕ_X is the fundamental function of X. Using successively condition (4), embedding (7), the embedding $L_\infty \subset X$, estimate (10), and the inequality $\alpha_0 \leqslant \delta$, we obtain

$$\begin{aligned}\phi_X(2^{-dn}) &\leqslant n^{-\delta} \Big\| \sum_{\jmath \in \mathcal{A} \cap \mathcal{B}_n} \mathbf{r}_\jmath \Big\|_X \leqslant n^{-\delta} D \int_0^1 \Big\| \sum_{\jmath \in \mathcal{A} \cap \mathcal{B}_n} r_\jmath(u) \mathbf{r}_\jmath \Big\|_X du \\ &\leqslant n^{-\delta} C_1 \int_0^1 \Big\| \sum_{\jmath \in \mathcal{A} \cap \mathcal{B}_n} r_\jmath(u) \mathbf{r}_\jmath \Big\|_\infty du \leqslant C_2 n^{-(\delta-1)/2} \leqslant C_3 n^{-(\alpha_0-1)/2}.\end{aligned}$$

Since this inequality holds for all sufficiently large $n \in \mathbb{N}$ and the function ϕ_X is quasiconcave, we have, for all $t \in [0,1]$,

$$\phi_X(t) \leqslant C \log^{-\gamma_0}\Big(\frac{e}{t}\Big),$$

where $\gamma_0 = (\alpha_0 - 1)/2 > 0$. Hence, $\Lambda(\log^{-\gamma_0}(e/t)) \subset \Lambda(\phi_X) \subset X$, and if $\gamma_0 > b/2$, that is, if $\alpha_0 > b+1$, then the required result holds. In the case $\gamma_0 \leqslant b/2$ (or, equivalently, $\alpha_0 \leqslant b+1$), we proceed as follows.

According to Blei's inequalities (see [13], Ch. VII, formula (9.30) and Ch. XIII, Corollary 29, or [14], formula (1.7)), for the same δ as above, all $p \geqslant 1$ and $u \in [0,1]$,

$$\Big\| \sum_{\jmath \in \mathcal{A} \cap \mathcal{B}_n} r_\jmath(u) \mathbf{r}_\jmath \Big\|_p \leqslant C p^{\beta/2} \Big(\sum_{\jmath \in \mathcal{A} \cap \mathcal{B}_n} (r_\jmath(u))^2 \Big)^{1/2} = C p^{\beta/2} n^{\delta/2}.$$

Therefore, by the extrapolation description (3) of the exponential Orlicz space $\mathrm{Exp}L^{2/\beta}$, we conclude that

$$\Big\| \sum_{\jmath \in \mathcal{A} \cap \mathcal{B}_n} r_\jmath(u) \mathbf{r}_\jmath \Big\|_{\mathrm{Exp}\, L^{2/\beta}} \leqslant C n^{\delta/2}.$$

Hence, the equality $\mathrm{Exp}\, L^{2/\beta} = \mathcal{M}(\log^{-\beta/2}(e/t))$ and the definition of the norm in Marcinkiewicz spaces (see § 2.1) imply that, for all $u \in [0,1]$,

$$\Big(\sum_{\jmath \in \mathcal{A} \cap \mathcal{B}_n} r_\jmath(u) \mathbf{r}_\jmath \Big)^*(t) \leqslant C n^{\delta/2} \log^{\beta/2}\Big(\frac{e}{t}\Big), \qquad 0 < t \leqslant 1.$$

Combining the last inequality with (9), we have, for all $u \in U_n$,

$$\Big(\sum_{\jmath\in\mathcal{A}\cap\mathcal{B}_n} r_\jmath(u)\mathbf{r}_\jmath\Big)^*(t) \leqslant Cn^{\delta/2}\min\Big\{n^{1/2}, \log^{\beta/2}\Big(\frac{e}{t}\Big)\Big\}, \qquad 0<t\leqslant 1. \tag{12}$$

Next, setting $\gamma_{k+1} = \gamma_0 + \gamma_k/\beta$, $k=0,1,\dots$, where still $\gamma_0 = (\alpha_0-1)/2$, let us show that, for each $k=0,1,\dots$,

$$\Lambda\Big(\log^{-\gamma_k}\Big(\frac{e}{t}\Big)\Big) \subset X. \tag{13}$$

This embedding holds for $k=0$, and so it suffices to verify that (13) with γ_k implies (13) with γ_{k+1}.

Indeed, from inequalities (11), (7) and (12) we have

$$\begin{aligned}
\phi_X(2^{-dn}) &\leqslant n^{-\delta}\Big\|\sum_{\jmath\in\mathcal{A}\cap\mathcal{B}_n}\mathbf{r}_\jmath\Big\|_X \leqslant Dn^{-\delta}\int_0^1\Big\|\sum_{\jmath\in\mathcal{A}\cap\mathcal{B}_n} r_\jmath(u)\mathbf{r}_\jmath\Big\|_X\,du\\
&\leqslant Cn^{-\delta}\Big(\int_{[0,1]\setminus U_n}\Big\|\sum_{\jmath\in\mathcal{A}\cap\mathcal{B}_n} r_\jmath(u)\mathbf{r}_\jmath\Big\|_\infty du + \int_U\Big\|\sum_{\jmath\in\mathcal{A}\cap\mathcal{B}_n} r_\jmath(u)\mathbf{r}_\jmath\Big\|_{\Lambda(\log^{-\gamma_k}(e/t))} du\Big)\\
&\leqslant C\cdot 2\Big(\frac{e}{d}\Big)^{-dn}\\
&\quad + C'n^{-\delta/2}\Big(\int_0^{e^{1-n^{1/\beta}}} n^{1/2}\,d\log^{-\gamma_k}\Big(\frac{e}{t}\Big) + \int_{e^{1-n^{1/\beta}}}^1 \log^{\beta/2}\Big(\frac{e}{t}\Big)\,d\log^{-\gamma_k}\Big(\frac{e}{t}\Big)\Big)\\
&\leqslant C''n^{-(\delta/2+\gamma_k/\beta-1/2)} \leqslant C''n^{-(\alpha_0/2+\gamma_k/\beta-1/2)} = C''n^{-\gamma_{k+1}}.
\end{aligned}$$

Hence, since the fundamental functions are quasiconcave, we arrive at (13) with γ_{k+1} in place of γ_k.

We next note that

$$\gamma_k = \gamma_0\sum_{i=0}^k \frac{1}{\beta^i} \to \frac{\beta\gamma_0}{\beta-1} \quad \text{as} \quad k\to\infty.$$

In addition, by the assumption $\alpha_0 > b+1-b/\beta$ and $\beta \geqslant 1$, and hence

$$\frac{\beta\gamma_0}{\beta-1} > \frac{1}{2}\Big(b-\frac{b}{\beta}\Big)\frac{\beta}{\beta-1} = \frac{b}{2}.$$

By the above relations, $\gamma_k > b/2$ for some sufficiently large k, and now the required result follows, as observed at the beginning of the proof. This proves Theorem 1.

In the case $b=1$, we get the following result.

Corollary 1. *Let X be a symmetric space and let $d\in\mathbb{N}$. Suppose that $\mathcal{A}\subset\Delta^d$ is an (α,β)-set with $\alpha+1/\beta>2$ such that, for some $D>0$ and any finite set $\mathcal{A}'\subset\mathcal{A}$,*

$$\Big\|\sum_{\jmath\in\mathcal{A}'}\mathbf{r}_\jmath\Big\|_X \leqslant D\int_0^1\Big\|\sum_{\jmath\in\mathcal{A}'} r_\jmath(u)\mathbf{r}_\jmath\Big\|_X\,du.$$

Then $\operatorname{Exp}L^2\subset X$.

In particular, this embedding holds if $\{\mathbf{r}_\jmath\}_{\jmath\in\mathcal{A}}$ *is an RUD sequence in* X *for some* $(\alpha-\varepsilon,\alpha+\varepsilon)$*-set* $\mathcal{A}$ *whenever* $\alpha>1$ *and* $\varepsilon>0$ *is sufficiently small.*

Theorem 2. *Let* X *be a symmetric space and let* $d\in\mathbb{N}$*. Assume that* $\mathcal{A}\subset\Delta^d$ *is an* (α,β)*-set,* $\alpha+1/\beta>2$*. Then the following conditions are equivalent:*

(a) $\{\mathbf{r}_\jmath\}_{\jmath\in\mathcal{A}}$ *is an RUD sequence in* X*;*

(b) $\{\mathbf{r}_\jmath\}_{\jmath\in\mathcal{A}}$ *is an unconditional basic sequence in* X*;*

(c) $\{\mathbf{r}_\jmath\}_{\jmath\in\mathcal{A}}$ *is equivalent in* X *to the canonical basis in* ℓ_2*, that is, for some constant* C_X*,*

$$C_X^{-1}\|\{a_\jmath\}_{\jmath\in\mathcal{A}}\|_{\ell_2}\leqslant\Big\|\sum_{\jmath\in\mathcal{A}}a_\jmath\mathbf{r}_\jmath\Big\|_X\leqslant C_X\|\{a_\jmath\}_{\jmath\in\mathcal{A}}\|_{\ell_2}. \tag{14}$$

In particular, if $\alpha>1$*, then for any* $(\alpha-\varepsilon,\alpha+\varepsilon)$*-set* $\mathcal{A}$*, where* $\varepsilon>0$ *is sufficiently small, conditions (a), (b) and (c) are equivalent.*

It is clear that we need to verify only the implication (a)⇒(c). However, this result is an immediate consequence of Corollary 1 and the following assertion.

Proposition 1. *Let* X *be a symmetric space such that* $\operatorname{Exp}L^2\subset X$*. Then there exists a constant* C' *such that, for each uniformly bounded* D*-RUD sequence* $\{x_j\}_{j\in\mathbb{N}}$ *from* X*, the following Khintchine type inequality holds:*

$$\Big\|\sum_{j\in\mathbb{N}}a_jx_j\Big\|_X\leqslant C'D\sup_{j\in\mathbb{N}}\|x_j\|_\infty\cdot\Big(\sum_{j\in\mathbb{N}}a_j^2\Big)^{1/2}.$$

Proof. It is known (see, for example, Lemma 3 in [9]) that, for every Orlicz function M and any measurable function $z=z(u,t)$ defined on $[0,1]\times[0,1]$,

$$\int_0^1\|z(u,\cdot)\|_{\mathrm{L}_M(\cdot)}\,du\leqslant 2\operatorname*{ess\,sup}_{t\in[0,1]}\|z(\cdot,t)\|_{\mathrm{L}_M(\cdot)}. \tag{15}$$

Therefore, from the conditions of the proposition, by applying the Khintchine inequality to the Rademacher system in the space $\operatorname{Exp}L^2$ (see [36], Ch. V, Theorem 8.7, or [6]), we have

$$\begin{aligned}\Big\|\sum_{j\in\mathbb{N}}a_jx_j\Big\|_X&\leqslant D\int_0^1\Big\|\sum_{j\in\mathbb{N}}r_j(u)a_jx_j\Big\|_X du\leqslant DC\int_0^1\Big\|\sum_{j\in\mathbb{N}}r_j(u)a_jx_j(\cdot)\Big\|_{\operatorname{Exp}L^2(\cdot)}du\\&\leqslant 2DC\operatorname*{ess\,sup}_{t\in[0,1]}\Big\|\sum_{j\in\mathbb{N}}r_j(\cdot)a_jx_j(t)\Big\|_{\operatorname{Exp}L^2(\cdot)}\leqslant C'D\operatorname*{ess\,sup}_{t\in[0,1]}\Big(\sum_{j\in\mathbb{N}}(a_jx_j(t))^2\Big)^{1/2}\\&\leqslant C'D\sup_{j\in\mathbb{N}}\|x_j\|_\infty\cdot\Big(\sum_{j\in\mathbb{N}}a_j^2\Big)^{1/2}.\end{aligned}$$

This proves Proposition 1, and, therefore, Theorem 2.

Theorem 2 illustrates that the difference in the behaviour of the Rademacher sequence $\{r_j\}$ and the chaos $\{r_{j_1}r_{j_2}\}_{j_1>j_2}$, which was mentioned in the introduction, is due to the different combinatorial dimensions of the index sets corresponding to these systems. Moreover, this result implies that unconditionality of a subsystem $\{\mathbf{r}_\jmath\}_{\jmath\in\mathcal{A}}$ of the chaos of any order d in a symmetric space X and its equivalence in X to the canonical basis in ℓ_2 are equivalent whenever the corresponding index set $\mathcal{A}$ has exact combinatorial dimension $\alpha > 1$.

For Orlicz spaces, Theorem 2 can be refined. Namely, if a set $\mathcal{A}$ has exact combinatorial dimension $\alpha > 1$, then the above conditions (a), (b) and (c) can be characterized in terms of certain embeddings.

Theorem 3. *Let L_M be an Orlicz space, $d \in \mathbb{N}$. Suppose that a set $\mathcal{A} \subset \Delta^d$ has exact combinatorial dimension $\alpha > 1$. Then the following conditions are equivalent:*

(i) $\{\mathbf{r}_\jmath\}_{\jmath\in\mathcal{A}}$ *is an RUD sequence in L_M;*

(ii) $\{\mathbf{r}_\jmath\}_{\jmath\in\mathcal{A}}$ *is an unconditional basic sequence in L_M;*

(iii) $\{\mathbf{r}_\jmath\}_{\jmath\in\mathcal{A}}$ *is equivalent in L_M to the canonical basis in ℓ_2, that is, for some constant C_M,*

$$C_M^{-1}\|\{a_\jmath\}_{\jmath\in\mathcal{A}}\|_{\ell_2} \leqslant \Big\|\sum_{\jmath\in\mathcal{A}} a_\jmath \mathbf{r}_\jmath\Big\|_{L_M} \leqslant C_M\|\{a_\jmath\}_{\jmath\in\mathcal{A}}\|_{\ell_2};$$

(iv) $L_M \supset \operatorname{Exp} L^{2/\alpha}$.

Proof. The equivalence of conditions (i), (ii) and (iii) is secured by Theorem 2. So, it only remains to verify that (iii) is equivalent to (iv).

Assume first that embedding (iv) holds. Applying again Blei's inequalities (see [13], Ch. VII, formula (9.30) and Ch. XIII, Corollary 29, or [14], formula (1.7)), we have, for all $p \geqslant 1$ and any sequence $\{a_\jmath\}_{\jmath\in A}$,

$$\Big\|\sum_{\jmath\in\mathcal{A}} a_\jmath \mathbf{r}_\jmath\Big\|_p \leqslant C(\alpha, d)p^{\alpha/2}\Big(\sum_{\jmath\in\mathcal{A}} a_\jmath^2\Big)^{1/2}.$$

Therefore, by the embedding $L_M \supset \operatorname{Exp} L^{2/\alpha}$ and the extrapolation description of the space $\operatorname{Exp} L^{2/\alpha}$ (see (3)), we have

$$\Big\|\sum_{\jmath\in\mathcal{A}} a_\jmath \mathbf{r}_\jmath\Big\|_{L_M} \leqslant C\Big\|\sum_{\jmath\in\mathcal{A}} a_\jmath \mathbf{r}_\jmath\Big\|_{\operatorname{Exp} L^{2/\alpha}} \leqslant C'\Big(\sum_{\jmath\in\mathcal{A}} a_\jmath^2\Big)^{1/2},$$

which gives the right-hand side inequality in (iii). The left-hand side of this inequality holds in each symmetric space X (because $X \subset L_1$, see also Lemma 6 in [9]). This proves the implication (iv)$\Rightarrow$(iii).

Now let us verify the implication (iii)$\Rightarrow$ (iv). By the assumption, the set $\mathcal{A}$ has exact combinatorial dimension α, and hence, for some constant $C > 0$ and each $n \in \mathbb{N}$, there exists a set $\mathcal{B}_n := B_1 \times B_2 \times \cdots \times B_d$ such that $|B_j| = n$, $j = 1, 2, \ldots, d$, and

$$C^{-1}n^\alpha \leqslant |\mathcal{A} \cap \mathcal{B}_n| \leqslant Cn^\alpha.$$

Now using (11) (with α instead of δ) and condition (iii), we have

$$\phi_{L_M}(2^{-dn}) \leqslant Cn^{-\alpha}\Big\|\sum_{\jmath\in\mathcal{A}\cap\mathcal{B}_n} \mathbf{r}_\jmath\Big\|_{L_M} \leqslant Cn^{-\alpha}C_M\Big(\sum_{\jmath\in\mathcal{A}\cap\mathcal{B}_n} 1\Big)^{1/2} \leqslant C'n^{-\alpha/2}.$$

Consequently, since ϕ_{L_M} is quasiconcave,

$$\phi_{L_M}(t) \leqslant C \log^{-\alpha/2}\left(\frac{e}{t}\right), \qquad t \in (0,1],$$

with some constant C. We have $\phi_{L_M}(t) = 1/M^{-1}(1/t)$, and hence by the last inequality,

$$\log^{\alpha/2}\left(\frac{e}{t}\right) \leqslant CM^{-1}\left(\frac{1}{t}\right),$$

or, equivalently,

$$M\left(C^{-1}\log^{\alpha/2}\left(\frac{e}{t}\right)\right) \leqslant \frac{1}{t}.$$

As a result, writing $C^{-1}\log^{\alpha/2}(e/t) = u$, we arrive at the inequality

$$M(u) \leqslant e^{(Cu)^{2/\alpha}-1} \quad \text{for} \quad u \geqslant 1.$$

By the definition of the norm in Orlicz spaces (see § 2.1), we have $L_M \supset \operatorname{Exp} L^{2/\alpha}$, as claimed. This completes the proof of Theorem 3.

§ 4. Concluding remarks

4.1. On the RUC property of uniformly bounded Bessel systems in symmetric spaces. According to Theorem 1, under certain conditions on density characteristics of an index set, the assumption that the corresponding subsystem of the Rademacher chaos has the RUD property in a symmetric space X implies that X is "far" from the space L_∞. In a certain sense, the opposite assertion is valid for the random unconditional convergence (RUC) property (see § 2.3) of such a subsystem. We obtain this result as a consequence of a more general fact related to uniformly bounded Bessel systems of functions. A similar assertion is known to hold under the extra conditions that $X \subset L_2$ and the system is orthonormal (see Proposition 2.1 in [31] and also Corollary 1.4 in [28]).

Recall that a bounded basic sequence $\{x_j\}_{j\in\mathbb{N}}$ in a Banach space X is a *Bessel* system if, for some constant $C(X)$ and any $a_j \in \mathbb{R}$, $j \in \mathbb{N}$,

$$\left(\sum_{j\in\mathbb{N}} a_j^2\right)^{1/2} \leqslant C(X)\left\|\sum_{j\in\mathbb{N}} a_j x_j\right\|_X.$$

Proposition 2. *Let X be a symmetric space such that $\operatorname{Exp} L^2 \subset X$. Then every uniformly bounded Bessel sequence $\{x_j\}_{j\in\mathbb{N}}$ has the RUC property in X.*

Proof. By the conditions of the proposition, inequality (15) for $L_M = \operatorname{Exp} L^2$ and using the Khintchine inequality in the space $\operatorname{Exp} L^2$ (see [36], Ch. V, Theorem 8.7, or [6]), we have

$$\begin{aligned}\int_0^1 \Big\|\sum_{j\in\mathbb{N}} r_j(u)a_jx_j\Big\|_X\,du &\leqslant C'\int_0^1\Big\|\sum_{j\in\mathbb{N}} r_j(u)a_jx_j(\,\cdot\,)\Big\|_{\operatorname{Exp}L^2(\,\cdot\,)}\,du\\ &\leqslant 2C'\operatorname*{ess\,sup}_{t\in[0,1]}\Big\|\sum_{j\in\mathbb{N}} r_j(\,\cdot\,)a_jx_j(t)\Big\|_{\operatorname{Exp}L^2(\,\cdot\,)} \leqslant C''\operatorname*{ess\,sup}_{t\in[0,1]}\Big(\sum_{j\in\mathbb{N}}(a_jx_j(t))^2\Big)^{1/2}\\ &\leqslant C''\sup_{j\in\mathbb{N}}\|x_j\|_\infty\cdot\Big(\sum_{j\in\mathbb{N}}a_j^2\Big)^{1/2} \leqslant C''C(X)\sup_{j\in\mathbb{N}}\|x_j\|_\infty\cdot\Big\|\sum_{j\in\mathbb{N}}a_jx_j\Big\|_X,\end{aligned}$$

proving Proposition 2.

Note that $\{\mathbf{r}_\jmath\}_{\jmath\in\Delta^d}$ is an uniformly bounded orthonormal sequence on $[0,1]$. Now from Proposition 2 we have the following result.

Corollary 2. *The system* $\{\mathbf{r}_\jmath\}_{\jmath\in\Delta^d}$ *is an RUC sequence in each symmetric space* X *with* $\operatorname{Exp}L^2\subset X$*.*

4.2. Asymptotic independence of a fractional Rademacher chaos. Let $d=3$, $\mathcal{A}=\{(i,j,i+j),\,1\leqslant i<j\}$. It is easily seen that $\mathcal{A}$ is a $(2,2)$-set. Therefore, by Theorem 3,

$$\Big\|\sum_{\jmath\in\mathcal{A}} a_\jmath\mathbf{r}_\jmath\Big\|_{\operatorname{Exp}L}\asymp\Big\|\{a_\jmath\}_{\jmath\in\mathcal{A}}\Big\|_{\ell_2}$$

and

$$\sup\Big\{\Big\|\sum_{\jmath\in E}a_\jmath\mathbf{r}_\jmath\Big\|_{\operatorname{Exp}L^\gamma}:\|\{a_\jmath\}_{\jmath\in E}\|_{\ell_2}\leqslant 1,\ E\subset\mathcal{A}\text{ is finite}\Big\}=\infty$$

for every $\gamma>1$. Moreover, if $\mathcal{A}_N:=\mathcal{A}\cap\{1,2,\dots,N\}^3$, where $N\in\mathbb{N}$, $N\geqslant 3$, then the sums

$$S_N:=|\mathcal{A}_N|^{-1/2}\sum_{\jmath\in\mathcal{A}_N}\mathbf{r}_\jmath$$

are normalized in L_2, and, by Theorem 1.5 in [14],

$$\sup_N\|S_N\|_p\asymp p,\qquad p\geqslant 1.$$

Consequently, (3) implies

$$\inf\Big\{\gamma:\ \sup_N\|S_N\|_{\operatorname{Exp}L^\gamma}=\infty\Big\}=1.$$

The last relation can be considered as a consequence of a certain “interdependence” of the functions $\mathbf{r}_\jmath$, $\jmath\in\mathcal{A}$. We claim that, at the same time, the sums S_N have asymptotically standard normal distribution corresponding to the space $\operatorname{Exp}L^2\subsetneqq\operatorname{Exp}L$. Hence the functions $\mathbf{r}_\jmath$, $\jmath\in\mathcal{A}$ are asymptotically independent like the usual Rademacher functions. This “divergence” in estimates for the moments of a Rademacher fractional chaos and its asymptotic behaviour was previously observed in [14]. To justify the last assertion, we will use Theorem 1.7 from [14].

Let

$$\mathcal{A}^*_{N,k} := \{(i,j,m) \in \mathcal{A}_N \colon k \in \{i,j,m\}\}, \qquad k \in \mathbb{N}.$$

We also consider the set $\mathcal{A}^\sharp_N \subset \mathcal{A}_N \times \mathcal{A}_N$ consisting of the pairs $((i,j,i+j),(k,l,k+l))$ of elements of the set $\mathcal{A}_N$ such that

$$\{i,j,i+j\} \cap \{k,l,k+l\} = \varnothing \tag{16}$$

and

$$\{i,j,i+j,k,l,k+l\} = \{i_1,j_1,i_1+j_1,k_1,l_1,k_1+l_1\} \tag{17}$$

for some $(i_1,j_1,i_1+j_1),(k_1,l_1,k_1+l_1) \in \mathcal{A}_N$ satisfying the conditions

$$(i_1,j_1,i_1+j_1) \neq (i,j,i+j) \quad \text{and} \quad (i_1,j_1,i_1+j_1) \neq (k,l,k+l). \tag{18}$$

To prove that the sums S_N have asymptotically standard normal distribution, it suffices to verify that

$$\lim_{N\to\infty} \max_k \frac{|\mathcal{A}^*_{N,k}|}{|\mathcal{A}_N|} = 0 \quad \text{and} \quad \lim_{N\to\infty} \frac{|\mathcal{A}^\sharp_N|}{|\mathcal{A}_N|^2} = 0$$

(see Theorem 1.7 in [14]). The first of these equalities is a consequence of the obvious estimates $|\mathcal{A}^*_{N,k}| \leqslant 3N$ and $|\mathcal{A}_N| \asymp N^2$. To verify the second claim it suffices to show that $\mathcal{A}^\sharp_N = \varnothing$.

Assume that $((i,j,i+j),(k,l,k+l)) \in \mathcal{A}^\sharp_N$, that is, (16) and (17) hold for some elements $(i_1,j_1,i_1+j_1),(k_1,l_1,k_1+l_1) \in \mathcal{A}_N$ satisfying (18). Let

$$V := \{i,j,i+j,k,l,k+l\} = \{i_1,j_1,i_1+j_1,k_1,l_1,k_1+l_1\}.$$

Then

$$\max\{x \colon x \in V\} = \max\{i+j,k+l\} = \max\{i_1+j_1,k_1+l_1\}$$

and

$$\Sigma_V = 2(i+j+k+l) = 2(i_1+j_1+k_1+l_1),$$

where Σ_V is the sum of all elements of the set V. Therefore, we either have $i+j = i_1+j_1, k+l = k_1+l_1$, or $i+j = k_1+l_1, k+l = i_1+j_1$, whence

$$\{i,j,k,l\} = \{i_1,j_1,k_1,l_1\}.$$

By the assumption, the numbers i, j, k, l, $i+j$, $k+l$ are pairwise distinct (see (16)), and so we have

$$i+k \neq i+j, \quad i+l \neq i+j, \quad j+k \neq i+j, \quad j+l \neq i+j, \quad k+l \neq i+j.$$

But hence the equality $i_1+j_1 = i+j$ gives $i_1 = i$, $j_1 = j$, which contradicts (18). Similarly, from the equality $i_1+j_1 = k+l$ we have $i_1 = k$, $j_1 = l$, which also contradicts (18).

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