Integr. Equ. Oper. Theory (2024) 96:25
https://doi.org/10.1007/s00020-024-02774-7
Published online August 23, 2024

Integral Equations and Operator Theory

C^*-Algebras Associated to Transfer Operators for Countable-to-One Maps

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Abstract. Our initial data is a transfer operator L for a continuous, countable-to-one map $\varphi : \Delta \to X$ defined on an open subset of a locally compact Hausdorff space X. Then L may be identified with a 'potential', i.e. a map $\varrho : \Delta \to X$ that need not be continuous unless φ is a local homeomorphism. We define the crossed product $C_0(X) \rtimes L$ as a universal C^*-algebra with explicit generators and relations, and give an explicit faithful representation of $C_0(X) \rtimes L$ under which it is generated by weighted composition operators. We explain its relationship with Exel–Royer's crossed products, quiver C^*-algebras of Muhly and Tomforde, C^*-algebras associated to complex or self-similar dynamics by Kajiwara and Watatani, and groupoid C^*-algebras associated to Deaconu–Renault groupoids. We describe spectra of core subalgebras of $C_0(X) \rtimes L$, prove uniqueness theorems for $C_0(X) \rtimes L$ and characterize simplicity of $C_0(X) \rtimes L$. We give efficient criteria for $C_0(X) \rtimes L$ to be purely infinite simple and in particular a Kirchberg algebra.

Mathematics Subject Classification. 47L30, 54H20; Secondary 37E99.

1. Introduction

Since 1970s transfer operators are indispensable tools in thermodynamical formalism and ergodic theory [7], and even earlier such operators, named averaging operators, played an important role in the study of Banach spaces $C(X)$ of continuous functions on a compact space X, see [49]. They are also crucial in the study of spectrum of weighted composition operators, see [2,3]. Transfer operators as a tool to construct C^*-algebras, were explicitly used for the first time by Exel in [18] to present Cuntz–Krieger algebras as crossed products associated to topological Markov chains. Since then a number of generalizations and modifications of such crossed products were introduced, see

This work was supported by the National Science Centre, Poland, Grant number 2019/35/B/ST1/02684.

for instance [9,11,21,22,44]. Their general structure as Cuntz–Pimsner algebras is now quite well-understood, see [10,39]. However, the detailed analysis of the associated C^*-algebras is usually limited to the case where the underlying mapping is a local homeomorphism on a compact Hausdorff space, see [9,11,15,22,23]. The exceptions are C^*-algebras associated to rational maps [27] or maps whose inverse branches form self-similar systems [28,29]. All these C^*-algebras can be viewed as crossed products by transfer operators for *finite-to-one* maps admitting at most finite number irregular points. However, in many problems there is a natural need to study transfer operators for partial continuous maps that are countable-to-one. This concerns in particular infinite graph C^*-algebras [9,11,39,51] or thermodynamic formalism for countable Markov shifts, interest in which has been growing in recent years, see [5,6,20,57]. In the present paper we give a general, comprehensive account of the main structural results for crossed products by transfer operators for arbitrary partial continuous maps that are countable-to-one.

More specifically we consider a continuous map $\varphi : \Delta \to X$ defined on an open subset Δ of a locally compact Hausdorff space X. We assume that $\varphi^{-1}(y)$ is countable for all $y \in \Delta$. Then every bounded transfer operator for φ is a map $L : C_0(\Delta) \to C_0(X)$ given by the formula

$$L(a)(y) = \sum_{x\in\varphi^{-1}(y)} \varrho(x)a(x)$$

where $\varrho : \Delta \to [0,\infty)$ is a map that we call a *potential*. A potential ϱ is in general only upper semi-continuous and the main role in our analysis is played by the following two sets:

$$\Delta_{\text{pos}} := \{x \in \Delta : \varrho(x) > 0\}, \quad \Delta_{\text{reg}} := \{x \in \Delta_{\text{pos}} : \varrho \text{ is continuous at } x\}.$$

So $\Delta_{\text{reg}} \subseteq \Delta_{\text{pos}} \subseteq \Delta \subseteq X$. As we show Δ_{reg} is an open subset of X and the restricted map $\varphi : \Delta_{\text{reg}} \to X$ is a local homeomorphism. We define the *crossed product* $C_0(X) \rtimes L$ as a universal C^*-algebra generated by the C^*-algebra $C_0(X)$ and weighted operators at, for $a \in C_0(\Delta)$, subject to relations

$$L(a) = t^*at,\ a \in C_0(\Delta), \quad a\sum_{i=1}^{n} u_i^K tt^* u_i^K = a,\ a \in C_c(\Delta_{\text{reg}})$$

where u_i^K's is a suitably normalized partition of unity on $K := \text{supp}(a)$, see (12) below. Apart from the case of covering maps on compact spaces treated in [23] this is the first general description of the crossed product $C_0(X) \rtimes L$ in terms of explicit relations coming from L. In other works the corresponding crossed product is usually defined and analyzed as the Cuntz–Pimsner algebra $\mathcal{O}_{M_L}$ associated to a C^*-correspondence M_L. We prove that $C_0(X) \rtimes L$ is isomorphic to $\mathcal{O}_{M_L}$ (Theorem 4.8). We do not know whether in general $C_0(X) \rtimes L$ can be naturally modeled by a topological quiver of Muhly and Tomforde [46]. One of our main structural result is the following the following version of (Cuntz–Krieger) uniqueness theorem (see Theorems 8.5, 8.7) that generalizes the corresponding results from [11,15,22,23].

Theorem A. *The following conditions are equivalent:*

(i) *Every representation of $C_0(X) \rtimes L$ is faithful provided it is faithful on $C_0(X)$.*

(ii) *The orbit representation of $C_0(X) \rtimes L$ on $\ell^2(X)$ is faithful; this representation sends function in $C_0(X)$ to operators of multiplication and the generator t to the weighted composition operator $Th := \sqrt{\varrho} h \circ \varphi$,*

(iii) *The map $\varphi : \Delta_{\mathrm{reg}} \to X$ is topologically free, that is the set of periodic points whose orbits are contained in Δ_{reg} has empty interior.*

If in addition $\Delta_{\mathrm{pos}} = \Delta_{\mathrm{reg}}$ the above conditions are further equivalent to

(iv) *$C_0(X)$ is a maximal abelian C^*-subalgebra of $C_0(X) \rtimes L$.*

In general we characterize faithful representations of $C_0(X) \rtimes L$ in terms of a canonical *generalized expectation* G for the inclusion $C_0(X) \subseteq C_0(X) \rtimes L$ (Theorem 5.9). We construct G using a *regular representation* of $C_0(X) \rtimes L$ on $\ell^2(X \rtimes \mathbb{Z})$. If $\Delta_{\mathrm{pos}} = \Delta_{\mathrm{reg}}$, then G is a genuine conditional expectation and $C_0(X) \rtimes L$ is naturally isomorphic to the C^*-algebra of the Renault–Deaconu groupoid for the partial local homeomorphism $\varphi : \Delta_{\mathrm{reg}} \to X$ (see Theorem 6.4). Then (iv) in Theorem A says that $C_0(X)$ is a Cartan subalgebra of $C_0(X) \rtimes L$ in the sense of Renault [55]. We show by example that if $\Delta_{\mathrm{pos}} \neq \Delta_{\mathrm{reg}}$, then topological freeness of $\varphi : \Delta_{\mathrm{reg}} \to X$ is not sufficient for maximal abeliannes of $C_0(X)$ in $C_0(X) \rtimes L$.

We say that L is *minimal* if there are no non-trivial open subsets $U \subseteq X$ such that $\varphi(U \cap \Delta_{\mathrm{pos}}) \subseteq U$ and $\varphi^{-1}(U) \cap \Delta_{\mathrm{reg}} \subseteq U$. As a corollary to Theorem A we get the following characterization of simplicity (see Theorem 8.11):

Theorem B. *If Δ_{reg} is infinite, then $C_0(X) \rtimes L$ is simple if and only if L is minimal.*

Inspired by notions of locally contractive groupoids [1] and contractive topological graphs [33] we define *contractive transfer operators*, see Definition 9.1. For such operators we get (see Theorem 9.5 and Corollary 9.6):

Theorem C. *If L is minimal and contractive, then $C_0(X) \rtimes L$ is purely infinite and simple. If in addition X is second countable, then $C_0(X) \rtimes L$ is a UCT-Kirchberg algebra (and so it is classifiable by its K-theory).*

We illustrate the power of Theorem C by showing that it covers and unifies all purely infinite results in [27,28], [1, Section 4], [19] (Examples 9.7, 9.8, 9.9).

Another fundamental C^*-algebra associated to L is the fixed point algebra of the canonical circle gauge action on $C_0(X) \rtimes L$. It is a direct limit $A_\infty = \overline{\bigcup_{n=0}^{\infty} A_n}$ of C^*-algebras

$$A_n = \overline{\operatorname{span}}\{a t^k t^{*k} b : a, b \in C_0(\Delta_k), k = 0, \ldots, n\}$$

where $\Delta_k = \varphi^{-k}(\Delta)$ is the natural domain for φ^k. The algebras A_n are interesting in their own right, see [36], and the C^*-algebra A_∞ has important dynamical interpretations. For special self-similar maps A_∞ was studied in [29]. When $\Delta_{\mathrm{reg}} = \Delta$, so that φ is a local homeomorphism, then A_∞ is a groupoid C^*-algebra of a generalized approximately proper equivalence relation on X. This is a crucial tool in the study of Gibbs states via the Radon–Nikodym problem [6,54]. Also stable C^*-algebras for irreducible Smale spaces are naturally Morita equivalent to algebras of the form A_∞ (see Remark 7.10 below). Putting $\Delta_{\mathrm{pos},n} := \{x \in \Delta_n : \prod_{i=0}^{n-1} \varrho(\varphi^i(x)) \neq 0\}$ we describe the spectra of A_n, $n \in \mathbb{N}$, and A_∞, as follows (see Proposition 7.2 and Theorems 7.6, 7.11):

Theorem D. *For each $n \in \mathbb{N}$ the algebra A_n is postliminary (Type I) and up to unitary equivalence all its irreducible representations are subrepresentations of the orbit representation on $\ell^2(X)$. Namely, we have a bijection*

$$\widehat{A}_n \cong \left(\bigsqcup_{k=0}^{n-1} \varphi^k(\Delta_{\mathrm{pos},k}) \setminus \Delta_{\mathrm{reg}} \right) \sqcup \varphi^n(\Delta_{\mathrm{pos},n}), \tag{1}$$

where a representation corresponding to $y \in \varphi^k(\Delta_{\mathrm{pos},k})$ is the restriction of the orbit representation to the subspace $\ell^2(\varphi^{-k}(x)) \subseteq \ell^2(X)$. The Jacobson topology on $\widehat{A}_n$ in general is finer than the pushout topology on the right-hand side of (1). But the two topologies coincide for instance when the potential ϱ is continuous, and if in addition X is second countable, then the primitive ideal space of A_∞ is homeomorphic to the quasi-orbit space:

$$\mathrm{Prim}(A_\infty) \cong X/\sim$$

where $x \sim y$ iff $\overline{\mathcal{O}(x)} = \overline{\mathcal{O}(y)}$ and the orbit of $x \in X$ is $\mathcal{O}(x) := \bigcup_{k=0, x\in\Delta_{\mathrm{pos},k}}^{\infty} \varphi^{-k}(\varphi^k(x))$.

The paper is organized as follows. In Sect. 2 we discuss transfer operators for partial maps and the properties of the associated potential ϱ. Covariant representations for transfer operators are introduced in Sect. 3. The crossed product $C_0(X) \rtimes L$ and its relationship with previous constructions modeled by Cuntz–Pimsner algebras are discussed in Sect. 4.

In Sect. 5, based on the well known gauge-invariance uniqueness for Cuntz–Pimsner algebras we prove faithfulness of the regular representation of $C_0(X) \rtimes L$, which in turn leads us to a generalized expectation-invariance uniqueness theorem. We use the latter in Sect. 6 to prove that Reanult–Deaconu groupoid C^*-algebras associated to a local homeomorphism φ is naturally isomorphic to $C_0(X) \rtimes L$. Section 7 is devoted to description of the spectrum of algebras A_n and A_∞ and it contains the proof of Theorem D. Section 8 introduces topological freeness for transfer operators and contains proof of Theorems A and B. Finally in Sect. 9 we give criteria for pure infiniteness for $C_0(X) \rtimes L$ (we prove Theorem C).

2. Transfer Operators for Partial Maps and Potentials

Throughout this paper $\varphi : \Delta \to X$ is a continuous map defined on an open subset Δ of a locally compact space X. We refer to (X, φ) as to a *partial dynamical system*. In addition we will fix a bounded transfer operator for (X, φ), which we will interpret as a *potential* for the system (X, φ). Namely, let us denote by $C_0(X)$ the C^*-algebra of continuous functions on X that vanish at infinity. We treat $C_0(\Delta)$ as an ideal in $C_0(X)$. By a *transfer operator* for (X,φ) we mean a positive linear map $L: C_0(\Delta)\to C_0(X)$ satisfying

$$L((a\circ\varphi)b)=aL(b),\quad a\in C_0(X),\quad b\in C_0(\Delta). \tag{2}$$

Remark 2.1. We could allow the transfer operator L to attain values in the bounded continuous functions $C_b(X)$, but then (2) forces L to take values in $C_0(X)$ anyway. Indeed, if $b\in C_c(\Delta)$ is compactly supported with the support K then taking $a\in C_c(X)$ such that $a|_{\varphi(K)}\equiv 1$ we get $L(b)=L((a\circ\varphi)b)=aL(b)\in C_c(X)$. Thus transfer operators map compactly supported functions to compactly supported ones.

Transfer operator could be defined in purely C^*-algebraic terms as follows. Let I be an ideal in a C^*-algebra A (by which we always mean a closed two-sided ideal). Let $\alpha: A\to M(I)$ be a non-degenerate $*$-homomorphism from A to the multiplier C^*-algebra $M(I)$ of I. Such maps are called *partial endomorphisms* of A in [22, Definition 1.1], [31, Definition 3.12]. A (bounded) *transfer operator for* α is a positive linear map $L: I\to A$ satisfying

$$L(\alpha(a)b)=aL(b),\quad a\in A,\quad b\in I. \tag{3}$$

Positivity implies that L is bounded and $*$-preserving. In addition the transfer equality (3) implies that $L(I)$ is an ideal. Transfer operators introduced in [22, Definition 1.2] are defined on a not necessarily closed ideal in I, and thus in general they are unbounded.

Having the triple (A,α,L) as above and assuming that $A=C_0(X)$, we necessarily have $I=C_0(\Delta)$, for an open set $\Delta\subseteq X$, and

$$\alpha(a)=a(\varphi(x)),\quad x\in\Delta,\quad a\in A,$$

for a continuous map $\varphi:\Delta\to X$. Accordingly, $M(I)=C_b(\Delta)$ consists of continuous bounded functions and $\alpha: C_0(X)\to C_b(\Delta)$. In particular, $\alpha: C_0(X)\to C_0(\Delta)\subseteq C_0(X)$ is an endomorphism of $C_0(X)$ if and only if the map $\varphi:\Delta\to X$ is *proper*, i.e. the preimage of every compact set in X is compact in Δ. Furthermore, denoting by $\mathcal{M}(X)$ the space of finite regular Borel measures on X equipped with the weak* topology, a transfer operator $L: C_0(\Delta)\to C_0(X)$ for α is of the form

$$L(a)(y)=\int_{\varphi^{-1}(y)} a(x)d\mu_y(x),\quad a\in C_0(\Delta),\quad y\in X, \tag{4}$$

where $X\ni y\longmapsto\mu_y\in\mathcal{M}(X)$ is a continuous map such that $\operatorname{supp}\mu_y\subseteq\varphi^{-1}(y)$ for every $y\in X$ and $\sup_{y\in X}\mu_y(X)=\|L\|<\infty$, cf., for instance, [2, 38, 39]. If the preimages of φ are countable, then this measure valued function can be replaced by a number valued function. We assume this throughout the paper.

Standing assumption

$$|\varphi^{-1}(y)|\leq\aleph_0\quad\text{for all}\quad y\in X. \tag{5}$$

Under this assumption, the measures $\{\mu_y\}_{y\in X}\subseteq \mathcal{M}(X)$ appearing in (4) are discrete and putting $\varrho(x):=\mu_{\varphi(x)}(\{x\})$, $x\in\Delta$, we get that the corresponding transfer operator is given by

$$L(a)(y)=\sum_{x\in\varphi^{-1}(y)}\varrho(x)a(x). \tag{6}$$

We refer to the map $\varrho: X\to[0,\infty)$ as to the *potential associated to* L, and we put

$$\Delta_{\mathrm{pos}}:=\Delta\setminus\varrho^{-1}(0)=\{x\in\Delta:\varrho(x)>0\}.$$

Obviously, every map admits a zero transfer operator (so that $\Delta_{\mathrm{pos}}=\emptyset$), but there is a large and important class of maps that admit a transfer operator with $\Delta_{\mathrm{pos}}=\Delta$. This concerns essentially all local homeomorphisms, see Theorem 6.4 below, and all open finite-to-one maps on compact spaces. This last claim follows from a result of Pavlov and Troitsky [47, Theorem 1.1]—we thank Magnus Goffeng for pointing this to us:

Theorem 2.2. (Pavilov and Troisky [47]) *Let $\varphi:\Delta\to X$ be a continuous surjection where Δ is a compact open subset of X. There exists a transfer operator $L: C(\Delta)\to C(X)$ with a strictly positive potential $\varrho:\Delta\to(0,+\infty)$ if and only if φ is an open map with $\sup_{x\in X}|\varphi^{-1}(x)|<\infty$.*

Proof. Under our assumptions the endomorphism $\alpha: C(X)\to C(\Delta)$, given by composition with φ, is a unital monomorphism – an inclusion. Conditional expectations E for the inclusion α are in bijective correspondence with transfer operators L for φ, given by $E=\alpha\circ L$. Thus the assertion follows from [47, Theorem 1.1] (in fact the 'if part' follows from the proof of [47, Theorem 4.3]). □

We fix a transfer operator L of the form (6). In general, ϱ has the following properties.

Proposition 2.3. *The potential ϱ is upper semi-continuous, and so ϱ is continuous at every point in $\varrho^{-1}(0)$. If $x_0\in\Delta_{\mathrm{pos}}=\Delta\setminus\varrho^{-1}(0)$, then the following are equivalent:*

(i) *ϱ is continuous at x_0,*
(ii) *φ is locally injective at x_0, i.e. there is open $U\subseteq\Delta$ with $x_0\in U$ such that $\varphi|_U: U\to X$ injective,*
(iii) *x_0 is a local homeomorphism point for φ, i.e. there is open U with $x_0\in U$ such that $\varphi: U\to\varphi(U)$ is a homeomorphism and $\varphi(U)$ is open in X.*

Moreover, φ restricted to Δ_{pos} is an open map.

Before we get into the proof of Proposition 2.3, we first prove a couple of lemmas.

Lemma 2.4. *Restriction of φ to Δ_{pos} is an open map.*

Proof. Every open set in $\Delta_{\rm pos}$ is of the form $U \cap \Delta_{\rm pos}$ where $U \subseteq \Delta$ is open in Δ. Let $y_0 \in \varphi(U \cap \Delta_{\rm pos})$ so that $y_0 = \varphi(x_0)$ for some $x_0 \in U \cap \Delta_{\rm pos}$. Take any continuous function $0 \le a \le 1$ supported on U and such that $a(x_0) = 1$. Then $\mu_{\varphi(x_0)}(a) \ge \varrho(x_0) > 0$ and $\mu_y(a) = 0$ for every $y \notin \varphi(U)$. Since the map $X \ni y \to \mu_y(a)$ is continuous the set $V := \{y \in X : \mu_y(a) > 0\}$ is open in X. Clearly, $y_0 = \varphi(x_0) \in V \subseteq \varphi(U \cap \Delta_{\rm pos})$. □

Lemma 2.5. *For any $x_0 \in \Delta$ and $\varepsilon > 0$ there is a neighbourhood U_0 of x_0 such that for any open $U \subseteq U_0$, with $x_0 \in U$, there is a neighbourhood V of $\varphi(x_0)$ such that*

$$\left| \sum_{x \in U \cap \varphi^{-1}(y)} \varrho(x) - \varrho(x_0) \right| < \varepsilon \quad \textit{for all} \quad y \in V. \tag{7}$$

Proof. Fix $\varepsilon > 0$. Since the measure $\mu_{\varphi(x_0)}$ is regular there is a neighbourhood U_1 of x_0 such that $\mu_{\varphi(x_0)}(U_1) < \mu_{\varphi(x_0)}(\{x_0\}) + \varepsilon$, which translates to

$$\sum_{x \in U_1 \cap \varphi^{-1}(\varphi(x_0))} \varrho(x) < \varrho(x_0) + \varepsilon.$$

Let U_0 be any neighbourhood of x_0 such that $\overline{U_0} \subseteq U_1$. Now for any neighbourhood $U \subseteq U_0$ of x_0 take two continuous functions such that

$$0 \le f_1, f_2 \le 1, \quad f_1(x) = \begin{cases} 1, & x \in U \\ 0, & x \notin U_0 \end{cases}, \quad f_2(x) = \begin{cases} 1, & x = x_0 \\ 0, & x \notin U \end{cases}.$$

Set $V = \{y : \mu_y(f_1) < \varrho(x_0) + \varepsilon \text{ and } \mu_y(f_2) > \varrho(x_0) - \varepsilon\}$. Clearly, $\varphi(x_0) \in V$ and for any $y \in V$ we have $\varrho(x_0) - \varepsilon < \mu_y(f_2) \le \sum_{x \in U \cap \varphi^{-1}(y)} \varrho(x) = \mu_y(U) \le \mu_x(f_1) < \varrho(x_0) + \varepsilon$. □

Corollary 2.6. *For any neighbourhood U of $x_0 \in \Delta$ and any $\varepsilon > 0$ there exists a continuous function $0 \le h \le 1$ supported on U such that $h(x) \equiv 1$ on an neighbourhood of x_0 and $\varrho(x_0) \le \max_{y \in X} \sum_{x \in \varphi^{-1}(y)} \varrho(x) h(x) < \varrho(x_0) + \varepsilon$.*

Proof. We may assume that $U \subseteq U_0$ where U_0 is as in Lemma 2.5 and then find V corresponding to U in this lemma. Take any continuous function $0 \le h \le 1$ supported on an open set contained with a boundary in $U \cap \varphi^{-1}(V)$ and such that $h(x) = 1$ on an open neighbourhood of x_0. Then

$$\varrho(x_0) \le \sum_{x \in \varphi^{-1}(\varphi(x_0))} \varrho(x) h(x) \le \max_{y \in X} \sum_{x \in \varphi^{-1}(y)} \varrho(x) h(x) = \|L(h)\|$$

and $\|L(h)\| = \max_{y \in V} \sum_{x \in \varphi^{-1}(y)} \varrho(x) h(x) \le \max_{y \in V} \sum_{x \in U} \varrho(x) < \varrho(x_0) + \varepsilon$. □

Proof of Proposition 2.3. Let $x_0 \in X$ and $\varepsilon > 0$. Let U and V be open sets as in Lemma 2.5. Then $U \cap \varphi^{-1}(V)$ is an open neighbourhood of x_0, and for any $x \in U \cap \varphi^{-1}(V)$ we have $\varrho(x) \le \sum_{y \in U \cap \varphi^{-1}(\varphi(x))} \varrho(y) < \varrho(x_0) + \varepsilon$. Hence ϱ is upper continuous at x_0.

Now let $x_0 \in \Delta \setminus \varrho^{-1}(0)$.

(i)⇒(ii). Suppose that ϱ is lower continuous at x_0. Then for any $\varepsilon < \varrho(x_0)/3$ there is a neighbourhood U of x_0 such that

$$\varrho(x) > \varrho(x_0) - \varepsilon > 0 \quad \text{for all} \quad x \in U. \tag{8}$$

By Lemma 2.5 we may assume that there is an open neighbourhood V of $\varphi(x_0)$ such that (7) holds. Then for $W := U \cap \varphi^{-1}(V)$ we get

$$\varrho(x_0) - \varepsilon < \mu_{\varphi(x)}(W) < \varrho(x_0) + \varepsilon \quad \text{for all} \quad x \in W.$$

We claim, that φ is injective on W. Indeed, assume on the contrary that W contains two distinct points x_1, x_2 such that $\varphi(x_1) = \varphi(x_2)$. Then by (8) we get

$$\varrho(x_0) + \varepsilon > \mu_{\varphi(x_1)}(W) \geq \varrho(x_1) + \varrho(x_2) > 2(\varrho(x_0) - \varepsilon).$$

which contradicts $\varepsilon < \varrho(x_0)/3$.

(ii)⇒(iii). This follows from Lemma 2.4.

(iii)⇒(i). Suppose that x_0 is a local homeomorphism point, and let U be a neighbourhood of x_0 such that $\varphi : U \to \varphi(U)$ is a homeomorphism. Let $\varepsilon > 0$. By Lemma 2.5 we may assume that there is a neighbourhood V of $\varphi(x_0)$ such that (7) holds. But for any x in $U \cap \varphi^{-1}(V)$ we have have $\varphi^{-1}(\varphi(x)) \cap U = \{x\}$ and thus

$$\varrho(x_0) - \varepsilon < \varrho(x) = \mu_{\varphi(x)}(\{x\}) = \mu_{\varphi(x)}(U) < \varrho(x_0) + \varepsilon.$$

Hence ϱ is continuous at x_0. □

3. Covariant Representations and Regular Points

Throughout the paper, we fix a transfer operator $L : C_0(\Delta) \to C_0(X)$ of the form (6) where $\varphi : \Delta \to X$ is a partial map and $\varrho : \Delta \to [0, \infty)$ is the associated potential. We write $A := C_0(X)$ and $I := C_0(\Delta)$, and let $\alpha : C_0(X) \to C_b(\Delta)$ be given by $\alpha(a) = a \circ \varphi$.

Definition 3.1. A *representation of the transfer operator* L is a pair (π, T) where $\pi : A \to B(H)$ is a non-degenerate representation and $T \in B(H)$ satisfies

$$\pi(L(a)) = T^*\pi(a)T, \quad a \in I = C_0(\Delta). \tag{9}$$

We say that (π, T) is *faithful* if π is faithful. We denote by

$$C^*(\pi, T) := C^*(\pi(A) \cup \pi(I)T)$$

the C^*-algebra generated by $\pi(A) \cup \pi(I)T$.

Remark 3.2. Without loss of generality, we could additionally assume in Definition 3.1 that $TH \subseteq \overline{\pi(I)H}$ (as composing T with the projection onto $\overline{\pi(I)H}$ does not affect (9) and the C^*-algebra $C^*(\pi, T)$). Assuming this we have $\|T\| \leq \|L\|^{\frac{1}{2}}$, with the equality when π is faithful. Indeed, since $L : I \to A$ is positive, we have $\|L\| = \lim_\lambda \|L(\mu_\lambda)\|$ for an approximate unit $\{\mu_\lambda\}$ in I, see for instance, [39, Lemma 2.1]. Hence

$$\|T\|^2 = \|T^*T\| = \lim_\lambda \|T^*\pi(\mu_\lambda)T\| = \lim_\lambda \|\pi(L(\mu_\lambda))\| \leq \lim_\lambda \|L(\mu_\lambda)\| = \|L\|$$

and the inequality is equality when π is injective. However, in what follows, we will not assume that $TH \subseteq \overline{\pi(I)H}$, as we will be mainly concerned with operators of the form $\pi(a)T$, for $a \in I$, and then we always have $\|\pi(a)T\| \leq \|a\|\|L\|^{\frac{1}{2}}$.

Remark 3.3. The C^*-algebra $C^*(\pi, T)$ is not affected if we replace Δ by any open set U such that $\Delta_{\text{pos}} \subseteq U \subseteq \Delta$, as then $\pi(C_0(\Delta))T = \pi(C_0(U))T$. Indeed, $\|\pi(a)T\|^2 = \|\pi(L(a^*a))\|$ and $\|L(a^*a)\| = \sup_{y \in X} \sum_{x \in \varphi^{-1}(y) \cap \Delta_{\text{pos}}} |a|^2(x)\varrho(x)$, so the norm of $\pi(a)T$ depends only on values of a on Δ_{pos}. In particular, we may always assume that $\Delta = \varphi^{-1}(\varphi(\Delta_{\text{pos}}))$, as the set $\varphi^{-1}(\varphi(\Delta_{\text{pos}}))$ is open because the map $\varphi : \Delta_{\text{pos}} \to X$ is open.

Lemma 3.4. *Let (π, T) be a representation of L. We have the following commutation relations*

$$\pi(b)T\pi(a) = \pi(b\alpha(a))T, \quad a \in A, \quad b \in I.$$

If in addition $TH \subseteq \overline{\pi(I)H}$ and φ is proper, then $T\pi(a) = \pi(\alpha(a))T$, $a \in A$.

Proof. Putting $c := \pi(b)T\pi(a)$ and $d := \pi(b\alpha(a))T$ one sees, that each of the expressions c^*d, d^*d, c^*c, d^*c is equal to $\pi\big(L(\alpha(a^*)b^*b\alpha(a))\big)$. Thus using the C^*-equality we get $\|c-d\|^2 = \|\big(c^*-d^*\big)\big(c-d)\| = \|c^*d+d^*d+c^*c-d^*c\| = 0$.

If φ is proper, then α takes values in $I = C_0(\Delta)$ (rather than in $M(I) = C_b(\Delta)$) and hence we may put $c := T\pi(a)$ and $d := \pi(\alpha(a))T$ in the calculations above. Then all the terms c^*d, d^*d, c^*c, d^*c are equal to $\pi(a^*)TT^*\pi(a)$. For instance, if $\{\mu_\lambda\}$ is an approximate unit in I, then

$$\begin{aligned} c^*d &= \pi(a^*)T^*\pi(\alpha(a))T = s\text{-}\lim_{\lambda} \pi(a^*)T^*\pi(\mu_\lambda)\pi(\alpha(a))T \\ &= s\text{-}\lim_{\lambda} \pi(a^*)\pi\big(L(\mu_\lambda)\alpha(a)\big) = s\text{-}\lim_{\lambda} \pi(a^*)\pi(L(\mu_\lambda))\pi(a) = \pi(a^*)TT^*\pi(a). \end{aligned}$$

Here s-lim stands for a limit in strong operator topology. □

Corollary 3.5. *If (π, T) is a representation of L, then*

$$\overline{\pi(I)TT^*\pi(I)} = \overline{\operatorname{span}}\{\pi(a)TT^*\pi(b) : a, b \in I\}$$

is a C^*-algebra, and so $\pi(A) \cap \overline{\pi(I)TT^*\pi(I)}$ is an ideal in $\pi(A)$.

Proof. By Lemma 3.4, $\pi(a)TT^*\pi(b) \cdot \pi(c)TT^*\pi(d) = \pi(a)TT^*\pi(\alpha(L(bc))d)$ for $a, b, c, d \in I$. Thus $\operatorname{span}\{\pi(a)TT^*\pi(b) : a, b \in I\}$ is a $*$-algebra. □

Remark 3.6. In view of Lemma 3.4, we have $\overline{\pi(I)TT^*\pi(I)} = \overline{\pi(I)T\pi(A)T^*\pi(I)}$, and if $TH \subseteq \pi(I)H$ and φ is proper, then $\overline{\pi(I)TT^*\pi(I)} = \overline{\pi(A)TT^*\pi(A)}$.

The spectrum of the ideal in Corollary 3.5 is related to the set of regular points that we define as follows.

Definition 3.7. The set of *regular points* for ϱ is

$$\Delta_{\text{reg}} := \{x \in \Delta : \varrho(x) > 0 \text{ and } \varrho \text{ is continuous at } x\}.$$

Clearly, Δ_{reg} is an open set, and by Proposition 2.3, a point $x \in \Delta$ is regular if and only if $\varrho(x) > 0$ and x is a local homeomorphism point for φ.

Remark 3.8. We have a hierarchy of sets $\Delta_{\text{reg}} \subseteq \Delta_{\text{pos}} \subseteq \Delta$ where Δ_{pos} need not be open nor closed in X. The map φ is open on Δ_{pos} and in addition locally injective on Δ_{reg}.

Proposition 3.9. *Let (π, T) be a faithful representation of L. Then*

$$\pi(A) \cap \overline{\pi(I)TT^*\pi(I)} \subseteq \pi(C_0(\Delta_{\text{reg}})).$$

Proof. To lighten the notation we will suppress π and we will write $A \subseteq B(H)$. Let us fix $a \in A$ such that $a \notin C_0(\Delta_{\text{reg}})$. That is, there is $x_0 \in X \backslash \Delta_{\text{reg}} \neq 0$ with $a(x_0) \neq 0$. We need to show that $a \notin \overline{ITT^*I}$ and to this end it suffices to show that for any $a_i, b_i \in I$, $i = 1, \ldots, N \in \mathbb{N}$, we have

$$\left\| a - \sum_{i=1}^{N} a_i TT^* b_i \right\| \geq |a(x_0)|. \tag{10}$$

We first show a weaker inequality, which holds for an arbitrary $y \in \Delta$ though,

$$\left\| a - \sum_{i=1}^{N} a_i TT^* b_i \right\| \geq |a(y)| - \sqrt{\varrho(y)} \sum_{i=1}^{N} \|a_i\| \|b_i\|. \tag{11}$$

Let $\varepsilon > 0$ and put $U := \{x \in \Delta : |a(x) - a(y)| < \varepsilon\}$. By Corollary 2.6 there is a continuous function $0 \leq h \leq 1$ supported on U such that $h(x) = 1$ on an open neighbourhood of y and

$$\varrho(y) \leq \|L(h^2)\| < \varrho(y) + \varepsilon.$$

Thus for any $b \in C_0(X)$ one has

$$\|hb^*T\|^2 = \|T^*bh\|^2 = \|L(|b|^2h^2)\| \leq \|b\|^2 \|L(h^2)\| \leq \|b\|^2(\varrho(y) + \varepsilon).$$

Using this we get

$$\begin{aligned} \left\| a - \sum_{i=1}^{N} a_i TT^* b_i \right\| &\geq \left\| h \left(a - \sum_{i=1}^{N} a_i TT^* b_i \right) h \right\| = \left\| ah^2 - \sum_{i=1}^{N} ha_i TT^* b_i h \right\| \\ &\geq \|ah^2\| - \sum_{i=1}^{N} \|ha_i T\| \|T^* b_i h\| \\ &\geq |a(y)| - \sqrt{(\varrho(y) + \varepsilon)} \sum_{i=1}^{N} \|a_i\| \|b_i\|. \end{aligned}$$

Passing with ε to zero, we get (11). Now we consider two cases.

(I). Suppose first that for each $\delta > 0$ every neighbourhood of x_0 contains a point x with $\varrho(x) < \delta$. Equivalently, there is a net $\{x_n\} \subseteq \Delta$ such that $x_n \to x_0$ and $\varrho(x_n) \to 0$. Applying (11) to $y = x_n$ we have

$$\left\| a - \sum_{i=1}^{N} a_i TT^* b_i \right\| \geq |a(x_n)| - \sqrt{\varrho(x_n)} \sum_{i=1}^{N} \|a_i\| \|b_i\|,$$

which by passing to the limit, gives (10).

(II). Finally, suppose that there is $\delta > 0$ and an open neighbourhood U of x_0 such that

$$\inf_{x\in U} \varrho(x) \geq \delta > 0.$$

Clearly it is enough to consider the case when $a(x_0) \neq 0$. Let $\varepsilon > 0$. We may assume that $U \subseteq \{x \in \Delta : |a(x) - a(x_0)| < \varepsilon\}$. Also, since $x_0 \notin \Delta_{\rm reg}$, ϱ is not continuous at x_0. Therefore, by Proposition 2.3, there exist two distinct points x_1, x_2 in U such that $\varphi(x_1) = \varphi(x_2)$. Let U_1, $U_2 \subseteq U$ be two open disjoint sets with $x_1 \in U_1$ and $x_2 \in U_2$. By Corollary 2.6, for each $i = 1, 2$, there are continuous functions $0 \leq h_i \leq 1$ supported on U_i such that $h_i(x_i) = 1$ and

$$\varrho(x_i) \leq \|L(h_i)\| < \varrho(x_i) + \varepsilon, \quad \text{and} \quad \varrho(x_i) \leq \|L(h_i^2)\| < \varrho(x_i) + \varepsilon.$$

Put $h := h_1 - \frac{\varrho(x_1)}{\varrho(x_2)} h_2$. Using that h is supported on U we get

$$\|ahT\| \geq \|hT\|(|a(x_0)| - \varepsilon).$$

On the other hand, $\|hT\|^2 = \|L(h^2)\| = \|L(h_1^2) + \left(\frac{\varrho(x_1)}{\varrho(x_2)}\right)^2 L(h_2^2)\| \geq \|L(h_1^2)\| \geq \varrho(x_1) \geq \delta$. Moreover, for any $b \in C_0(X)$ we have

$$\begin{aligned}\|T^*bhT\| &= \|L(hb)\| \leq \|L(h)\| \cdot \|b\| = \left\| L(h_1) - \frac{\varrho(x_1)}{\varrho(x_2)} L(h_2) \right\| \cdot \|b\| \\ &\leq \left((\varrho(x_1) + \varepsilon) - \frac{\varrho(x_1)}{\varrho(x_2)} \varrho(x_2) \right) \cdot \|b\| = \varepsilon \cdot \|b\|.\end{aligned}$$

Using all this, we get

$$\begin{aligned}\left\| a - \sum_{i=1}^N a_i TT^* b_i \right\| &\geq \frac{\left\| \left(a - \sum_{i=1}^N a_i TT^* b_i\right)(hT) \right\|}{\|hT\|} \\ &\geq \frac{\|ahT\|}{\|hT\|} - \sum_{i=1}^N \frac{\|a_iT\| \cdot \|T^*b_ihT\|}{\|hT\|} \\ &\geq (|a(x_0)| - \varepsilon) - \varepsilon \sum_{i=1}^N \frac{\|a_iT\| \cdot \|b_i\|}{\sqrt{\delta}}.\end{aligned}$$

Passing with ε to zero, we get (10). □

Definition 3.10. We say that a representation (π, T) of L is *covariant* if

$$\pi(C_0(\Delta_{\rm reg})) \subseteq \overline{\pi(I)TT^*\pi(I)}.$$

Thus a faithful representation (π, T) is covariant iff $\pi(C_0(\Delta_{\rm reg})) = \overline{\pi(I)TT^*\pi(I)}$.

Remark 3.11. Every transfer operator admits a faithful covariant representation (see Example 3.14). If $\Delta_{\rm reg}$ is non-empty, then there are faithful representations that are not covariant. Indeed, if (π, T) is any representation of L on a Hilbert space H, then putting $\widetilde{H} := H \otimes \ell^2(\mathbb{N})$, $\widetilde{\pi} := \pi \otimes \mathrm{id}$ and $\widetilde{T} := T \otimes U$ where U is the unilateral shift on $\ell^2(\mathbb{N})$, we get a representation $(\widetilde{\pi}, \widetilde{T})$ of L with $\widetilde{\pi}(A) \cap \overline{\widetilde{\pi}(I)\widetilde{T}\widetilde{T}^*\widetilde{\pi}(I)} = 0$, because UU^* is a non-trivial projection.

3.1. Characterizations of Covariant Representations

Let K be a compact subset of Δ_{reg}. Then we may find a finite cover $\{U_i\}_{i=1}^n$ of K such that $\bigcup_{i=1}^n U_i$ is contained in a compact subset of Δ_{reg} and $\varphi|_{U_i}$ is injective for every $i = 1, \ldots, n$. Take a partition of unity $\{v_i\}_{i=1}^n \subseteq C_0(\Delta_{\text{reg}})$ on K subordinated to $\{U_i\}_{i=1}^n$. Then

$$u_i^K := \sqrt{\frac{v_i}{\varrho}}, \quad i = 1, \ldots, n, \tag{12}$$

are well defined functions in $C_c(\Delta_{\text{reg}})$ because ϱ is bounded away from zero on $\bigcup_{i=1}^n U_i$. We will use these functions to characterize covariant representations.

Proposition 3.12. *Let (π, T) be a representation of L. The following are equivalent:*

(i) (π, T) *is covariant.*
(ii) *for every $a \in C_c(\Delta_{\text{reg}})$ supported on a set where φ is injective we have $\pi(a)TT^*\pi(u) = \pi(a)$ for some $u \in C_0(X)$.*
(iii) *for every $a \in C_c(\Delta_{\text{reg}})$ supported on a set where φ is injective we have $\pi(a)TH = \pi(a)H$.*
(iv) *For every $a \in C_c(\Delta_{\text{reg}})$ supported on K we have*
$$\pi(a)\sum_{i=1}^n \pi(u_i^K)TT^*\pi(u_i^K) = \pi(a). \tag{13}$$
(v) *for every $x_0 \in \Delta_{\text{reg}}$ and $\varepsilon > 0$ there is a neighbourhood U of x_0 such that for every $a, b \in C_0(U)$ with $\|a\|, \|b\| \le 1$ we have*
$$\|\pi(a)TT^*\pi(b) - \varrho(x_0)\pi(ab)\| < \varepsilon.$$

The above conditions hold whenever $\overline{\pi(I)TH} = H$ (which is equivalent to $\overline{\pi(A)TH} = H$ when φ is proper).

Proof. Clearly, (ii) implies (iii). Since $C_c(\Delta_{\text{reg}})$ is dense in $C_0(\Delta_{reg})$ and every element in $C_c(\Delta_{\text{reg}})$ is a finite sum of functions supported on sets where φ is injective, we see that (ii) also implies (i). For converse implications, let $a \in C_c(\Delta_{\text{reg}})$ have support K such that $\varphi|_K$ is injective and let $u \in C_c(\Delta_{\text{reg}})$ be such that $u|_K = (\varrho|_K)^{-1}$. For every $b \in C_0(\Delta)$ and $x \in \Delta_{\text{reg}}$ we have $a(x)\alpha(L(ub))(x) = a(x)\sum_{t\in\varphi^{-1}(\varphi(x))} \varrho(t)u(t)b(t) = a(x)b(x)$. Hence

$$\Big(\pi(a)TT^*\pi(u)\Big)\pi(b)T = \pi(a\alpha(L(ub)))T = \pi(a)\pi(b)T.$$

Thus $\pi(a)TT^*\pi(u) = \pi(a)$ whenever $\pi(a)$ is determined by its action on $\pi(I)TH$. Both (i) and (iii) imply this. Indeed, if (i) holds then $\pi(a) \in \overline{\pi(I)TT^*\pi(I)}$, and if we assume (iii) we get

$$\pi(a)H = \pi(a)TH = \pi(a)T\pi(A)H = \pi(a)\pi(\alpha(A))TH = \pi(a)\pi(I)TH.$$

Hence (i), (ii), (iii) are equivalent and they follow from the condition $\pi(I)TH = H$. If φ is proper, then $\pi(I)TH = \pi(A\alpha(A))TH = \pi(A)T\pi(A)H = \pi(A)TH$.

Since $C_c(\Delta_{\text{reg}})$ is dense in $C_0(\Delta_{\text{reg}})$, (iv) readily implies (i). Conversely, if we assume (i), then for every $a \in C_c(\Delta_{\text{reg}})$ the operator $\pi(a) \in \overline{\pi(I)TT^*\pi(I)}$

is determined by its action on $\pi(I)TH$. Moreover, for every $b \in I$ we have $a\sum_{i=1}^{n} u_i^K \alpha(L(u_i^K b)) = ab$. Thus

$$\left(\pi(a)\sum_{i=1}^{n}\pi(u_i^K)TT^*\pi(u_i^K)\right)\pi(b)T = \pi\left(a\sum_{i=1}^{n}u_i^K\alpha(L(u_i^K b))\right)T. \\ = \pi(a)\pi(b)T.$$

This implies (13). Hence (i)⇒(iv).

(iii) ⇒ (v). Let U be a neighbourhood of $x_0 \in \Delta_{\mathrm{reg}}$ such that $\varphi|_U$ is injective and $U \subseteq \{x \in \Delta_{\mathrm{reg}} : |\varrho(x) - \varrho(x_0)| < \varepsilon\}$. Take any $a, b \in C_0(U)$ with $\|a\|, \|b\| \le 1$. Note that $\varrho a \in C_0(U)$ and $\|\varrho ab - \varrho(x_0)ab\| < \varepsilon$. The argument in the proof that (iii) implies (i), shows that $\pi(a)TT^*\pi(b) = \pi(\varrho ab)$. Hence (v) holds.

(v) ⇒ (iii). Let $a \in C_c(\Delta_{\mathrm{reg}})$ have support K such that $\varphi|_K$ is injective. Without loss of generality we may assume that $\|a\| \le 1$. By (v) and compactness of K there is a partition of unity $\{u_i\}_{i=1}^n$ on K subordinate to an open cover $\{U_i\}_{i=1}^n$ of K such that for every $i = 1, \ldots, n$ there is a point $x_i \in U_i$ such that for every $b \in C_0(U_i)$, $\|b\| \le 1$, we have $\|\pi(u_i a)T^*T\pi(b) - \varrho(x_i)\pi(ab)\| < \varrho(x_i)/2$. Clearly, a satisfies (iii) iff each $u_i a$ satisfies (iii). Hence we may assume that $a \in C_0(U)$ where $\varphi|_U$ is injective and there is $x_0 \in U$ such that

$$\|\pi(a)T^*T\pi(b) - \varrho(x_0)\pi(ab)\| < \varrho(x_0)/2,$$

for any $b \in C_0(U)$, $\|b\| \le 1$. Now let $\{\mu_\lambda\}$ be an approximate unit in $C_0(U)$ and let $P := \text{s-}\lim \pi(\mu_\lambda)$ be the projection given by the strong limit. Then we have

$$\|PT^*TP - \varrho(x_0)P\| \le \varrho(x_0)/2.$$

Thus $\|1/\varrho(x_0)PT^*TP - P\| \le 1/2 < 1$ and therefore the operator $1/\varrho(x_0)$ $PT^*TP : PH \to PH$ is invertible. In particular, $P_U TH = P_U H$ and this gives $\pi(a)TH = \pi(a)P_U TH = \pi(a)P_U H = \pi(a)H$. □

Remark 3.13. Assume $X = \Delta_{\mathrm{reg}}$. Equivalently, $\varphi : X \to X$ is a local homeomorphism and $\varrho > 0$ is strictly positive. Then condition (iv) in Proposition 3.12 reduces to

$$\sum_{i=1}^{n}\pi(u_i^X)TT^*\pi(u_i^X) = 1,$$

which is the condition identified by Exel and Vershik in [23]. Also conditions in Proposition 3.12 are equivalent to the condition $\overline{\pi(A)TH} = H$, which is called axiom (A3) in [3].

Example 3.14. (Orbit representation) There is a natural faithful covariant representation (π_o, T_o) of L on the Hilbert space $\ell^2(X)$. We will call it the *orbit representation*. Namely we define a faithful representation $\pi_o : C_0(X) \to B(\ell^2(X))$ by

$$(\pi_o(a)h)(x) := a(x)h(x), \quad a \in C_0(X), h \in \ell^2(X).$$

Let $\{\mathbb{1}_x\}_{x\in X}$ be the standard orthonormal basis of $\ell^2(X)$. Since $\sum_{x\in\varrho^{-1}(y)}\varrho(x)\leq \|L\|$, $y\in X$, there is $T_o\in B(\ell^2(X))$ such that $T_o\mathbb{1}_y:=\sum_{x\in\varphi^{-1}(y)}\sqrt{\varrho(x)}\mathbb{1}_x$, $y\in X$. Its adjoint is given by $T_o^*\mathbb{1}_x=\sqrt{\varrho(x)}\mathbb{1}_{\varphi(x)}$, for $x\in\Delta$, and $T_o^*\mathbb{1}_x=0$ for $x\notin\Delta$. Equivalently,

$$(T_oh)(x)=\begin{cases}\sqrt{\varrho(x)}h(\varphi(x)), & x\in\Delta\\ 0, & x\notin\Delta,\end{cases}\qquad (T_o^*h)(y)=\sum_{x\in\varphi^{-1}(y)}\sqrt{\varrho(x)}h(x),$$

for $h\in\ell^2(X)$. Clearly (π_o,T_o) is a faithful representation of L. To see that it is covariant we show condition (ii) in Proposition 3.12. Let $a\in C_c(\Delta_{\text{reg}})$ be supported on a set K such that $\varphi|_K$ is injective and let $u\in C_c(\Delta_{\text{reg}})$ be such that $u|_K=(\varrho|_K)^{-1}$. Then $(\pi_o(a)T_oT_o^*\pi_o(u)h)(x)=a(x)\sqrt{\varrho(x)}\left(\sum_{t\in\varphi^{-1}(\varphi(x))}\sqrt{\varrho(t)}u(t)h(t)\right)=a(x)h(x)=(\pi_o(a)h)(x)$. Hence $\pi_o(a)=\pi_o(a)T_oT_o^*\pi_o(u)$.

4. The Crossed Product

Recall that $L:C_0(\Delta)\to C_0(X)$ is a transfer operator for a partial map $\varphi:\Delta\to X$, $A=C_0(X)$, $I=C_0(\Delta)$, and $\alpha:A\to M(I)$ is given by composition with φ. Let us consider a universal $*$-algebra $\mathcal{A}(L)$ generated by $C_c(X)$ (viewed as a $*$-algebra) and an element t subject to relations

$$L(a)=t^*at,\quad atb=a\alpha(b)t,\quad \text{for all}\quad a\in C_c(\Delta),b\in C_c(X), \tag{14}$$

and for every compact $K\subseteq\Delta_{\text{reg}}$ and $a\in C_c(\Delta_{\text{reg}})$ supported on K

$$a\sum_{i=1}^{n}u_i^Ktt^*u_i^K=a \tag{15}$$

where u_i^K's are given by (12). Note that these relations are satisfied by operators coming from covariant representations of L, see Lemma 3.4 and Proposition 3.12.

Definition 4.1. The *algebraic crossed-product* $C_c(X)\rtimes_{\text{alg}}L$ is the $*$-subalgebra of $\mathcal{A}(L)$ generated by $C_c(X)$ and $C_c(\Delta)t$.

To describe the structure of $C_c(X)\rtimes_{\text{alg}}L$ we need to iterate partial transfer operators. Let

$$\Delta_n:=\varphi^{-n}(X),\quad I_n:=C_0(\Delta_n),\quad n\in\mathbb{N}.$$

We put $\Delta_0:=X$ and $I_0:=A=C_0(X)$. So Δ_n is a natural domain for the partial map φ^n; the composition $\underbrace{\varphi\circ\cdots\circ\varphi}_{n\text{times}}$ makes sense on Δ_n. Define $\alpha^n:A\to M(I_n)$ to be the partial endomorphism of A given by composition with $\varphi^n:\Delta_n\to X$. Having the map $\varrho:\Delta\to[0,\infty)$, that defines L via (6), for each $n\in\mathbb{N}$ we define $\varrho_n:\Delta_n\to[0,\infty)$ by

$$\varrho_n(x):=\prod_{i=0}^{n-1}\varrho(\varphi^i(x)),\quad x\in\Delta_n.$$

We also put $\varrho_0 \equiv 1$. Then the formula

$$L^n(a)(y) := \sum_{x\in\varphi^{-n}(y)} \varrho_n(x)a(x), \quad a \in C_0(\Delta_n), y \in X,$$

defines a transfer operator $L^n : I_n \to A$ for $\alpha^n : A \to M(I_n)$. To describe the maps L^n and α^n more algebraically, note that

$$I_n^0 := \mathrm{span}\{a_1\alpha(a_2\alpha(\cdots a_n)\cdots) : a_1,\ldots,a_n \in C_c(\Delta)\} \tag{16}$$

is a dense $*$-subalgebra of I_n (we put $I_0^0 := C_c(X)$). Thus L^n and α^n are determined by the following formulas, for $a_1,\ldots,a_n \in C_c(\Delta)$ and $a \in C_c(X)$:

$$L^n\Big(a_1\alpha(a_2\alpha(\cdots a_n)\cdots)\Big) = L(L(\cdots L(L(a_1)a_2)a_3\cdots)a_n) \tag{17}$$

$$a_1\alpha(a_2\alpha(\cdots a_n)\cdots)\cdot\alpha^n(a) = a_1\alpha(a_2\alpha(\cdots a_n\alpha(a)\cdots). \tag{18}$$

Lemma 4.2. *The algebraic crossed-product is the following linear span*

$$C_c(X) \rtimes_{\mathrm{alg}} L = \mathrm{span}\{at^n t^{*m} b : a \in I_n^0, b \in I_m^0, n, m \in \mathbb{N}_0\}. \tag{19}$$

Moreover, for all $n, m, k, l \in \mathbb{N}_0$ and $a \in I_n^0, b \in I_m^0, c \in I_k^0, d \in I_l^0$, we have

$$(at^n t^{*m} b)\cdot(ct^k t^{*l} d) = \begin{cases} at^n t^{*m-k+l}\alpha^l(L^k(bc))d & m \geq k, \\ a\alpha^n(L^m(bc))t^{k-m+n}t^{*l}d & m < k. \end{cases} \tag{20}$$

Proof. Using (17) and (18) we get that (14) generalizes to

$$L^n(a) = t^{*n}at^n, \quad at^n b = a\alpha^n(b)t^n, \quad \text{for all} \quad a \in I_n^0, b \in A, n \in \mathbb{N}. \tag{21}$$

For instance, for $n = 2$, and $a_1, a_2 \in I$, we have

$$L^2\Big(a_1\alpha(a_2)\Big) \overset{(17)}{=} L(L(a_1)a_2) \overset{(14)}{=} t^*(t^*a_1ta_2)t \overset{(14)}{=} t^{*2}a_1\alpha(a_2)t^2,$$

$$\Big(a_1\alpha(a_2)\Big)t^2 b \overset{(14)}{=} a_1ta_2tb \overset{(14)}{=} a_1ta_2\alpha(b)t = a_1\alpha(a_2\alpha(b))t^2 \overset{(18)}{=} \Big(a_1\alpha(a_2)\Big)\cdot\alpha^2(b)t^2.$$

Using (21) one readily gets (20). In turn (20) implies that the self-adjoint linear space $\mathrm{span}\{at^n t^{*m}b : a \in I_n^0, b \in I_m^0, n, m \in \mathbb{N}_0\}$ is closed under multiplication. Hence it is a $*$-algebra, and clearly it is generated by $I_0^0 \cup I_1^0$. This proves (19). □

By universality every covariant representation (π, T) of L induces (uniquely) a representation $\pi \rtimes T$ of the $*$-algebra $C_c(X) \rtimes_{\mathrm{alg}} L$ where $\pi \rtimes T(a) = \pi(a)$, $a \in C_c(X)$, and $\pi\rtimes T(at) = aT$ for $a \in C_c(\Delta)$. Namely, $\pi\rtimes T(\sum_{i=1}^n a_i t^{n_i} t^{*m_i} b_i) = \sum_{i=1}^n a_i T^{n_i} T^{*m_i} b_i$ for $a_i \in I_{n_i}^0$, $b_i \in I_{m_i}^0$, $i = 1,\ldots,n$. We put

$$\|x\|_{\max} := \sup\{\|\pi \rtimes T(x)\| : (\pi, T) \text{ is a covariant representation for } L\}.$$

It is easily verified that $\|\cdot\|_{\max}$ is a C^*-seminorm (a submultiplicative seminorm satisfying the C^*-equality). It is finite because $\|\sum_{i=1}^n a_i t^{n_i} t^{*m_i} b_i\|_{\max} \leq \sum_{i=1}^n \|a_i\|\|b_i\|(\|L^{n_i}\|\|L^{m_i}\|)^{\frac{1}{2}}$, cf. Remark 3.2. Restriction $\|\cdot\|_{\max}$ to $C_c(X)$ coincides with the unique C^*-norm on A, because there exists a faithful covariant representation, see Example 3.14. In other words, the (self-adjoint and two-sided) ideal

$$\mathcal{N} := \{x \in C_c(X) \rtimes_{\mathrm{alg}} L : \|x\|_{\max} = 0\}$$

intersects $C_c(X)$ trivially.

Definition 4.3. The *crossed product* of A by the transfer operator L is the C^*-algebra $A \rtimes L$ obtained by the Hausdorff completion of $C_c(X) \rtimes_{\mathrm{alg}} L$ in $\|\cdot\|_{\max}$:

$$A \rtimes L = \overline{C_c(X) \rtimes_{\mathrm{alg}} L/\mathcal{N}}^{\|\cdot\|_{\max}}.$$

Remark 4.4. Since $C_c(X) \cap \mathcal{N} = \{0\}$, we may and we will treat $C_c(X)$ as a $*$-subalgebra of $A \rtimes L$. The closure of $C_c(X)$ in $A \rtimes L$ will be identified with A. We will also abuse the notation and write at^n, $a \in I_n^0$, for their images in $A \rtimes L$. In fact we extend this notation to any $a \in I_n = C_0(\Delta_n)$ by writing at^n for the limit in $A \rtimes L$ of a sequence $a_n t^n$ where $\{a_n\}_{n=1}^{\infty} \subseteq I_n^0$ converges uniformly to a. So by Lemma 4.2 we have

$$A \rtimes L = \overline{\mathrm{span}}\{at^n t^{*m} b : a \in I_n, b \in I_m, n, m \in \mathbb{N}_0\}.$$

Proposition 4.5. *Assume that $A \rtimes L \subseteq B(H)$ is represented in a faithful and non-degenerate way on a Hilbert space H. The crossed product $A \rtimes L$ is the universal C^*-algebra for covariant representations of L:*

- (i) *$A \rtimes L$ contains A as a C^*-subalgebra, and is generated by A and It for $t \in B(H)$ such that $L(a) = t^*at$, $a \in I$ and $C_0(\Delta_{\mathrm{reg}}) \subseteq \overline{Itt^*I}$.*
- (ii) *Every covariant representation (π, T) of L induces a representation $\pi \rtimes T$ of $A \rtimes L$ where $\pi \rtimes T(a) = \pi(a)$, $a \in A$, and $\pi \rtimes T(at) = \pi(a)T$, $a \in I$.*

Every C^*-algebra possessing properties (i), (ii) is isomorphic to $A \rtimes L$ by an isomorphism which is identity on A.

Proof. (i) and (ii) follow by construction. To see the last part, assume that $C = C^*(A \cup Is) \subseteq B(K)$ is a C^*-algebra, represented on a Hilbert space K, that satisfies analogues of (i), (ii). Then (ii) for $A \rtimes L$ and C give $*$-epimorphisms $\Psi : A \rtimes L \to C$ and $\Phi : C \to A \rtimes L$ which clearly are inverse to each other. □

Remark 4.6. Proposition 4.5 shows that $A \rtimes L$ depends only on $L : I \to A$, or equivalently on $\varrho : \Delta \to X$ (it depends only on ϱ up to continuous factors, see Corollary 5.6 below).

4.1. Cuntz–Pimsner Picture and Other Constructions

Let $L : I \to A$ be a transfer operator for the partial endomorphism $\alpha : A \to M(I)$. The C^*-correspondence M_L associated to L, cf. [18,22], is a Hausdorff completion of the A-bimodule I where $a \cdot \xi \cdot b = ax\alpha(b)$, for $\xi \in I$, $a, b \in A$, in the A-valued pre-inner product given by $\langle \xi, \eta \rangle_A := L(\xi^* \eta)$, $\xi, \eta \in I$. A representation of M_L is a pair (π, ψ) where $\pi : A \to B(H)$ is a non-degenerate representation and $\psi : M_L \to B(H)$ is a (necessarily linear) map such that $\pi(a)\psi(\xi)\pi(b) = \psi(a\xi b)$ and $\psi(\xi)^*\psi(\eta) = \pi(\langle \xi, \eta \rangle_A)$ for $a, b \in A$, $\xi, \eta \in M_L$.

Lemma 4.7. *Every representation (π, ψ) of M_L comes from a representation (π, T) of L in the sense that $\psi(q(\xi)) = \pi(\xi)T$, $\xi \in I$, where $q : I \to M_L$ is the canonical quotient map. This gives a bijective correspondence between representations (π, ψ) of M_L and representations (π, T) of L satisfying $TH \subseteq \overline{\pi(I)H}$.*

Proof. If (π, T) is a representation of L, then $\psi(q(\xi)) := \pi(\xi)T$, $\xi \in I$, is well defined because $\|\pi(\xi)T\|^2 = \|\pi(L(\xi^*\xi))\| \leq \|L(\xi^*\xi)\| = \|q(\xi)\|$, and clearly, (π, ψ) is a representation of M_L. Let (π, ψ) be a representation of M_L and let $\{\mu_\lambda\}$ be an approximate unit in $I = C_0(\Delta)$. We claim that the net of operators $T_\lambda := \psi(q(\mu_\lambda))$ is strongly Cauchy. Indeed, let $h \in H$ and $\lambda \leq \lambda'$, in the directed set Λ. Then

$$\|(T_\lambda - T_{\lambda'})h\|^2 = \langle h, L\,(\mu_\lambda - \mu_{\lambda'})^2)h\rangle \leq \langle h, L(\mu_\lambda - \mu_{\lambda'})h\rangle.$$

Since the net $\{L(\mu_\lambda)\}_{\lambda\in\Lambda}$ is strongly convergent the last expression tends to zero. Hence $T := \text{s-}\lim_{\lambda\in\Lambda} T_\lambda$ defines a bounded operator. For every $a \in C_0(\Delta)$ we have

$$T^*aT = \text{s-}\lim_{\lambda\in\Lambda} T_\lambda^* a T_\lambda = \lim_{\lambda\in\Lambda} L(\mu_\lambda a \mu_\lambda) = L(a).$$

Thus (π, T) is a representation of L satisfying $TH \subseteq \overline{\pi(I)H}$. □

Theorem 4.8. *The crossed product $A \rtimes L$ is naturally isomorphic with Katsura's Cuntz–Pimsner algebra $\mathcal{O}_{M_L}$. In particular, $A \rtimes L$ is always nuclear, and satisfies the Universal Coefficient Theorem (UCT) if A is separable (equivalently X is second countable).*

Proof. By [32, Propositions 3.3 and 4.9] there is the largest ideal J_{M_L} in A such that for every faithful representation (π, ψ) of M_L we have

$$\{a \in A : \pi(a) \in \overline{\psi(M_L)\psi(M_L)^*}\} \subseteq J_{M_L}.$$

The faithful representation (π, ψ) of M_L is called *covariant* if the above inclusion is an equality. Hence by Lemma 4.7 and Propositions 3.9 (and Example 3.14) we have $J_{M_L} = C_0(\Delta_{\text{reg}})$ and we have a bijective correspondence between covariant representations (π, ψ) of M_L and covariant representations (π, T) of L satisfying $TH \subseteq \overline{\pi(I)H}$. By definition $\mathcal{O}_{M_L}$ is generated by the range of a universal covariant representation of M_L. By Proposition 4.5 and Remark 3.2, $A \rtimes L$ is generated by a universal covariant representations (π, T) of L satisfying $TH \subseteq \overline{\pi(I)H}$. This gives a natural isomorphism $A \rtimes L \cong \mathcal{O}_{M_L}$, cf. the last part of Proposition 4.5.

Since A is commutative (and hence nuclear), $A \rtimes L \cong \mathcal{O}_{M_L}$ is nuclear by [32, Corollary 7.4]. If A is separable, then satisfies the UCT by [32, Proposition 8.8]. □

Remark 4.9. We have seen in the above proof that Katsura's ideal J_{M_L} for M_L is $C_0(\Delta_{\text{reg}})$. It also follows from the proof of Lemma 4.7 that the C^*-correspondence M_L is naturally isomorphic to $\overline{It}$ with operations coming from the C^*-algebra $A \rtimes L$.

Corollary 4.10. *The crossed product $A \rtimes L$ is naturally isomorphic to the crossed product $\mathcal{O}(A, \alpha, L)$ by the partial endomorphism α defined in [22].*

Proof. The crossed product $\mathcal{O}(A, \alpha, L)$ in [22] is defined to be $\mathcal{O}_{M_L}$. □

Corollary 4.11. *If $\Delta = X$ and $\varrho > 0$ on a dense subset of X, then $A \rtimes L$ is naturally isomorphic to the Exel's crossed product $A \rtimes_{\alpha,L} \mathbb{N}$ [18], generalized to the non-unital case in [11].*

Proof. The assumptions mean that $\alpha : A \to A$ is non-degenerate and L is faithful. Thus the assertion follows from [39, Proposition 4.9]. □

We naturally associate to L a *topological correspondence* in the sense of [13, Definition 2.1], see also [14, Subsection 9.3]. The underlying topological directed graph (E^0, E^1, s, r) is the graph of φ:

$$E^0 := X, \quad E^1 := \Delta, \quad r(x) := x, \quad s(x) := \varphi(x).$$

It is equipped with the continuous family of measures $\mu = \{\mu_y\}_{y\in X}$ along fibers of φ given by $\mu_y(a) := L(a)(y)$, $a \in C_c(X)$. Note that we only have $\operatorname{supp} \mu_y \subseteq s^{-1}(y)$, $y \in X$. Thus the topological correspondence $\mathcal{Q} := (X, \Delta, id, \varphi, \mu)$ is a *topological quiver* in the sense of [46] iff $\operatorname{supp} \mu_y = s^{-1}(y)$, $y \in X$ iff $\Delta = \Delta_{\text{pos}}$ (note that we use a convention where s and r play the opposite role in [46]).

Lemma 4.12. *The C^*-correspondence $M_{\mathcal{Q}}$ associated to the topological correspondence $\mathcal{Q} = (X, \Delta, id, \varphi, \mu)$ in* [13, Definition 2.4], *cf.* [46, 3.1], *coincides with M_L.*

Proof. This follows immediately from the constructions (definitions). □

Corollary 4.13. *If $\Delta = \Delta_{\text{pos}}$, so that $\mathcal{Q} = (X, \Delta, id, \varphi, \mu)$ is a topological quiver, then the crossed product $A \rtimes L$ is naturally isomorphic to the quiver C^*-algebra associated to $\mathcal{Q}$ by Muhly and Tomforde* [46].

Proof. By definition the quiver C^*-algebra is the Cuntz–Pimsner algebra of $M_{\mathcal{Q}}$, which by Lemma 4.12 is equal to M_L. Hence the assertion follows from Theorem 4.8. □

Remark 4.14. Note that if Δ_{pos} is locally compact, then $\mathcal{Q}_{\text{pos}} := (X, \Delta_{\text{pos}}, id, \varphi, \mu)$ is a topological quiver. Moreover, if $\Delta_{\text{pos}} \subseteq \Delta$ is open, we may apply Corollary 4.13 to the restricted map $\varphi : \Delta_{\text{pos}} \to X$, to conclude that $A \rtimes L$ is the quiver algebra associated to $\mathcal{Q}_{\text{pos}}$. If Δ_{pos} is closed in Δ and Δ is normal, one may show that the C^*-correspondences M_L and $M_{\mathcal{Q}_{\text{pos}}}$ are isomorphic and hence $A \rtimes L$ is again the quiver algebra of $\mathcal{Q}_{\text{pos}}$. We do not know whether $A \rtimes L$ has a natural topological quiver model in general.

Example 4.15. (Maps on Riemann surfaces) Let $\varphi : \Delta \to X$ be a non-constant holomorphic map defined on an open connected subset Δ of a Riemann surface X (so that Δ is a Riemann surface as well). Let $x \in \Delta$. By branching lemma, φ locally at x looks like $z \to z^d$, and then $m(x) := d \in \mathbb{N}$ is called the multiplicity of φ at x. In particular, $\varphi^{-1}(y)$ is a discrete subset of Δ, for every $y \in X$. Assume that φ is proper. Then it is surjective and the number $d := \sum_{x\in\varphi^{-1}(y)} m(x)$, called the *degree* of φ, does not depend on $y \in X$ and is finite. In particular,

$$L(a)(y) := \sum_{x\in\varphi^{-1}(y)} m(x)a(x), \quad a \in C_0(\Delta)$$

defines a transfer operator for φ, and $\|L\| = d$. If $\Delta = X = \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere, then φ is a rational function $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ and the crossed product $C(\widehat{\mathbb{C}}) \rtimes L$ is isomorphic to the C^*-algebra $\mathcal{O}_R(\widehat{\mathbb{C}})$ associated to R in [27] (which is $\mathcal{O}_{M_L}$ by definition). If R is of degree at least two and has an exceptional point, then R is conjugated either to a polynomial or a map $z \to z^d$ for some $d \in \mathbb{Z}\backslash\{0\}$, [4, Theorem 4.1.2]. The rational map R (and the transfer operator L) restricts to the *Julia set* J_R and *Fatou set* F_R and the crossed products $C(J_R) \rtimes L$ and $C_0(F_R) \rtimes L$ to the C^*-algebras studied in [27].

Example 4.16. (Branched coverings with finite system of branches) Let $\varphi : \Delta \to X$ be a continuous surjective partial map such that φ^{-1} has a *finite system of branches*, i.e. there is a finite collection of partial maps $\{\gamma_i\}_{i=1}^N$ such that each $\gamma_i : X \to \Delta$ is continuous injective and $\varphi^{-1}(y) = \{\gamma_i(y) : i = 1, \ldots, N\}$. Then $\varrho(x) := |\{i : x \in \gamma_i(X)\}|$ defines a potential for φ as clearly

$$L(a)(y) = \sum_{i=1}^{N} a(\gamma_i(y)) = \sum_{x\in\varphi^{-1}(y)} \varrho(x)a(x), \quad a \in C_0(\Delta),$$

defines a transfer operator $L : C_0(\Delta) \to C_0(X)$ for φ. If $X = \Delta$ is compact and γ_i is a proper contraction, for all i, the crossed product $A \rtimes L$ is naturally isomorphic to the C^*-algebra associated to the *self-similar set* X in [28] (it is defined there as $\mathcal{O}_{M_L}$). The model example is the *tent map* $\varphi : [0, 1] \to [0, 1]$ where $\varphi(x) = 1 - |1 - 2x|$ and $L(a)(y) = a(\frac{y}{2}) + a(1 - \frac{y}{2})$.

If a map has infinitely many branches one may define a transfer operator by using a scaling function that will make the sums converge:

Example 4.17. Let $X = [0, 1]$, $\Delta = (0, 1]$ and $\varphi(x) = \sin\frac{1}{x}$. On may define a transfer operator for φ by the formula $L(a)(y) = 2^{[y=\pm 1]} \sum_{x\in\varphi^{-1}(y)} e^{-1/x}a(x)$.

5. Invariance Uniqueness Theorems and the Regular Representation

An important consequence of universality of $A \rtimes L$ is that it is equipped with a *circle gauge action* $\gamma : \mathbb{T} \to \mathrm{Aut}(A \rtimes L)$. Namely, for each $\lambda \in \mathbb{T}$ the pair $(\mathrm{id}_A, \lambda t)$ may be treated as covariant representation of L. Hence by Proposition 4.5(ii) there is a $*$-epimorphism $\gamma_\lambda : A \rtimes L \to A \rtimes L$ such that

$$\gamma_\lambda|_A = \mathrm{id}_A, \quad \text{and} \quad \gamma_\lambda(at) = \lambda at, \ a \in I.$$

Moreover, we clearly have $\gamma_1 = \mathrm{id}_{A\rtimes L}$ and $\gamma_{\lambda_1} \circ \gamma_{\lambda_2} = \gamma_{\lambda_1\lambda_2}$ for $\lambda_1, \lambda_2 \in \mathbb{T}$. Thus $\gamma : \mathbb{T} \to \mathrm{Aut}(A \rtimes L)$ is a group homomorphism. Its fixed points form a C^*-algebra $A_\infty := \{x \in A \rtimes L : \gamma_\lambda(x) = x \text{ for all } \lambda \in \mathbb{T}\}$. We call A_∞ the *core C^*-subalgebra* of $A \rtimes L$. It is well known, see, for instance [51, Proposition 3.2], that the formula $E(x) := \int_{\mathbb{T}} \gamma_\lambda(x)\, d\lambda$ defines a faithful conditional expectation onto A_∞. That is, E is norm one projection onto A_∞, which is necessarily a completely positive A_∞-bimodule map, see [60, III, Theorem 3.4, IV, Corollary 3.4]. Faithfulness here means that $E(a^*a) = 0$ implies $a = 0$ for all $a \in A \rtimes L$.

Proposition 5.1. *We have*

$$A_\infty = \overline{\text{span}}\{at^n t^{*n} b : a, b \in I_n, n \in \mathbb{N}_0\},$$

and the conditional expectation $E : A \rtimes L \to A_\infty$ is the unique contractive projection onto A_∞ such that $E(at^n t^{*m} b) = 0$ for $n \neq m$ ($a \in I_n$, $b \in I_m$).

Proof. We have $E(at^n t^{*m} b) = at^n t^{*m} b \int_{\mathbb{T}} \lambda^{n-m}\, d\lambda$ which is zero when $n \neq m$ and $at^n t^{*n} b$ when $n = m$. This determines E and implies that $A_\infty = \overline{\text{span}}\{at^n t^{*n} b : a, b \in I_n, n \in \mathbb{N}_0\}$. □

The gauge-invariance uniqueness for Cuntz–Pimsner algebras implies the following

Theorem 5.2. *Let (π, T) be a faithful covariant representation of L and let $C^*(\pi, T)$ be the C^*-algebra generated by $\pi(A) \cup \pi(I)T$. Then $\pi \rtimes T$ is faithful on the core subalgebra A_∞ of $A \rtimes L$, and the following conditions are equivalent:*

- (i) *$\pi \rtimes T$ is an isomorphism, i.e. $A \rtimes L \cong C^*(\pi, T)$;*
- (ii) *$C^*(\pi, T)$ is equipped with a circle gauge action, i.e. a group homomorphism $\gamma : \mathbb{T} \to \text{Aut}(C^*(\pi, T))$ where $\gamma_z|_{\pi(A)} = \text{id}_{\pi(A)}$ and $\gamma_z(\pi(a)T) = z\pi(a)T$, for $z \in \mathbb{T}$, $a \in I$;*
- (iii) *There is a conditional expectation from $C^*(\pi, T)$ onto $(\pi \rtimes T)(A_\infty) \subseteq C^*(\pi, T)$ that annihilates all the operators of the form $\pi(a)T^m T^{*n}\pi(b)$ with $n \neq m$, $a \in I_m$, $b \in I_n$.*

Proof. Faithfulness of $\pi \rtimes T$ on A_∞ follows from Theorem 4.8 and [32, Theorem 6.4]. Implications (i)⇒(ii)⇒(iii) are obvious, cf. Proposition 5.1. Assume (iii) and denote by E_π the conditional expectation from $C^*(\pi, T)$ onto $(\pi \rtimes T)(A_\infty)$. Then $E_\pi \circ \pi \rtimes T = \pi \rtimes T \circ E$, and this composite map is faithful because E is faithful and $\pi \rtimes T$ is faithful on the range of E. This implies that $\pi \rtimes T$ is faithful on the whole of $A \rtimes L$. □

Corollary 5.3. *For each $n \in \mathbb{N}$, $A \rtimes L^n$ is naturally isomorphic to the C^*-subalgebra of $A \rtimes L$ generated by $A \cup I_n t^n$.*

Proof. By (21) we see that (id, t^n) is a faithful representation of L^n into $A \rtimes L$. We use Proposition 3.12(ii) to show that the representation (id, t^n) is covariant. To this end note that the set of regular points for ϱ_n is $\Delta^n_{\text{reg}} = \{x \in \Delta_n : x, \varphi(x), \ldots, \varphi^{n-1}(x) \in \Delta_{\text{reg}}\}$. Let $a \in C_c(\Delta^n_{\text{reg}})$ with support K contained in an open set $U \subseteq \Delta^n_{\text{reg}}$ where $\varphi^n|_U$ is injective. Put $a_0 := a$ and for each $k = 1, \ldots, n-1$ let $a_k \in C_0(\varphi^k(U))$ be such that $a_k|_{\varphi^k(K)} \equiv 1$, so that then we have $a = \prod_{k=0}^{n-1} \alpha^k(a_k)$. For each $k = 0, \ldots, n-1$, the map $\varphi|_{\varphi^k(U)}$ is injective. Hence by Proposition 3.12(ii) there is $u_k \in C_0(X)$ such that $a_k = a_k t t^* u_k$. In particular, since $a_0 = a \in C_0(\Delta_n)$ we may assume that $u_0 \in C_0(\Delta_n)$. Then $u := \prod_{k=0}^{n-1} \alpha^k(u_k)$ is well defined and using (21) we get

$$at^n t^{*n} u = a_0(ta_1 t \cdots a_{n-1} t t^* u_{n-1} \cdots t^* u_1 t^*) u_0 = \prod_{k=0}^{n-1} \alpha^k(a_k) = a.$$

Hence (id, t^n) is a covariant representation of L^n. It is equipped with a circle gauge action. Indeed, if $\gamma : \mathbb{T} \to \mathrm{Aut}(A \rtimes L)$ is the gauge action γ^n on $A \rtimes L$, then the desired gauge action on $\overline{\mathrm{span}}\{at^{nk}t^{*nl}b : a \in I_{nk}, b \in I_{nl}, n, m \in \mathbb{N}_0\}$ can be defined by the formula $\gamma^n_\lambda(b) := \gamma_z(b)$, for any $\lambda \in \mathbb{T}$ and $z \in \mathbb{T}$ such that $z^n = \lambda$. Hence we have the natural isomorphism $A \rtimes L^n \cong \overline{\mathrm{span}}\{at^{nk}t^{*nl}b : a \in I_{nk}, b \in I_{nl}, n, m \in \mathbb{N}_0\}$ by Theorem 5.2. □

5.1. Regular Representation and Generalized Expectations

The orbit representation (π_o, T_o) defined in Example 3.14 in general does give a faithful representation of $A \rtimes L$. Tensoring it with the regular representation λ of $\mathbb{Z}$ guarantees that:

Definition 5.4. The *regular representation of the transfer operator* L is the pair $(\tilde{\pi}, \tilde{T})$ where $\tilde{\pi} : C_0(X) \to B(H)$ and $\tilde{T} \in B(H)$ act on $H := \ell^2(X) \otimes \ell^2(\mathbb{Z}) \cong \ell^2(X \times \mathbb{Z})$ by $\tilde{\pi} = \pi_o \otimes id_{\ell^2(\mathbb{Z})}$ and $\tilde{T} = T_o \otimes \lambda$. Thus using the standard orthonormal basis $\{\mathbb{1}_{x,n}\}_{x\in X, n\in\mathbb{Z}}$ of H we have

$$\tilde{\pi}(a)\mathbb{1}_{x,n} = a(x)\mathbb{1}_{x,n}, \quad \tilde{T}\mathbb{1}_{y,n} = \sum_{x\in\varphi^{-1}(y)} \sqrt{\varrho(x)}\mathbb{1}_{x,n+1}.$$

Proposition 5.5. *The regular representation* $(\tilde{\pi}, \tilde{T})$ *is a faithful covariant representation of* L *that extends to a faithful representation* $\tilde{\pi} \rtimes \tilde{T}$ *of* $A \rtimes L$*, so* $A \rtimes L \cong C^*(\tilde{\pi}, \tilde{T})$.

Proof. Using that (π_o, T_o) is a faithful covariant representation of L, one readily concludes that $(\tilde{\pi}, \tilde{T})$ is also a faithful covariant representation of L. By Theorem 5.2, to prove that $\tilde{\pi} \rtimes \tilde{T}$ is faithful it suffices to show that $C^*(\tilde{\pi}, \tilde{T})$ has the appropriate gauge action. To this end, for each $z \in \mathbb{T}$ we define a unitary operator $U_z \in B(\ell^2(X \times \mathbb{Z}))$ by the formula $U_z\mathbb{1}_{x,n} := z^n\mathbb{1}_{x,n}$, $x \in X$, $n \in \mathbb{Z}$. Putting $\gamma_z(b) := U_z b U_z^*$, $b \in C^*(\tilde{\pi}, \tilde{T})$, we get $\gamma_z|_{\pi(A)} = \mathrm{id}_{\pi(A)}$ and $\gamma_z(\pi(a)\tilde{T}) = z\pi(a)\tilde{T}$ for $z \in \mathbb{T}$ and $a \in I$. Hence $\gamma : \mathbb{T} \to \mathrm{Aut}(C^*(\tilde{\pi}, \tilde{T}))$ is the desired homomorphism. □

Corollary 5.6. *Let* L *and* L' *be transfer operators for a fixed partial map* $\varphi : \Delta \to X$ *and let* $\varrho, \varrho' : \Delta \to [0, +\infty)$ *be the corresponding potentials. Assume that there is a continuous strictly positive map* $\omega : \Delta :\to (0, \infty)$ *such that* $\varrho' = \varrho\omega$*. Then*

$$C^*(\tilde{\pi}, \tilde{T}) = C^*(\tilde{\pi}, \tilde{T}'),$$

where $(\tilde{\pi}, \tilde{T})$ *and* $(\tilde{\pi}, \tilde{T}')$ *are regular representations of* L *and* L' *respectively. Thus* $C_0(X) \rtimes L$ *and* $C_0(X) \rtimes L'$ *are naturally isomorphic.*

Proof. For any $a \in C_c(\Delta)$ we have $a\omega^{\frac{1}{2}}, a\omega^{-\frac{1}{2}} \in C_c(\Delta)$, $\tilde{\pi}(a)\tilde{T}' = \tilde{\pi}(a\omega^{\frac{1}{2}})\tilde{T}$ and $\tilde{\pi}(a)\tilde{T} = \tilde{\pi}(a\omega^{-\frac{1}{2}})\tilde{T}'$. Hence $\tilde{\pi}(C_c(\Delta))\tilde{T}' = \tilde{\pi}(C_c(\Delta))\tilde{T}$ which implies $\tilde{\pi}(C_0(\Delta))\tilde{T}' = \tilde{\pi}(C_0(\Delta))\tilde{T}$ and this gives the assertion. □

Using the regular representation we prove the existence of a canonical faithful completely positive map from $C_0(X) \rtimes L$ to the C^*-algebra $\mathcal{B}(X)$ of all bounded Borel complex valued maps on X. We denote by $\delta_{i,j}$ the Kronecker symbol.

Lemma 5.7. *There is a faithful completely positive map $G : C_0(X) \rtimes L \to \mathcal{B}(X)$ such that*

$$G(at^k t^{*l} b) = \delta_{k,l} \cdot ab\varrho_k$$

for all $a \in I_k, b \in I_l$ and $k, l \in \mathbb{N}_0$. In particular, G is a (genuine) conditional expectation from $C_0(X) \rtimes L$ onto $C_0(X)$ iff $\varrho : \Delta \to [0, +\infty)$ is continuous.

Proof. In view of Proposition 5.5 we may identify $A \rtimes L$ with $C^*(\tilde{\pi}(A) \cup \tilde{\pi}(I)\tilde{T})$. Let $P_{x,n}$ be the one-dimensional orthogonal projection onto the subspace spanned by $\mathbb{1}_{x,n} \in H := \ell^2(X) \otimes \ell^2(\mathbb{Z})$, for $(x, n) \in X \times \mathbb{Z}$. Since the projections $\{P_{x,n}\}_{(x,n)\in X\times\mathbb{Z}}$ are pairwise orthogonal and sum up, in the strong topology, to the identity operator, we get that the formula

$$G(b) := \sum_{(x,n)\in X\times\mathbb{Z}} P_{x,n} b P_{x,n}, \quad b \in B(H),$$

defines a faithful, completely positive, contractive map (the series is strongly convergent). For any $a \in I_k, b \in I_l$, $k, l \in \mathbb{N}_0$ $(x, n) \in X \times \mathbb{Z}$ we get

$$\begin{aligned} G(at^k t^{*l} b)\mathbb{1}_{x,n} &= P_{x,n} a t^k t^{*l} b \mathbb{1}_{x,n} = P_{x,n} a t^k b(x)\sqrt{\varrho_l(x)}\mathbb{1}_{\varphi^l(x),n-l} \\ &= P_{x,n} \sum_{t\in\varphi^{-k}(\varphi^l(x))} a(t)\sqrt{\varrho_k(t)} b(x)\sqrt{\varrho_l(x)}\mathbb{1}_{t,n+k-l} \\ &= \delta_{k,l} \cdot a(x)b(x)\varrho_k(x)\mathbb{1}_{x,n} = \delta_{k,l} \cdot (ab\varrho_k)\mathbb{1}_{x,n}. \end{aligned}$$

□

Remark 5.8. The above map G is an identity on $A = C_0(X) \subseteq \mathcal{B}(X)$. Therefore G is a *generalized expectation* for the C^*-inclusion $A \subseteq A \rtimes L$ in the sense of [42, Definition 3.1].

Theorem 5.9. *Let (π, T) be a faithful covariant representation of L. Then $\pi \rtimes T : A \rtimes L \to C^*(\pi, T)$ is faithful if and only if there is a bounded linear map $F : C^*(\pi, T) \to \mathcal{B}(X)$ such that $F(\pi(a)T^k T^{*l}\pi(b)) = \delta_{k,l} \cdot ab\varrho_k$, for all $a \in I_k, b \in I_l, k, l \in \mathbb{N}_0$.*

Proof. If $\pi \rtimes T$ is faithful, then F exists by Lemma 5.7. Conversely, if F exists, then for any $b \in A \rtimes L$ with $\pi \rtimes T(b) = 0$, we have $G(b^*b) = F(\pi \rtimes T(b^*b)) = F(\pi \rtimes T(b)^*\pi \rtimes T(b)) = F(0) = 0$, which by faithfulness of G implies that $b = 0$. Hence $\pi \rtimes T$ is faithful. □

6. Local Homeomorphisms and the Groupoid Model

In this section we assume that $\varphi : \Delta \to X$ is a *local homeomorphism*. We show that the groupoid C^*-algebra associated in [53] to (X, φ) is naturally isomorphic to the crossed product of $C_0(X)$ by a transfer operator. We first discuss the existence of a transfer operator for (X, φ).

Lemma 6.1. (cf. [22, Lemma 2.1]) *Let $\varphi : \Delta \to X$ be a local homeomorphism. For any continuous function $\varrho : \Delta \to [0, +\infty)$ with $\sup_{y\in X} \sum_{x\in\varphi^{-1}(y)} \varrho(x) < \infty$, the formula $L(a)(y) = \sum_{x\in\varphi^{-1}(y)} \varrho(x)a(x)$ defines a transfer operator*

$L : C_0(\Delta) \to C_0(X)$ for φ. Moreover every transfer operator for φ is of the above form (even if we drop our standing assumption (5)).

Proof. For each $a \in C_0(\Delta)$ and $y \in Y$ we have $|L(a)(y)| \le \sum_{x\in\varphi^{-1}(y)} |\varrho(x)a(x)| \le \|a\| \cdot M$ where $M := \sup_{y\in X} \sum_{x\in\varphi^{-1}(y)} \varrho(x)$. Hence L is a well defined bounded linear operator from $C_0(\Delta)$ to the space of bounded functions on X. Clearly, L is positive and satisfies the transfer identity (2). Thus it suffices to show that $L(a)$ is continuous on X for any $a \in C_c(\Delta)$, see Remark 2.1(1). Let K be the compact support of a and take any $y \in X$. If $y \notin \varphi(K)$, then $L(a)(y) = 0$ and as $X\setminus\varphi(K)$ is open, $L(a)$ is continuous at y. Assume then that $y \in \varphi(K)$. Since φ is a local homeomorphism, $\varphi^{-1}(y) \cap K$ is finite, and we may find pairwise disjoint, non-empty open sets $\{U_i\}_{i=1}^n$ covering $\varphi^{-1}(y) \cap K$ and such that $\varphi|_{U_i}$ is injective for any $i = 1, \dots n$. By [22, Lemma 2.1 claim] we may find open $V \subseteq \bigcap_{i=1}^n \varphi(U_i)$ containing y and such that $\varphi^{-1}(V) \cap \left(K\setminus \bigcup_{i=1}^n U_i\right) = \emptyset$. So $\varphi^{-1}(V) \cap K \subseteq \bigcup_{i=1}^n U_i$. Using this we see that $L(a)|_V = \sum_{i=1}^n (\varrho \circ \varphi|_{U_i}^{-1}) \cdot (a \circ \varphi|_{U_i}^{-1})$. Since the latter sum is finite and involves only continuous functions, $L(a)|_V$ is continuous. This finishes the proof of the first part.

For the second part note that since φ is a local homeomorphism, for every $y \in X$, $\varphi^{-1}(y)$ is discrete. Hence the measures in (4) have to be discrete, here we do not need our standing assumption (5). Thus every transfer operator L for φ is of the form (6) and the associated potential ϱ is continuous by Proposition 2.3. In addition, $\sup_{y\in X} \sum_{x\in\varphi^{-1}(y)} \varrho(x) = \|L\| < \infty$. $\square$

The important question is whether we can find a *strictly positive* continuous $\varrho : \Delta \to (0, +\infty)$ with $\sup_{y\in X} \sum_{x\in\varphi^{-1}(y)} \varrho(x) < \infty$. Note that a necessary condition for this is our standing assumption (5). We answer this question in the affirmative in two important cases.

Example 6.2. If $\varphi : \Delta \to X$ is a proper local homeomorphism, then for each $y \in X$ the preimage $\varphi^{-1}(y)$ is finite and in fact the map $X \ni y \mapsto |\varphi^{-1}(y)| \in \mathbb{N}_0$ is continuous (locally constant), see [11, Lemma 2.2] where it is assumed that $\Delta = \varphi(\Delta) = X$ but the proof works in our setting. Thus putting $\varrho(x) := |\varphi^{-1}(\varphi(x))|^{-1}$, $x \in \Delta$, we get a continuous strictly positive function $\varrho > 0$ such that $\sum_{x\in\varphi^{-1}(y)} \varrho(x) = 1$ for every $y \in \varphi(\Delta)$. The corresponding transfer operator is given by the formula

$$L(a)(y) = \frac{1}{|\varphi^{-1}(y)|} \sum_{x\in\varphi^{-1}(y)} a(x), \quad a \in C_0(\Delta). \tag{22}$$

If φ is not proper, (22) fails to define a transfer operator even if we assume $\sup_{y\in X} |\varphi^{-1}(y)| < \infty$ (the function $L(a)(y)$ may be discontinuous).

Example 6.3. Let $\varphi : \Delta \to X$ be any local homeomorphism, but assume that there is a partition of unity $\{f_n\}_{n=1}^\infty$ subordinated to a countable cover $\{U_n\}_{n=1}^\infty$ of Δ such that $\varphi|_{U_n}$ is injective. Such a partition exists if Δ is second countable or more generally if Δ is σ-compact. Then

$$\varrho(x) = \sum_{n=1}^\infty \frac{1}{2^n} f_n(x), \quad x \in \Delta,$$

defines a continuous strictly positive function $\varrho : \Delta \to (0, +\infty)$ such that for every $y \in X$ we have $\sum_{x\in\varphi^{-1}(y)} \varrho(x) \le 1$. Thus ϱ yields a transfer operator.

For an introduction to the theory of étale, locally compact, Hausdorff groupoids we recommend [59]. For any such a groupoid $\mathcal{G}$ the *groupoid C^*-algebra* $C^*(\mathcal{G})$ is the maximal C^*-completion of the $*$-algebra $C_c(\mathcal{G})$ with operations:

$$(f * g)(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} f(\gamma_1)g(\gamma_2) \quad \text{and} \quad f^*(\gamma) = \overline{f(\gamma^{-1})},$$

where $\gamma, \gamma_1, \gamma_2 \in \mathcal{G}$, $f, g \in C_c(\mathcal{G})$. Then the embedding $C_c(\mathcal{G}) \subseteq C_0(\mathcal{G})$ extends to a contractive embedding $C^*(\mathcal{G}) \subseteq C_0(\mathcal{G})$, so that we may view elements of $C^*(\mathcal{G})$ as functions on $\mathcal{G}$ and the formulas for algebraic operations remain valid. Also, identifying X with $\mathcal{G}^0 := \{(x, 0, x) : x \in X\}$, $C_0(X) \subseteq C^*(\mathcal{G})$ is a non-degenerate C^*-subalgebra and there is a conditional expectation F from $C^*(\mathcal{G})$ onto $C_0(X)$ given by restriction of functions. This conditional expectation F is faithful if $\mathcal{G}$ is amenable.

The *transformation groupoid* or *Renault–Deaconu groupoid* associated to (X, φ) is an étale, amenable, locally compact, Hausdorff groupoid, see [53], where

$$\mathcal{G} := \{(x, n-m, y) : n, m \in \mathbb{N}_0, x \in \Delta_n, y \in \Delta_m, \varphi^n(x) = \varphi^m(y)\},$$

the groupoid structure is given by $(x, n, y)(y, m, z) := (x, n+m, z)$, $(x, n, y)^{-1} := (y, -n, x)$, and the topology is defined by the basic open sets $\{(x, n-m, y) : (x, y) \in U \times V, \varphi^n(x) = \varphi^m(y)\}$ where $U \subseteq \Delta_n$, $V \subseteq \Delta_m$ are open sets such that $\varphi^m|_U$ and $\varphi^n|_V$ are injective. For full local homeomorphisms on compact spaces the isomorphism in the following theorem is well known, see [3, 21, 23].

Theorem 6.4. *Assume $\varphi : \Delta \to X$ is a local homeomorphism and let $\varrho : \Delta \to (0, +\infty)$ be any strictly positive continuous map with $\sup_{y\in X} \sum_{x\in\varphi^{-1}(y)} \varrho(x) < \infty$ (such a map always exists when φ is proper or Δ is σ-compact). Then $L(a)(y) = \sum_{x\in\varphi^{-1}(y)} \varrho(x)a(x)$ is a well defined transfer operator $L : C_0(\Delta) \to C_0(X)$ for φ, and we have an isomorphism*

$$C^*(\mathcal{G}) \cong C_0(X) \rtimes L$$

where $\mathcal{G}$ is the Renault–Deaconu groupoid associated to φ. This isomorphism is determined by the formula

$$\Phi(a_n \otimes b_m) := a_n {\varrho_n}^{-\frac{1}{2}} t^n t^{*m} {\varrho_m}^{-\frac{1}{2}} b_m, \quad a_n \in C_c(\Delta_n), \quad b \in C_c(\Delta_m), n, m \in \mathbb{N}_0,$$

where $(a_n \otimes b_m)(x, k, y) = \delta_{k,n-m} \cdot a_n(x)b_m(y)$.

Proof. Let us assume that $C^*(\mathcal{G}) \subseteq B(H)$ is represented in a faithful and non-degenerate way on some Hilbert space H. Let $\{\mu_\lambda\}_{\lambda\in\Lambda} \subseteq C_c(\Delta)$ be an approximate unit in I and consider the net of functions $\{T_\lambda\}_{\lambda\in\Lambda} \in C_c(\mathcal{G})$ given by $T_\lambda(x, 1, \varphi(x)) = \mu_\lambda(x)\varrho(x)^{\frac{1}{2}}$ and $T_\lambda(x, n, y) = 0$ if $(n, y) \neq (1, \varphi(x))$. We claim that $\{T_\lambda\}_{\lambda\in\Lambda}$ is strongly Cauchy. Indeed, let $a \in A$, $h \in H$ and

$\lambda \leq \lambda'$, in the directed set Λ. We have $T_\lambda^* a T_{\lambda'} = L(\mu_\lambda a \mu_{\lambda'})$ in the $*$-algebra $C_c(\mathcal{G})$. Thus

$$\begin{aligned}\|(T_\lambda - T_{\lambda'})ah\|^2 &= \langle h, L\left(\alpha(a^*)(\mu_\lambda - \mu_{\lambda'})^2\alpha(a)\right)h\rangle \\ &\leq \langle h, a^* L(\mu_\lambda - \mu_{\lambda'})ah\rangle \\ &= \langle ah, L(\mu_\lambda - \mu_{\lambda'})ah\rangle.\end{aligned}$$

Since the net $\{L(\mu_\lambda)\}_{\lambda\in\Lambda}$ is strongly convergent the last expression tends to zero. Hence $T := \text{s-}\lim_{\lambda\in\Lambda} T_\lambda$ defines a bounded operator. For every $a \in C_0(\Delta)$ we have

$$T^* a T = \text{s-}\lim_{\lambda\in\Lambda} T_\lambda^* a T_\lambda = \lim_{\lambda\in\Lambda} L(\mu_\lambda a \mu_\lambda) = L(a).$$

If a is supported on a set K such that $\varphi|_K$ is injective, then taking $u \in C_c(\Delta_{\text{reg}})$ such that $u|_K = (\varrho|_K)^{-1}$ we get $aTT^*u = \text{s-}\lim_{\lambda\in\Lambda} aT_\lambda T_\lambda u = \lim_{\lambda\in\Lambda} \mu_\lambda a = a$. Hence (id, T) is a covariant representation of L by Proposition 3.12. Thus we have a $*$-homomorphism $\text{id}\times T : C_0(X) \rtimes L \to B(H)$. It takes values in $C^*(\mathcal{G})$ because if $a \in C_c(\Delta)$, then $aT \in C_c(\mathcal{G})$ where $aT(x,k,y) = \delta_{(k,y),(1,\varphi(x))} \cdot a(x)\varrho(x)^{\frac{1}{2}}$. More generally, one readily checks that for $a_n \in C_c(\Delta_n)$, $b \in C_c(\Delta_m)$ we have $a_n {\varrho_n}^{-\frac{1}{2}} T^n T^{*m} {\varrho_m}^{-\frac{1}{2}} b_m = a_n \otimes b_m \in C_c(\mathcal{G})$. Since functions $a_n \otimes b_m$ span $C_c(\mathcal{G})$ we conclude that $\text{id}\times T : C_0(X) \rtimes L \to C^*(\mathcal{G})$ is a surjective $*$-homomorphism that intertwines the conditional expectations $G : C_0(X) \rtimes L \to C_0(X)$ and $F : C^*(\mathcal{G}) \to C_0(X)$. Hence $\text{id}\times T$ is an isomorphism by Corollary 5.9. Its inverse is as described in the assertion. $\square$

Example 6.5. (Deaconu–Muhly C^*-algebras associated with branched coverings) We consider a slightly more general situation than in [16] and by a *branched self-covering* we mean a continuous open and surjective map $\varphi : X \to X$ of a locally compact, σ-compact space, for which there is a closed set $S \subseteq X$ such that $\varphi|_{X\setminus S}$ is a local homeomorphism. The C^*-algebra $DM(X,\varphi)$ associated to σ in [16] is by definition the C^*-algebra of the Renault–Deaconu groupoid associated to the partial local homeomorphism $\varphi : X\setminus S \to X$. Thus by Theorem 6.4 we have

$$DM(X,\varphi) \cong C_0(X) \rtimes L.$$

where $L : C_0(X) \to C_0(X)$ is any transfer operator for $\varphi : X \to X$ given by continuous $\varrho : X \to [0,+\infty)$ with $S = \varrho^{-1}(0)$. If in addition, S has empty interior, then $C_0(X) \rtimes L$ is Exel's crossed product.

Example 6.6. (Graph C^*-algebras) Let $E = (E^0, E^1, r, s)$ be a countable directed graph ($r, s : E^1 \to E^0$ are range and source maps). The *boundary space* $\partial E = E^\infty \cup E_s^* \cup E_{inf}^*$ of E, cf. [61], [9, Subsection 4.1] or [39], as a set consists of all infinite paths and of finite paths that start in sources or in infinite emitters. It is a locally compact Hausdorff space with topology generated by cylinder sets and their complements. The one-sided *topological Markov shift* associated to E is the map $\sigma : \partial E\setminus E^0 \to \partial E$ defined, for $\mu = \mu_1\mu_2\cdots \in \partial E\setminus E^0$, by the formulas

$$\sigma(\mu) := \mu_2\mu_3\cdots \text{ if } \mu \notin E^1, \quad \text{and} \quad \sigma(\mu) := s(\mu_1) \text{ if } \mu = \mu_1 \in E^1.$$

This is a countable-to-one local homeomorphism. So we have a partial endomorphism $\alpha : C_0(\partial E) \to M(C_0(\partial E\setminus E^0))$. One may always find strictly positive numbers $\lambda = \{\lambda_e\}_{e\in E^1}$, such that the formula

$$L(a)(\mu) = \sum_{e\in E^1,\, e\mu\in\partial E} \lambda_e\, a(e\mu)$$

defines a bounded map $L : C_0(\partial E\setminus E^0) \to C_0(\partial E)$ ([39, Proposition 5.4] characterizes when this happens), and then L is a transfer operator for σ. By [39, Theorem 5.6] we then also have

$$C^*(E) \cong C_0(\partial E) \rtimes L,$$

where $C^*(E)$ is the graph C^*-algebra - the universal C^*-algebra generated by partial isometries $\{s_e : e \in E^1\}$ and mutually orthogonal projections $\{p_v : v \in E^1\}$ such that $s_e^* s_e = p_{s(e)}$, $s_e s_e^* \leq p_{r(e)}$ and $p_v = \sum_{r(e)=v} s_e s_e^*$ whenever the sum is finite.

Example 6.7. (Exel–Laca C^*-algebras) Let I be any set and let $\mathbb{A} = \{A(i,j)_{i,j\in I}\}$ be a $\{0,1\}$-matrix over I with no identically zero rows. The Exel–Laca algebra $\mathcal{O}_{\mathbb{A}}$ is the universal C^*-algebra generated by partial isometries $\{s_i : i \in I\}$ with commuting initial projections and mutually orthogonal range projections satisfying $s_i^* s_i s_j s_j^* = A(i,j) s_j s_j^*$ and

$$\prod_{i\in E} s_i^* s_i \prod_{j\in F} (1 - s_j^* s_j) = \sum_{k\in I} \prod_{i\in E} A(i,k) \prod_{j\in F} (1 - A(j,k)) s_k s_k^*$$

whenever $E, F \subseteq I$ are finite sets such that $\prod_{i\in E} A(i,k) \prod_{j\in F}(1 - A(j,k))$ is non zero only for finitely many $k \in I$.

For any word $\alpha = \alpha_1 \cdots \alpha_n$ in I admissible by $\mathbb{A}$ we put $s_\alpha = s_{\alpha_1} \cdots s_{\alpha_n}$. Then

$$\mathcal{D}_{\mathbb{A}} := \overline{\operatorname{span}} \left\{ s_\alpha \big(\prod_{i\in E} s_x^* s_x \big) s_\alpha^* : E \subseteq I \text{ is a finite set}, \alpha \text{ is a finite word} \right\}$$

is a commutative C^*-subalgebra of $\mathcal{O}_{\mathbb{A}}$. The spectrum X of this algebra is a second countable totally disconnected space described in [20], as a certain subset of $\{0,1\}^{\mathbb{F}}$ where $\mathbb{F}$ is a free group generated by I. It is also described in [53] as a spectrum of a certain Boolean algebra that models a Markov shift. In particular, there is a naturally associated partial local homeomorphism $\varphi : \Delta \to X$ defined on an open dense subset $\Delta \subseteq X$. The space of infinite admissible words

$$X_{\mathbb{A}} := \{\omega \in I^{\mathbb{N}} : A(\omega_n, \omega_{n+1}) = 1 \text{ for } n \in \mathbb{N}\}$$

embeds naturally into X (and is dense in X when $\mathbb{A}$ is irreducible), in a way that

$$\varphi(\omega_1\omega_2\cdots) = \omega_2\omega_3\cdots, \quad \text{for} \quad \omega \in X_{\mathbb{A}} \cap \Delta.$$

Moreover, by [53, Proposition 4.8] $\mathcal{O}_{\mathbb{A}}$ is isomorphic to the C^*-algebra $C^*(\mathcal{G})$ of the Renault–Deaconu groupoid associated to φ. Thus by Theorem 6.4, $\mathcal{O}_{\mathbb{A}}$ is isomorphic to the crossed product $C_0(X) \rtimes L$ for a certain transfer operator L for φ. Such an isomorphism is described in [22, Proposition 2.13] for an unbounded transfer operator, and one can make the operator bounded by choosing appropriate potential ϱ. For instance, as in Example 6.6, it suffices to choose positive numbers $\lambda = \{\lambda_i\}_{i\in I}$, such that the formula

$$T_\lambda := \sum_{i\in I} \sqrt{\lambda_i} s_i$$

converges strictly in $\mathcal{O}_{\mathbb{A}}$, cf. [39, Proposition 5.4]. Then $L(a) := T_\lambda(a)T_\lambda^*$ defines a bounded transfer operator for φ and $\mathcal{O}_{\mathbb{A}} \cong C_0(X) \rtimes L$.

7. Spectra of the Core Subalgebras

We now proceed to the analysis of the internal structure of the core subalgebra A_∞ of $A \rtimes L$. The fundamental fact is that $A_\infty = \overline{\bigcup_{n\in\mathbb{N}_0} A_n}$ is a direct limit of algebras that can be further decomposed into 'liminary pieces'. Namely, for each $n \in \mathbb{N}_0$ we put

$$\begin{aligned} K_n : &= \overline{I_n t^n t^{*n} I_n}, \quad A_n := K_0 + K_1 + \dots + K_n \\ &= \overline{\operatorname{span}} \left\{ a t^k t^{*k} b : a, b \in I_k, k = 0, \dots, n \right\}. \end{aligned}$$

Proposition 7.1. *For each $n \in \mathbb{N}_0$, A_n and K_n are C^*-subalgebras of A_∞. Moreover, $\overline{K_n K_m} = K_m$ for $n \le m$ and $A_n \cap K_{n+1} = K_n \cap K_{n+1} = \overline{I_n t^n C_0(\Delta_{\mathrm{reg}}) t^{*n} I_n}$.*

Proof. By Remark 4.9, $J_{M_L} = C_0(\Delta_{\mathrm{reg}})$ and we have an isomorphism of C^*-correspondences $M_L \cong \overline{It}$. By Corollary 5.3, this implies that for any $n \in \mathbb{N}$ we have $M_{L^n} \cong \overline{I_n t^n}$, and so also $M_{L^n} \cong M_L^{\otimes n}$, see (16), (17) and (18). In particular, K_n is naturally isomorphic with compact operators on M_{L^n}, and the assertion follows from the corresponding facts for Cuntz–Pimsner algebras, see, for instance, [32, Lemma 5.4, Propositions 5.9, 5.11]. ☐

Recall that $I_n = C_0(\Delta_n)$, where Δ_n is the domain of φ^n, and $I_0 = A = C_0(X)$. We put

$$\Delta_{\mathrm{pos},n} := \Delta_n \backslash \varrho_n^{-1}(0) = \{x \in \Delta_n : \varrho_n(x) > 0\}, \quad n \in \mathbb{N}_0,$$

which is the natural domain for the n-th iterate of the partial map $\varphi|_{\Delta_{\mathrm{pos}}}$ where $\Delta_{\mathrm{pos}} := \{x \in \Delta : \varrho(x) > 0\}$. Using the transfer identity, we see that the closure of $L^n(I_n)$ is an ideal in A. Its spectrum is

$$\widehat{L^n(I_n)} = \{y \in X : \varphi^{-n}(y) \backslash \varrho_n^{-1}(0) \neq \emptyset\} = \varphi^n(\Delta_{\mathrm{pos},n}),$$

and in particular, this set is open in X. For any positive function $\rho : \Omega \to (0, \infty)$ we denote by $\ell^2(\Omega, \rho)$ the weighted ℓ^2-space consisting of those functions $f : \Omega \to \mathbb{C}$ for which $\|f\|_2 := \left(\sum_{x\in\Omega} |f(x)|^2 \rho(x)\right)^{1/2} < \infty$. This is a Hilbert space unitarily isomorphic to $\ell^2(\Omega)$ via the map $\ell^2(\Omega, \rho) \ni \mathbb{1}_x \mapsto \sqrt{\rho(x)} \mathbb{1}_x \in \ell^2(\Omega)$.

Proposition 7.2. *For each $n \in \mathbb{N}$ the algebra K_n is liminary (in fact it has a continuous trace) and up to unitary equivalence all its irreducible representations are subrepresentations of the orbit representation. Moreover, we have a homeomorphism*

$$\widehat{K}_n \cong \varphi^n(\Delta_{\mathrm{pos},n})$$

under which the representation π_y^n of K_n corresponding to $y \in \varphi^n(\Delta_{\mathrm{pos},n})$ acts on $H_y^n := \ell^2(\varphi^{-n}(y)\backslash\varrho_n^{-1}(0), \varrho_n)$ and is defined by

$$\pi_y^n(at^nt^{*n}b)h = a \cdot \left(\sum_{x\in\varphi^{-n}(y)} \varrho_n(x)b(x)h(x)\right),$$

for $a, b \in I_n$ and $h \in H_y^n$. The map $U \mapsto \overline{I_nt^nC_0(U)t^{*n}I_n}$ is a bijection between open subsets of $\varphi^n(\Delta_{\mathrm{pos},n})$ and ideals in K_n.

Proof. As in the proof of Proposition 7.1, we may identify M_{L^n} with $\overline{I_nt^n}$ and then K_n is identified with the algebra of compact operators on M_{L^n}. Hence $\overline{I_nt^n}$ is a Morita–Rieffel equivalence bimodule between $K_n = \overline{I_nt^nt^{*n}I_n}$ and $\overline{t^{*n}I_nt^n} = \overline{L^n(I_n)} = C_0(\varphi^n(\Delta_{\mathrm{pos},n}))$. Such an equivalence preserves spectra and a number of other properties, see [52]. In particular K_n has a continuous trace, because $C_0(\varphi^n(\Delta_{\mathrm{pos},n}))$ has it. The ideal in K_n corresponding via the equivalence $M_{L^n} = \overline{I_nt^n}$ to an ideal $C_0(U)$ in $C_0(\varphi^n(\Delta_{\mathrm{pos},n}))$ is $\overline{\langle M_{L^n}, C_0(U)M_{L^n}\rangle} = \overline{I_nt^nC_0(U)t^{*n}I_n}$. This correspondence extends to a homeomorphism $\widehat{K}_n \cong \varphi^n(\Delta_{\mathrm{pos},n})$ where the representation π_y of K_n corresponding to $y \in \varphi^n(\Delta_{\mathrm{pos},n})$ acts on the Hilbert space $H_y := M_{L^n} \otimes_{\mathrm{ev}_y} \mathbb{C}$ which by construction is the Hausdorff completion of the algebraic tensor product $C_0(\Delta_n) \otimes \mathbb{C}$ in seminorm coming from the sesqui-linear form determined by

$$\langle c_1 \otimes \lambda_1, c_2 \otimes \lambda_2\rangle = \overline{\lambda_1}L(c_1^*c_2)(y)\lambda_2 = \sum_{x\in\varphi^{-n}(y)} \varrho_n(x)\overline{\lambda_1 c_1(x)}\lambda_2 c_2(x).$$

Then π_y is determined by the formula $\pi_y(at^nt^*b)[c \otimes \lambda] = [aL^n(bc) \otimes \lambda]$, for $a, b, c \in C_0(\Delta_n)$, $\lambda \in \mathbb{C}$. Using this one readily sees that the map $[c \otimes 1] \mapsto c|_{\varphi^{-n}(y)\backslash\varrho_n^{-1}(0)}$ determines a unitary $H_y \cong H_y^n = \ell^2(\varphi^{-n}(y)\backslash\varrho_n^{-1}(0), \varrho_n)$ that intertwines π_y and π_y^n. Moreover, the subspace $G_y := \ell^2\left(\varphi^{-n}(y)\backslash\varrho_n^{-1}(0)\right)$ of $\ell^2(X)$ is invariant under the action of $\pi_o \rtimes T_o(K_n)$, because for $a \in A$ and $x \in \varphi^{-n}(y)\backslash\varrho_n^{-1}(0)$ we have

$$\pi_o(a)\mathbb{1}_x = a(x)\mathbb{1}_x, \quad T_o^nT_o^{*n}\mathbb{1}_x = \sum_{x'\in\varphi^{-n}(y)\backslash\varrho_n^{-1}(0)} \sqrt{\varrho_n(x')\varrho_n(x)}\mathbb{1}_{x'}.$$

Thus we have a subrepresentation $\sigma_y : K_n \to B(G_y)$ of $\pi_o \rtimes T_o|_{K_n}$ where $\sigma_y(at^*tb) = \pi_o(a)T_oT_o^*\pi_o(b)|_{G_y}$. The canonical isomorphism $G_y \cong H_y^n$ is a unitary equivalence between σ_y and π_y^n. □

Remark 7.3. If $\Delta = X$ is compact, then $t \in A \rtimes L$ and the unique extension of π_y^n to $A + K_n$ is defined by the formulas $\pi_y^n(a)h = a \cdot h$, $\pi_y^n(t^nt^{*n})h =$

$\left(\sum_{x\in\varphi^{-n}(y)} \varrho_n(x)h(x)\right)\cdot 1$, for $a \in A$ and $h \in H^n_y$. Thus $\pi^n_y(a)$ is a multiplication operator and $\pi^n_y(t^n t^{*n})$ is a rank one operator whose range consists of constant functions.

Having a continuous map $f : U \to Y$ defined on an open subset U of a topological space X, we may *attach* X *to* Y *along* f to get the space $X \cup_f Y := (X \sqcup Y)/(x \sim f(x)$ for all $x \in U)$ equipped with the quotient topology. This is the *pushout* of $f : U \to Y$ and the inclusion map $U \subseteq X$. We may always identify $X \cup_f Y$ with the disjoint union $X \cup_f Y := (X\setminus U) \sqcup Y$ where the second summand Y is open in $X \cup_f Y$ and if the map f is open, then the open sets in $X \cup_f Y$ can be identified with pairs of open sets $V \subseteq X$, $W \subseteq Y$ satisfying $\varphi^{-1}(W) = V \cap U$ (then the corresponding open set in $X \cup_f Y$ is $V\setminus U \sqcup W$). We use this construction to describe the spectrum of the C^*-algebras A_n, as $A_{n+1} = A_n + K_n$ may be viewed as a pushout of A_n and K_{n+1} via the C^*-algebra $A_n \cap K_{n+1}$.

Lemma 7.4. *For each $n \in \mathbb{N}$, we have continuous bijection form $\widehat{K_n + K_{n+1}}$ onto the pushout of $\varphi^n(\Delta_{\mathrm{pos},n})$ and $\varphi^{n+1}(\Delta_{\mathrm{pos},n+1})$ along the partial homeomorphism $\varphi : \varphi^n(\Delta_{\mathrm{pos},n}) \cap \Delta_{\mathrm{reg}} \to \varphi^{n+1}(\Delta_{\mathrm{pos},n+1})$. We have a continuous bijection*

$$\widehat{K_n + K_{n+1}} \xrightarrow{\cong} \varphi^n(\Delta_{\mathrm{pos},n})\setminus\Delta_{\mathrm{reg}} \sqcup \varphi^{n+1}(\Delta_{\mathrm{pos},n+1}), \tag{23}$$

where the topology on the right hand side consists of sets $U_n\setminus\Delta_{\mathrm{reg}} \sqcup U_{n+1}$ where $U_n \subseteq \varphi^n(\Delta_{\mathrm{pos},n})$, $U_{n+1} \subseteq \varphi^{n+1}(\Delta_{\mathrm{pos},n+1})$ are open and $\varphi^{-1}(U_{n+1}) = U_n \cap \Delta_{\mathrm{reg}}$.

Proof. Since K_{n+1} is an ideal in $K_n + K_{n+1}$ we may identify $\widehat{K}_{n+1} \cong \varphi^{n+1}(\Delta_{\mathrm{pos},n+1})$ with an open subset of $\widehat{K_n + K_{n+1}}$. Its complement is naturally identified with the spectrum of the quotient $K_n/(K_{n+1} \cap K_n) \cong (K_n + K_{n+1})/K_{n+1}$. By Proposition 7.1, $K_n \cap K_{n+1} = \overline{I_n t^n C_0(\Delta_{\mathrm{reg}}) t^{*n} I_n}$ is an ideal in K_n. Hence using the homeomorphisms from Proposition 7.2 we may identify $\widehat{K}_{n+1}$ with $\varphi^{n+1}(\Delta_{\mathrm{pos},n+1})$ and $K_n \cap K_{n+1}$ with $\varphi^n(\Delta_{\mathrm{pos},n}) \cap \Delta_{\mathrm{reg}}$. Accordingly, we get the bijection (23), which restricts to homeomorphisms $\varphi^{n+1}(\Delta_{\mathrm{pos},n+1})$ and $\widehat{K_n + K_{n+1}}\setminus\widehat{K}_{n+1} \cong \varphi^n(\Delta_{\mathrm{pos},n})\setminus\Delta_{\mathrm{reg}}$. Any representation π that is in $\widehat{K_n + K_{n+1}}\setminus\widehat{K}_{n+1}$ is a representation of K_n that vanishes on K_{n+1}. Every ideal in K_n is of the form $\overline{It^n C_0(U) t^{*n} I}$, and the ideal in $K_n + K_{n+1}$ generated by the latter is

$$\overline{It^n C_0(U) t^{*n} I} + \overline{It^{n+1} C_0(\varphi(U \cap \Delta_{\mathrm{reg}}) t^{*n+1} I}.$$

Hence the bijection (23) becomes continuous if $\widehat{L^n(I_n)}\setminus\Delta_{\mathrm{reg}} \sqcup \widehat{L^{n+1}(I_{n+1})}$ is equipped with the pushout topology. □

The pushout topology on the right hand side of (23) is always T_0, and the continuous bijection (23) might be a homeomorphism even when this topology is non-Hausdorff, see [29] and Example 7.8 below. However, in general the pushout topology is weaker than the topology of the spectrum $\widehat{K_n + K_{n+1}}$, and a general description of the topology of the latter requires more than just the pushout data:

Example 7.5. Let us consider $A_1 = A + K_1 = K_0 + K_1$ associated to the transfer operator $L(a)(y) = a(\frac{y}{2})$ for the tent map $\varphi : [0,1] \to [0,1]$, $\varphi(x) = 1 - |1-2x|$. Then $\varrho = \mathbb{1}_{[0,\frac{1}{2}]}$, $\Delta_{\mathrm{pos}} = [0, \frac{1}{2}]$ and $\Delta_{\mathrm{reg}} = [0, 1/2)$. So as sets we have

$$\widehat{A}_1 \cong X \setminus \Delta_{\mathrm{reg}} \sqcup \varphi(\Delta_{\mathrm{pos}}) = [1/2, 1] \sqcup [0, 1].$$

The pushout topology on the right hand side is the usual one with the only exception that neighbourhoods of 1/2 in the first summand contain sets of the form $[1/2, 1/2+\varepsilon) \sqcup (1-\varepsilon, 1)$. So in particular the pushout topology is not Hausdorff in this case (it is T_0 though). The topology on $\widehat{A}_1$ is larger and in fact $\widehat{A}_1$ is homeomorphic to the direct union of two closed intervals. Indeed, the operator tt^* in the regular representation becomes the multiplication operator by the characteristic function $\mathbb{1}_{[0,\frac{1}{2}]}$. So A_1 is generated by $A = C[0,1]$ and $tt^* = \mathbb{1}_{[0,\frac{1}{2}]}$ and $A_1 = Att^* \oplus A(1 - tt^*) = C[0, 1/2] \oplus C[1/2, 1]$. The extra open set in $\widehat{A}_1$ (not seen by the pushout topology) comes from the ideal generated by the element $1 - tt^*$ which is neither in $K_0 = A$ nor in K_1. Thus the precise description of $\widehat{A}_1$ seems to require some additional algebraic data that is difficult to pin down.

Theorem 7.6. *Let $\varrho : \Delta \to [0, +\infty)$ be a potential associated to a transfer operator $L : C_0(\Delta) \to C_0(X)$ for $\varphi : \Delta \to X$. For each $n \in \mathbb{N}$ the algebra A_n is postliminary and we have a natural bijection*

$$\widehat{A}_n \xrightarrow{\cong} \left(\bigsqcup_{k=0}^{n-1} \varphi^k(\Delta_{\mathrm{pos},k}) \setminus \Delta_{\mathrm{reg}} \right) \sqcup \varphi^n(\Delta_{\mathrm{pos},n}). \tag{24}$$

More specifically, for every irreducible representation π of A_n there is a maximal $k \le n$ with $\pi(K_k) \neq 0$ and a unique $y \in \varphi^k(\Delta_{\mathrm{pos},k})$ ($y \in \varphi^k(\Delta_{\mathrm{pos},k}) \setminus \Delta_{\mathrm{reg}}$ if $k < n$) such that $\pi \cong \pi_y^k$ where π_y^k is a representation of A_n on $\ell^2(\varphi^{-k}(y) \setminus \varrho_k^{-1}(0), \varrho_k)$ determined by

$$\pi_y^k(at^i t^{*i} b)h = a \cdot \left(\sum_{x \in \varphi^{-i}(y)} \varrho_i(x) b(x) h(x) \right), \quad a, b \in I_i, \quad i = 1, \ldots, k,$$

and $\pi_y^k(K_i) = 0$ for all $k < i \le n$. If we equip the right hand side of (24) with the the topology that consists of sets $\left(\bigsqcup_{k=0}^{n-1} U_k \setminus \Delta_{\mathrm{reg}} \right) \sqcup U_n$ where U_k is an open subset of $\varphi^k(\Delta_{\mathrm{pos},k})$, for $k = 0, \ldots, n$, and $U_k \cap \Delta_{\mathrm{reg}} = \varphi^{-1}(U_{k+1})$ for $k < n$, then (24) is continuous and its inverse is continuous when restricted to each direct summand.

Proof. We prove this by induction. The assertion holds for $n = 1$ by Lemma 7.4. Assume that for certain n we have $\widehat{A}_n \cong \left(\bigsqcup_{k=0}^{n-1} \varphi^k(\Delta_{\mathrm{pos},k}) \setminus \Delta_{\mathrm{reg}} \right) \sqcup \varphi^n(\Delta_{\mathrm{pos},n})$ as in the assertion. Here $\bigsqcup_{k=0}^{n-1} \varphi^k(\Delta_{\mathrm{pos},k}) \setminus \Delta_{\mathrm{reg}}$ corresponds to the closed set $\widehat{A}_n \setminus \widehat{K}_n$ and $\widehat{K}_n \cong \varphi^n(\Delta_{\mathrm{pos},n})$ is the homeomorphism from Proposition 7.2.

By Proposition 7.1, $K_n + K_{n+1}$ is an ideal in A_{n+1}. The corresponding open subset of $\widehat{A}_{n+1}$ is $\widehat{K_n + K_{n+1}} \cong \widehat{L^n(I_n)} \setminus \Delta_{\mathrm{reg}} \sqcup \widehat{L^{n+1}(I_{n+1})}$ as described in Lemma 7.4. Its complement $\widehat{A}_{n+1}\setminus\widehat{K_n+K_{n+1}} \cong \widehat{A}_n\setminus\widehat{K}_n \cong \left(\bigsqcup_{k=0}^{n-1}\widehat{L^k(I_k)}\setminus\Delta_{\text{reg}}\right)$. Since $A_n\cap K_{n+1} = K_n\cap K_{n+1}$, see Proposition 7.1, we conclude that the topology on $\widehat{A}_{n+1} \cong \left(\bigsqcup_{k=0}^{n}\widehat{L^k(I_k)}\setminus\Delta_{\text{reg}}\right)\sqcup \widehat{L^{n+1}(I_{n+1})}$ is as described in the assertion. The ranges of all representations in $\widehat{A}_n$ contain compact operators. Hence A_n is postliminary. □

Remark 7.7. If π is an irreducible representation of A_n, there is a 'dynamical procedure' of determining y and k for which $\pi\cong\pi_y^k$. Namely, the set $Z := \{x\in X : a(x)\neq 0 \text{ implies } \pi(a)\neq 0 \text{ for all } a\in A\}$ is closed and there is $k\leq n$ such that $\varphi^k(Z)$ is a singleton. If there is a minimal $k<n$ such that $\varphi^k(Z)=\{y\}\notin\Delta_{\text{reg}}$, then $\pi\cong\pi_y^k$. Otherwise $\pi\cong\pi_y^n$ where $\varphi^n(Z)=\{y\}$.

Example 7.5 shows that the continuous bijection (24) in general fails to be a homeomorphism. Obviously, it is a homeomorphism when the pushout topology on the right hand side of (24) is Hausdorff, and less obviously, when ϱ is continuous, see Theorem 7.11 below. This may also happen in a non-continuous and non-Hausdorff case:

Example 7.8. Let us consider the standard transfer operator $L(a)(y)=\frac{1}{2}[a(\frac{y}{2})+a(1-\frac{y}{2})]$ for the tent map $\varphi:[0,1]\to[0,1]$, $\varphi(x)=1-|1-2x|$. Then $\varrho=\frac{1}{2}\mathbb{1}_{X\setminus\{\frac{1}{2}\}}+\mathbb{1}_{\{\frac{1}{2}\}}$, $\Delta_{\text{pos}}=X=[0,1]$ and $\Delta_{\text{reg}}=X\setminus\{\frac{1}{2}\}=[0,\frac{1}{2})\cup(\frac{1}{2},1]$. Accordingly,

$$\widehat{A}_n\cong\bigcup_{k=0}^{n-1}\{\pi_{1/2}^k\}\cup\{\pi_x^n : x\in[0,1]\}$$

where the pushout topology on the right hand side can be described as follows: $\{\pi_x^n : x\in[0,1]\}$ is an open set homeomorphic to $[0,1]$ and each $\pi_{1/2}^k$ has a basis of neighbourhoods of the form $\{\pi_{1/2}^k\}\cup\{\pi_x^n : x\in(0,\varepsilon)\}$, if $k<n-1$, and $\{\pi_{1/2}^{n-1}\}\cup\{\pi_x^n : x\in(1-\varepsilon,1)\}$, if $k=n-1$ (so $\pi_{1/2}^k$, $k<n-1$, cannot be separated from π_0^n and $\pi_{1/2}^{n-1}$ can not be separated from π_1^n). This topology coincides with the standard topology of $\widehat{A}_n$, as using the regular representation one can see that A_n is naturally isomorphic the C^*-subalgebra of $C([0,1],M_{2^n}(\mathbb{C}))$ consisting continuous matrix valued functions a satisfying

$$a(1)\in M_{2^{n-1}}(\mathbb{C})\oplus M_{2^{n-1}}(\mathbb{C}),$$
$$a(0)\in M_{2^{n-1}+1}(\mathbb{C})\oplus M_{2^{n-2}}(\mathbb{C})\oplus\cdots\oplus M_2(\mathbb{C})\oplus\mathbb{C}.$$

In particular, representations π_1^n and $\pi_{1/2}^{n-1}$ are of dimension $2^{n-1}=|\varphi^{-n}(1)|$, and π_0^n is of dimension $2^{n-1}+1=|\varphi^{-n}(0)|$. This example is covered by the main result of [29].

The algebra A_∞ as a rule is not postliminary (the example of Glimm algebras shows that A_∞ will usually be antiliminary, cf. [48, Theorem 6.5.7]). Accordingly, one can not hope to describe the spectrum $\widehat{A}_\infty$ completely in a reasonable way. However, the inductive limit of spaces $\widehat{A}_n$ will give a dense subset of $\widehat{A}_\infty$, and when the continuous maps (24) are opens one can use them to describe the *primitive ideal space* $\mathrm{Prim}(A_\infty)$ of A_∞. We show how to do it in the case when ϱ is continuous.

7.1. The Case of a Continuous Potential

When ϱ is continuous then A_∞ has a natural étale groupoid model. The groupoid in question is a motivating example in the theory of approximately proper equivalence relations [54], which recently has been generalized to cover partial local homeomorphisms [6]. Namely, assume that $\varphi : \Delta \to X$ is a local homeomorphism. For each $n \in \mathbb{N}$ consider the equivalence relation

$$R_n := \{(x, y) \in \Delta_n \times \Delta_n : \varphi^n(x) = \varphi^n(y)\}$$

as an étale groupoid with the product topology inherited from $\Delta_n \times \Delta_n$. Then we get a generalized approximately proper (in short GAP) equivalence relation on $X \times X$

$$R := \bigcup_{n=0}^{\infty} R_n,$$

equipped with the *inductive limit topology*, i.e. $U \subseteq R$ is open iff $U \cap R_n$ is open for every $n \in \mathbb{N}$, see [6, Proposition 5.6]. Similarly, each finite union $\bigcup_{k=0}^{n} R_k$ becomes an étale groupoid. All these groupoids are amenable (see the proof of [53, Proposition 2.4]).

Proposition 7.9. *Assume $\varphi : \Delta \to X$ is a local homeomorphism and let $\varrho : \Delta \to (0, +\infty)$ be any strictly positive continuous map with $\sup_{y\in X} \sum_{x\in\varphi^{-1}(y)} \varrho(x) < \infty$. Equivalently, fix a transfer operator $L : C_0(\Delta) \to C_0(X)$ with continuous potential $\varrho > 0$. We have natural isomorphisms*

$$K_n \cong C^*(R_n), \quad A_n \cong C^*\left(\bigcup_{k=0}^{n} R_k\right), \quad n \in \mathbb{N}, \quad A_\infty \cong C^*(R),$$

where $K_n := \overline{I_n t^n t^{*n} I_n}$, $A_n := K_0 + \cdots + K_n$, $A_\infty = \overline{\bigcup_{n=0}^{\infty} A_n}$ are core subalgebras of $A \rtimes L$, and R is the GAP relation associated to φ.

Proof. We view R and $\bigcup_{k=0}^{n} R_k$ as open subgroupoids of $\mathcal{G}$ with the same unit space X. Then the inclusions $C_c(\bigcup_{k=0}^{n} R_k) \subseteq C_c(R) \subseteq C_c(\mathcal{G})$ extend to $*$-homomorphisms $C^*(\bigcup_{k=0}^{n} R_k) \to C^*(R) \to C^*(\mathcal{G})$ that intertwine the canonical faithful conditional expectations onto $C_0(X)$ (the groupoids in question are amenable). Hence the aforementioned $*$-homomorphisms are faithful and we may write $C^*(\bigcup_{k=0}^{n} R_k) \subseteq C^*(R) \subseteq C^*(\mathcal{G})$. Now it is immediate that the isomorphism from Theorem 6.4 restricts to isomorphisms $C^*(\bigcup_{k=0}^{n} R_k) \cong A_n$, $C^*(R) \cong A_\infty$.

The groupoid R_n can be viewed as the restriction of $\bigcup_{k=0}^{n} R_k$ to an open invariant subset $\Delta_n \subseteq X$. Hence $C^*(R_n)$ can be identified with an ideal in $C^*(\bigcup_{k=0}^{n} R_k)$ generated by $C_0(\Delta)$, see [59, Proposition 4.3.2]. Restriction of the isomorphism $C^*(\bigcup_{k=0}^{n} R_k) \cong A_n$ gives $C^*(R_n) \cong K_n$. $\square$

Remark 7.10. By [62], all irreducible Smale spaces $(\tilde{X}, \tilde{\varphi})$ with totally disconnected stable sets are inverse limits for certain finite-to-one continuous surjections $\varphi : X \to X$ satisfying Wieler's axioms. By Proposition 7.9 and

[17, Theorem 5.6], if φ is an open Wieler map, then the stable algebra S and the stable Ruelle algebra R_s of the Smale space $(\tilde{X}, \tilde{\varphi})$, cf. [50], are Morita-equivalent to the algebras A_∞ and $A \rtimes L$, respectively.

The groupoids R_n, $\bigcup_{k=0}^n R_k$, R are not only amenable but also *principle* (all the isotropy groups are trivial). In particular, inclusions $C_0(X) \subseteq C^*(\bigcup_{k=0}^n R_k) \cong A_n, C^*(R) \cong A_\infty$ are C^*-*diagonals* in the sense of Kumjian [37], and the ideals in A_n and A_∞ correspond to open invariant sets in $\bigcup_{k=0}^n R_k$ and R, respectively (see [59, Theorem 4.3.3] or [8, Corollary 3.12]). Also the primitive ideal spaces can be identified with quasi-orbit spaces, see [8, Corollary 3.19] or [41, Theorem 7.17]. This allows us to improve Theorem 7.6 in the case when ϱ is continuous, as follows.

Theorem 7.11. *If $\varrho : \Delta \to [0, +\infty)$ is continuous, then the continuous bijection in (24) is a homeomorphism, and we have natural homeomorphisms*

$$\widehat{A}_n \cong \mathrm{Prim}(A_n) \cong X/\sim_n .$$

where $x \sim_n y$ iff there is $k \le n$ such that $x, y \in \Delta_{\mathrm{pos},k}$ and $\varphi^k(x) = \varphi^k(y)$. If in addition X is second countable, we have a homeomorphism

$$\mathrm{Prim}(A_\infty) \cong X/\sim$$

where $x \sim y$ iff $\overline{\mathcal{O}_R(x)} = \overline{\mathcal{O}_R(y)}$ and $\mathcal{O}_R(x) := \bigcup_{k=0, x\in\Delta_{\mathrm{pos},k}}^\infty \varphi^{-k}(\varphi^k(x))$ is the orbit of $x \in X$.

Proof. Since ϱ is continuous we may assume that Δ is equal to $\Delta_{\mathrm{pos}} = \Delta_{\mathrm{reg}}$, so that $\varrho > 0$ and we may apply isomorphisms from Proposition 7.9. The orbits of x for the groupoid $R_{[0,n]} := \bigcup_{k=0}^n R_k$ are given by $\mathcal{O}_n(x) = \bigcup_{k=0, x\in\Delta_{\mathrm{pos},k}}^n \varphi^{-k}(\varphi^k(x))$. We have a bijection

$$\left(\bigsqcup_{k=0}^{n-1} \varphi^k(\Delta_{\mathrm{pos},k}) \setminus \Delta_{\mathrm{reg}}\right) \sqcup \varphi^n(\Delta_{\mathrm{pos},n}) \xrightarrow{\cong} X/\sim_n$$

that sends a point y in k-th summand to $\varphi^{-k}(y)$, which is an $R_{[0,n]}$-orbit. Using this bijection one checks that open $R_{[0,n]}$-invariant subsets of X correspond to open sets in the pushout topology of the right-hand side of (24). Since ideals in $A_n \cong C^*(\bigcup_{k=0}^n R_k)$ correspond bijectively to open $R_{[0,n]}$-invariant sets, this gives the first part of the assertion (we have $\widehat{A}_n \cong \mathrm{Prim}(A_n)$ because A_n is postliminary, in fact Type I_0).

The second part follows because $\mathcal{O}_R(x)$ is the orbit of x under the groupoid R and the primitive ideal space $\mathrm{Prim}(A_\infty) \cong \mathrm{Prim}(C^*(R))$ is homeomorphic to the quasi-orbit space for R by [8, Corollary 3.19]. □

8. Topological Free Transfer Operators and Simplicity

A full map on locally compact Hausdorff space is called topologically free if the set of its periodic points has empty interior. We will introduce topological freeness for a partial map $\varphi : \Delta \to X$ by reducing it to the case of a full map. Namely, we will restrict φ to its *essential domain* $\Delta_\infty := \bigcap_{n=1}^\infty \Delta_n \cap \varphi^n(\Delta_n)$,

$\Delta_n = \varphi^{-n}(\Delta)$, $n \in \mathbb{N}$, which gives a full map $\varphi : \Delta_\infty \to \Delta_\infty$, see [40, Definition 3.1]. As a starting step of defining topological freeness for transfer operators we analyze this notion for open partial maps.

Definition 8.1. A partial continuous open map $\varphi : \Delta \to X$ is *topologically free* if the set of periodic points for $\varphi : \Delta_\infty \to \Delta_\infty$ has empty interior in Δ_∞.

Lemma 8.2. *Suppose that $\varphi : \Delta \to X$ is a partial continuous open map of X. The following conditions are equivalent:*

(i) *φ is topologically free;*
(ii) *for every $n \in \mathbb{N}$ the set $\{x \in \Delta_n : \varphi^n(x) = x\}$ has empty interior in X;*
(iii) *for every $k, l \in \mathbb{N}_0$ with $l < k$, $\{x \in \Delta_k : \varphi^k(x) = \varphi^l(x)\}$ has empty interior in X.*

Proof. Note that $\{x \in \Delta_n : \varphi^n(x) = x\} \subseteq \Delta_\infty$ is closed in Δ_∞, because Δ_∞ is Hausdorff and φ is continuous. Moreover, Δ_∞ is a Baire space, as it is a G_δ subset of the locally compact Hausdorff space X. Thus $\{x \in \Delta_\infty : \exists_{n\in\mathbb{N}}\, \varphi^n(x) = x\} = \bigcap_{n\in\mathbb{N}}\{x \in \Delta_n : \varphi^n(x) = x\}$ has empty interior if and only if each of the intersected sets has empty interior. This proves (i)⇔(ii).

The implication (ii)⇒(iii) is immediate. For the converse assume that there is a non-empty open set $U \subseteq \{x \in \Delta_k : \varphi^k(x) = \varphi^l(x)\}$ where $l < k$. Then $V := \varphi^l(U)$ is non-empty open set contained in $\{x \in \Delta_n : \varphi^n(x) = x\}$ where $n := k - l \in \mathbb{N}$. □

If we consider a transfer operator L associated to a partial map $\varphi : \Delta \to X$, then the natural 'domain of openness' for φ is Δ_{pos}, see Proposition 2.3. The next lemma shows that in the definition of topological freeness we can equally-well use the smaller set Δ_{reg}.

Lemma 8.3. *If L is a transfer operator for a map $\varphi : \Delta \to X$, then $\varphi : \Delta_{\text{reg}} \to X$ is topologically free if and only if $\varphi : \Delta_{\text{pos}} \to X$ is topologically free.*

Proof. Since $\Delta_{\text{reg}} \subseteq \Delta_{\text{pos}}$, topological freeness of $\varphi : \Delta_{\text{pos}} \to X$ implies topological freeness of $\varphi : \Delta_{\text{reg}} \to X$. Assume now that $\varphi : \Delta_{\text{pos}} \to X$ is not topologically free. So there is a non-empty open set $U \subseteq \Delta_{\text{pos},n} = \Delta_n \setminus \varrho_n^{-1}(0)$, for some $n \in \mathbb{N}$, such that for each $x \in U$ we have $x = \varphi^n(x)$. Then φ is injective on each of the sets U, $\varphi(U)$, …, $\varphi^{n-1}(U)$ and since they are contained in $\Delta_{\text{pos}} = \Delta \setminus \varrho^{-1}(0)$ it follows from Proposition 2.3 that they are in fact contained in Δ_{reg}. Therefore $\varphi : \Delta_{\text{reg}} \to X$ is not topologically free. □

The foregoing observations naturally lead to the following definition, which agrees with the version of topological freeness suggested in [14, Example 9.14].

Definition 8.4. We say that the *transfer operator L is topologically free* if the restricted open map $\varphi : \Delta_{\text{reg}} \to X$ is topologically free.

Theorem 8.5. *Let L be a transfer operator for a partial map $\varphi : \Delta \to X$. The following conditions are equivalent:*

(i) *Every faithful covariant representation* (π, T) *of* L *extends to a faithful representation* $\pi \rtimes T$ *of the crossed product* $A \rtimes L$.
(ii) A *detects ideals in* $A \rtimes L$, *i.e.* $A \cap N \neq \{0\}$ *for any non-zero ideal* N *in* $A \rtimes L$.
(iii) *The orbit representation* (π_o, T_o) *introduced in Example* 3.14 *extends to a faithful representation* $\pi_o \rtimes T_o$ *of the crossed product* $A \rtimes L$.
(iv) *The map* $\varphi : \Delta_{\mathrm{reg}} \to X$ *is topologically free.*

Proof. The equivalence (i)⇔(ii) is straightforward and implication (i)⇒(iii) is trivial. To prove (iii)⇒(iv) assume $\varphi : \Delta_{\mathrm{reg}} \to X$ is not topologically free. Then there is a non-empty open set $U \subseteq \Delta_n$, such that $\varphi^n|_U = id|_U$ and $\varphi^k(U) \subseteq X_{\mathrm{reg}}$ for $k = 0, \dots, n-1$. In particular, ϱ_n is continuous and non-zero at every point in U. Thus for any non-zero $a \in C_c(U) \subseteq A$ we have $a\sqrt{\varrho_n} \in A$. Using the regular representation one readily calculates that $at^n - a\sqrt{\varrho_n}$ is a non-zero element of $A \rtimes L \cong C^*(\widetilde{\pi}(A) \cup \widetilde{\pi}(I)\widetilde{T})$, but $(\pi_o \rtimes T_o)(at^n - a\sqrt{\varrho_n}) = 0$. Hence $\pi_o \rtimes T_o$ is not faithful.

Implication (iv)⇒(i) follows from [14, Example 9.14] as by Lemma 8.3 condition described there is equivalent to topological freeness as we define it. □

Topological freeness for homeomorphisms appeared already in the work of Zeller–Meier, see [63, Proposition 4.14], who used it to characterize when $C_0(X)$ is maximal abelian in the associated crossed product. This result was generalized to crossed products by local homeomorphisms by Carlsen and Silvestrov [15]. However, it seems that there is no obvious generalization of this fact if we allow irregular points (discontinuity points of ϱ). More specifically, let A' denote the commutant of $A = C_0(X)$ in $A \rtimes L$. So A is maximal abelian in $A \rtimes L$ iff $A = A'$. Using the generalized expectation G introduced in Lemma 5.7 we clearly have

$$C_0(X) \subseteq \{b \in A \rtimes L : G(b) = b\} \subseteq C_0(X)'.$$

It turns out that when ϱ is discontinuous already the first inclusion might be proper.

Example 8.6. For the tent map $\varphi : [0,1] \to [0,1]$ where $\varphi(x) = 1 - |1-2x|$ and $\varrho = 1\!\!1_{[0,\frac{1}{2}]}$ we have the transfer operator $L(a)(y) = a(\frac{y}{2})$. Using the regular representation one readily calculates that

$$G(t^n t^{*n}) = t^n t^{*n} = 1\!\!1_{[0,\frac{1}{2^n}]}.$$

Accordingly, $C_0(X) \subsetneq \{b \in A \rtimes L : G(b) = b\}$ as the latter contains some functions that are discontinuous at points $\frac{1}{2^n}$, $n \in \mathbb{N}$. Hence $C_0(X) \neq C_0(X)'$ even though the map φ is topologically free.

For the sake of completeness we will generalize the main result of [15] (to partial, not necessarily surjective maps on locally compact spaces, and arbitrary continuous ϱ). The proof is based on Renault's characterization of Cartan subalgebras [55], see also [43, 7.2].

Theorem 8.7. *Suppose that the transfer operator L is given by a continuous ϱ. Then the equivalent conditions in Theorem 8.5 are further equivalent to each of the following:*

(i) *The C^*-algebra $C_0(X)$ is maximal abelian in $C_0(X) \rtimes L$.*
(ii) *$C_0(X)$ is a Cartan subalgebra of $C_0(X) \rtimes L$, in the sense of [55].*
(iii) *The partial map $\varphi : \Delta_{\text{reg}} = \Delta_{\text{pos}} \to X$ is topologically free.*

Proof. The transfer operator $L : I \to A$ for $\varphi : \Delta \to X$ restricts to the transfer operator $L_{\text{reg}} : C_0(\Delta_{\text{reg}}) \to A$ for the partial homeomorphism $\varphi : \Delta_{\text{reg}} \to X$. Since we assume ϱ is continuous we get $\Delta_{\text{reg}} = \Delta \setminus \varrho^{-1}(0)$ and the crossed products $A \rtimes L$ and $A \rtimes L_{\text{reg}}$ are naturally isomorphic (their regular representations coincide, see Proposition 5.5). Thus by Theorem 6.4 we may identify $A \rtimes L$ with the groupoid C^*-algebra $C^*(\mathcal{G})$ of the Renault–Deaconu groupoid $\mathcal{G}$ associated to $\varphi : \Delta_{\text{reg}} \to X$. By the work of Renault's [55], $C_0(X)$ is maximal abelian in $C^*(\mathcal{G})$ if and only if $C_0(X)$ is a Cartan subalgebra of $\mathcal{G}$ if and only if the groupoid $\mathcal{G}$ is effective (Renault considered second countable groupoids but his theory works without this assumption, see [43, 7.2]) By [55, Corollary 3.3] and [53, Proposition 2.3] the groupoid $\mathcal{G}$ is effective if and only if the map $\varphi : \Delta_{\text{reg}} \to X$ is topologically free (local homeomorphisms satisfying (iii) in Lemma 8.2 are called essentially free in [53]). This proves the desired equivalence. □

8.1. Simplicity

The following definition and lemma are compatible with [22, Definition 3.1 and Proposition 3.2] where partial local homeomorphisms are considered.

Definition 8.8. Let $U \subseteq X$. We say U is *positively invariant* if $\varphi(U \cap \Delta_{\text{pos}}) \subseteq U$, U is *negatively invariant* if $\varphi^{-1}(U) \cap \Delta_{\text{reg}} \subseteq U$, and U is *invariant* if it is both positively and negatively invariant. We say that L is *minimal* if there are no non-trivial open invariant sets.

Lemma 8.9. *Let U be an open subset of X and put $J := C_0(U)$ and recall that $I = C_0(\Delta)$.*

(i) *U is positively invariant iff $L(J \cap I) \subseteq J$;*
(ii) *U is negatively invariant iff $L^{-1}(J) \cap C_c(\Delta_{\text{reg}}) \subseteq J$ iff $L^{-1}(J) \cap C_0(\Delta_{\text{reg}}) \subseteq J$;*

Proof. (i). If U is positively invariant, then for any $y \notin U$ we have $\varphi^{-1}(y) \cap \Delta_{\text{pos}} = \emptyset$, which implies $L(a)(y) = 0$ for any $a \in J \cap I$. Thus $L(J \cap I) \subseteq J$. If U is not positively invariant, then there is $x \in U \setminus \varrho^{-1}(0)$ such that $\varphi(x) \notin U$. Taking any positive $a \in I \cap J$ with $a(x) \neq 0$, we get $L(a)(\varphi(x)) = \sum_{t \in \varphi^{-1}(\varphi(x))} \varrho(t) a(t) > 0$ which shows that $L(J \cap I) \not\subseteq J$.

(ii). This follows from the equalities $C_c(\varphi^{-1}(U) \cap \Delta_{\text{reg}}) = L^{-1}(J) \cap C_c(\Delta_{\text{reg}})$ and $C_0(\varphi^{-1}(U) \cap \Delta_{\text{reg}}) = L^{-1}(J) \cap C_0(\Delta_{\text{reg}})$, which are readily verified. □

Example 8.10. (Directed graphs with one circuit) Suppose that $\varphi : X \to X$ is a *directed graph with one circuit*, i.e. X is a countable discrete set and there is a point $x \in X$ such that for any $y \in X$ we have $\varphi^n(y) = x$ for some $n \geq 1$, cf. [12]. In particular, x is a periodic point and φ is a local homeomorphism. Take any $\varrho : X \to (0, +\infty)$ with $\sup_{y\in X} \sum_{x\in\varphi^{-1}(y)} \varrho(x) < \infty$, so that it defines a transfer operator L for φ. Then $X = \Delta_{\mathrm{pos}} = \Delta_{\mathrm{reg}}$ and the transfer operator L is minimal but φ is not topologically free. Hence, $A \rtimes L$ is not simple by Theorem 8.5.

Theorem 8.11. *The following conditions are equivalent:*

(i) *the crossed product $A \rtimes L$ is simple;*
(ii) *L is minimal and $\varphi : \Delta_{\mathrm{reg}} \to X$ is topologically free;*
(iii) *L is minimal and $\varphi : \Delta_{\mathrm{reg}} \to X$ is not a directed graph with one circuit.*

Proof. We first show that (i)⇔(ii). If $\varphi : \Delta_{\mathrm{reg}} \to X$ is not topologically free, then $A \rtimes L$ is not simple by Theorem 8.5. So let us assume that $\varphi : \Delta_{\mathrm{reg}} \to X$ is topologically free. Then for any non-zero ideal N in $A \rtimes L$ the ideal $J := A \cap N$ in A is non-zero. Hence $J = C_0(U)$ for some non-empty open set U. Note that $L(J \cap I) = t^* INIt \subseteq K$, so $L(J \cap I) \subseteq J$. Also for any $a \in L^{-1}(J) \cap C_c(\Delta_{\mathrm{reg}})$, using (15), we obtain

$$\begin{aligned} a &= \sum_{i,j=1}^{n} u_i^K tt^* u_i^K a u_j^K tt^* u_j^K = \sum_{i,j=1}^{n} u_i^K tL\left(u_i^K a u_j^K\right) t^* u_j^K \\ &= \sum_{i,j=1}^{n} u_i^K tL\left(\alpha\left(u_i^K \circ \varphi|_{U_i}^{-1}\right) a \alpha\left(u_j^K \circ \varphi|_{U_j}^{-1}\right)\right) t^* u_j^K \\ &= \sum_{i,j=1}^{n} u_i^K t\left(u_i^K \circ \varphi|_{U_i}^{-1}\right) L(a) \left(u_j^K \circ \varphi|_{U_j}^{-1}\right) t^* u_j^K \in \overline{ItJt^*I} \subseteq N, \end{aligned}$$

so $a \in J$. Hence U is an invariant set by Lemma 8.9. If $U \neq X$ then $N \neq A \rtimes L$ and $A \rtimes L$ is not simple. Conversely, if $N \neq A \rtimes L$ then $U \neq X$ because otherwise N would contain A and $A \rtimes L = A(A \rtimes L)$ would be N.

Implication (ii)⇒(iii) is clear and to prove the converse assume that L is minimal but $\varphi : \Delta_{\mathrm{reg}} \to X$ is not topologically free. Then there is a non-empty open set $U \subseteq \Delta_{\mathrm{reg},n}$ for some $n \geq 1$ such that $\varphi^n|_U = id$ and $U, \varphi(U), \ldots, \varphi^{n-1}(U)$ are pairwise disjoint. For any disjoint open sets V_1, $V_2 \subseteq U$ the sets $U_i := \bigcup_{m\in\mathbb{N}, k=0,\ldots,n} \varphi^{-m}(\varphi^k(V_i))$ are disjoint open and invariant. Thus minimality of L forces $U = \{x\}$ to be a singleton and $\varphi : \Delta_{\mathrm{reg}} \to X$ to be a directed graph with one circuit. □

Remark 8.12. If $\Delta = \Delta_{\mathrm{reg}}$, then $E := (X, \Delta, \mathrm{id}, \varphi)$ is a topological graph in the sense of Katsura [33] and Theorem 8.11 could be deduced from [34, Theorem 8.12]. In this regular case, one could also get simplicity criteria for $A \rtimes L$ using the Renault–Deaconu groupoid model, see [42] and references therein.

9. Locally Contractive Transfer Operators and Pure Infiniteness

The following definition is inspired by [35, Definition 2.7].

Definition 9.1. We say that an open set $V \subseteq X$ is *contracting* if there are pairwise disjoint, non-empty open sets $U_k \subseteq \Delta_{\mathrm{reg},n_k} \cap V$ for $k = 1, \dots, m$, $n_k \geq 1$, such that

$$V \not\subseteq \overline{\bigcup_{k=1}^{m} U_k} \quad \text{and} \quad \overline{V} \subseteq \bigcup_{k=1}^{m} \varphi^{n_k}(U_k).$$

We say that L is *contracting* if $\Delta = \Delta_{\mathrm{pos}}$ and there is $x_0 \in \Delta$ such that every neighbourhood of x_0 contains a contracting open set and $\overline{\bigcup_{n=0}^{\infty} \varphi|_{\Delta_{\mathrm{reg}}}^{-n}(x_0)} = X$.

If $\Delta = \Delta_{\mathrm{reg}}$, then L is contractive iff the topological graph $E := (X, \Delta, \mathrm{id}, \varphi)$ is contractive in the sense of [35], and we could use [35, Theorem A] to show that $A \rtimes L$ is purely infinite simple whenever L is minimal and contractive. A formally weaker result could be obtained using a groupoid model and [1, Proposition 2.4], as L is contractive if the Renault–Deaconu groupoid $\mathcal{G}$ is minimal and locally contractive in the sense of [1], but the converse is not clear. We will now prove a general result allowing irregular points, that is admitting the case when $\Delta_{\mathrm{pos}} \neq \Delta_{\mathrm{reg}}$.

Lemma 9.2. *If there is a contracting precompact open set V, then there are $b, c \in A \rtimes L \setminus \{0\}$ satisfying $b^*bb = b$, $b^*bc = c$, $b^*c = 0$ and $ab = b$ for all $a \in C_0(X)$ which is 1 on V.*

Proof. Since $U_k \subseteq \Delta_{\mathrm{reg},n_k}$ and $\overline{V} \subseteq \bigcup_{k=1}^{m} \varphi^{n_k}(U_k)$ we may find $a_k \in C_c(U_k)^+$, for $k = 1, \dots, m$, such that $g = \sum_{k=1}^{m} L^{n_k}(a_k) \in C_0(X)$ is 1 on V. Indeed, each $U_k = \bigcup_{i=1}^{m_k} U_k^i$ is a union of open sets where $\varphi^{n_k}|_{U_k^i}$ is a homeomorphism onto its range. Taking a partition of unity $\{h_k^i\}_{k=1,i=1}^{m,m_k} \subseteq C_0(X)$ on $\overline{V}$ subordinated to $\{\varphi^{n_k}(U_k^i)\}_{k=1,i=1}^{m,m_k}$ we may put $a_k := \sum_{i=1}^{m_k} \varrho_{n_k}^{-1} \cdot h_k^i \circ (\varphi^{n_k}|_{U_k^i})^{-1}$. Now define $b = \sum_{k=1}^{m} \sqrt{a_k} t^{n_k}$. Then for $a \in C_0(X)$ which is 1 on V we have $ab = b$. Since the sets $\{U_k\}_{k=1}^{m}$ are pairwise disjoint we get $b^*b = \sum_{k,l=1}^{m} t^{*n_k} \sqrt{a_k}\sqrt{a_l} t^{n_l} = \sum_{k=1}^{m} t^{*n_k} a_k t^{n_k} = g$. Since g is 1 on $\overline{V}$, we have $b^*bb = b$. Take any non-zero $c \in C_c(V \setminus \overline{\bigcup_{k=1}^{m} U_k})$, then clearly, $b^*bc = c$ and $b^*c = 0$. □

Proposition 9.3. *If L is minimal and there is a contracting open set, then $A \rtimes L$ is simple and contains an infinite projection.*

Proof. If there is a contracting set, then $\varphi : \Delta_{\mathrm{reg}} \to X$ can not be a directed graph with one circuit. Hence $A \rtimes L$ is simple by Theorem 8.11. By Lemma 9.2 there is $b \in A \rtimes L$ with $b^*bb = b$, $b^*b \neq bb^*$. Such elements are called *scaling*, and a simple C^*-algebra has an infinite projection if and only if it has a scaling element, see [30, Proposition 4.2]. □

Lemma 9.4. *Assume that $\varphi : \Delta_{\mathrm{reg}} \to X$ is topologically free, $\Delta = \Delta_{\mathrm{pos}}$ and there is $x_0 \in \Delta$ such that $\overline{\bigcup_{n=0}^{\infty} \varphi^{-n}(x_0) \cap \Delta_{\mathrm{reg},n}} = X$. For any non-zero positive $b \in A \rtimes L$ there is $d \in A \rtimes L$ and $a \in A$ which is 1 on some neighbourhood of x_0 and $\|d^*bd - a\| < 1/2$.*

Proof. Put $\varepsilon := \|E(b)\|/5$ where E is the conditional expectation onto the core A_∞. Choose a positive $b_0 \in \operatorname{span}\{at^n t^{*m} c : a \in I_n, c \in I_m, n, m \in \mathbb{N}_0\}$ such that $\|b - b_0\| < \varepsilon$. Since $\Delta = \Delta_{\mathrm{pos}}$, in view of Lemma 8.3, we see that $\varphi : \Delta \to X$ is topologically free, and this implies that the topological quiver $\mathcal{Q} = (X, \Delta, id, \varphi, \mu)$ satisfies condition (L). By Corollary 4.13 the C^*-algebra associated to $\mathcal{Q}$ is isomorphic to $A \rtimes L$. Hence we may apply [46, Proposition 6.14] to conclude that there is $d_0 \in A \rtimes L$ and $a_0 \in C_0(X)^+$ such that $\|d_0\| \le 1$, $\|a_0\| = \|E(b_0)\|$ and $\|d_0^* b_0 d_0 - a_0\| < \varepsilon$. Since $\|a_0\| = \|E(b_0)\| > \|E(b)\| - \varepsilon = 4\varepsilon$, the open set

$$U := \{x \in X : a_0(x) > 4\varepsilon\}$$

is non-empty. As $\bigcup_{n=0}^{\infty} \varphi^{-n}(x_0) \cap \Delta_{\mathrm{reg},n}$ is dense in X there is $n \in \mathbb{N}$ and an open subset $U_0 \subseteq U \cap \Delta_{\mathrm{reg},n}$ such that $\varphi^n|_{U_0}$ is a local homeomorphism onto an open neighbourhood V_0 of x_0. Take $c_0 \in C_c(V_0)^+$, $\|c_0\| \le 1$, such that c_0 is 1 on an open neighbourhood $V \subseteq V_0$ of x_0. Then $c := (a_0 \varrho_n)^{-1} \cdot c_0 \circ (\varphi^n|_{U_0})^{-1} \in C_c(U_0)$ is such that $a := L^n(c a_0 c)$ is 1 on V and $\|ct^n\|^2 = \|L^n(c^2)\| \le \max_{x \in U_0} a_0(x)^{-1} < (4\varepsilon)^{-1}$. Thus putting $d := d_0 c t^n$ we get

$$\begin{aligned}\|d^* b d - a\| &= \|t^{*n} c d_0^* b d_0 c t^n - t^{*n} c a_0 c t^n\| \le \|d_0^* b d_0 - a_0\| \cdot \|ct^n\|^2 \\ &\le (\|d_0^*(b - b_0) d_0\| + \|d_0^* b_0 d_0 - a_0\|) \cdot \|ct^n\|^2 < (\varepsilon + \varepsilon)(4\varepsilon)^{-1} < 1/2.\end{aligned}$$

□

Theorem 9.5. *If L is minimal and contractive, then $A \rtimes L$ is purely infinite and simple.*

Proof. By Proposition 9.3, $A \rtimes L$ is simple and contains an infinite projection p. Hence it suffices to show that for each non-zero positive $b_0 \in A \rtimes L$, the hereditary C^*-subalgebra $\overline{b_0(A \rtimes L)b_0}$ generated by b_0 contains a projection equivalent to the infinite projection p. To this end, note that by Theorem 8.11, $\varphi : \Delta_{\mathrm{reg}} \to X$ is topologically free. Let $x_0 \in \Delta$ be such that $\overline{\bigcup_{n=0}^{\infty} \varphi|_{\Delta_{\mathrm{reg}}}^{-n}(x_0)} = X$ and every neighbourhood of x_0 contains a non-empty contracting open set. By Lemma 9.4 there is $d \in A \rtimes L$ and $a \in A$ which is 1 on a neighbourhood V of x_0 and $\|d^* b_0 d - a\| < 1/2$. We may take V to be a precompact open contracting set. By Lemma 9.2 there are non-zero $b, c \in A \rtimes L$ such that

$$b^* b b = b, \quad b^* b c = c, \quad b^* c = 0 \quad \text{and} \quad ab = b. \tag{25}$$

Since $A \rtimes L$ is simple and $c^* c \neq 0$ there are $b_1, \dots, b_l \in A \rtimes L$ such that $p = \sum_{k=1}^{l} b_k^* c^* c b_k$, see [30, Lemma 4.1]. Set $e := \sum_{k=1}^{l} b^k c b_k$. Then using (25) we get

$$e^* a e = e^* e = \sum_{k,i=1}^{l} b_k c^* b^{k*} b^i c b_i = \sum_{k=1}^{l} b_k c^* c b_k = p.$$

In particular, $\|e\| = 1$. Using all these we get

$$\|e^* d^* b_0 d e - p\| = \|e^*(d^* b_0 d e - a) e\| < 1/2.$$

Let f be the characteristic function of the interval $(\frac{1}{2}, \frac{3}{2})$. Then $p_0 := f(e^* d^* b_0 d e)$ is a well defined projection with $\|p_0 - e^* d^* b_0 d e\| \le 1/2$, cf. [56, Lemma 2.2.4].

Hence $\|p_0 - p\| < 1$ and therefore p_0 and p are equivalent, cf. [56, Proposition 2.2.5]. Then $q := f(\sqrt{b_0}dee^*d^*\sqrt{b_0})$ is a projection in $\overline{b_0(A \rtimes L)b_0}$ which is equivalent to p_0 and hence to the infinite projection p. ☐

Corollary 9.6. *If X is second countable and L is minimal and contractive, then $A \rtimes L$ is a Kirchberg algebra, i.e. a simple, separable, nuclear, purely infinite C^*-algebra satisfying the UCT.*

Proof. Combine Theorems 9.5 and 4.8. ☐

Example 9.7. (C^*-algebras of rational maps) Let $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree at least two, and let $X = \Delta = J_R$ be the Julia set for R. The transfer operator $L : C(J_R) \to C(J_R)$ for $\varphi : J_R \to J_R$, considered in Example 4.15, is minimal and contractive. Indeed, $\Delta_{\text{pos}} = J_R$ is uncountable and $\Delta_{\text{pos}} \setminus \Delta_{\text{reg}}$ is finite by [4, Corollary 2.7.2, Theorem 4.2.4]. Moreover, for any open $V \subseteq J_R$ there is $n \in \mathbb{N}$ such that $R^n(V) = J_R$, by [4, Theorem 4.2.5], and $\overline{\bigcup_{n=0}^{\infty} R^{-n}(z)} = J_R$ for every $z \in J_R$, by [4, Theorem 4.2.7]. So one may find $z_0 \in J_R$ whose inverse orbit $\bigcup_{n=0}^{\infty} R^{-n}(z)$ does not contain any critical point, and any open neighbourhood of z_0 contains a contractive open set. Hence $C(J_R) \rtimes L$ is simple and purely infinite. This recovers [27, Theorem 3.8] as a special case of Theorem 9.5. Note that the Fatou set $F_R = \widehat{\mathbb{C}} \setminus J_R$ is open and invariant for $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Thus $C(\widehat{\mathbb{C}}) \rtimes L$ is simple if and only if $\widehat{\mathbb{C}} = J_R$.

As a by product we also recover the main result of [24]. Namely, let μ^L be the Lyubich measure. It is a φ-invariant regular probability measure whose support is J_R. Denoting by T_φ the composition operator on $L_2(\mu^L)$ and identifying $C(J_R)$ with operators of multiplication on $L_2(\mu^L)$ we get $L(a) = d \cdot T_\varphi^* a T_\varphi$ where d is the degree of R, see [45, Lemma, p. 366]. Thus using Proposition 3.12 one gets that $(id, d^{1/2}T_\varphi)$ is covariant representation of $L : C(J_R) \to C(J_R)$ and therefore the simple C^*-algebra $C(J_R) \rtimes L$ is isomorphic to the C^*-subalgebra of $B(L_2(\mu^L))$ generated by $C(J_R)$ and the composition operator T_φ.

Example 9.8. (Branched expansive coverings) Consider the transfer operator from Example 4.16, where $\varphi : X \to X$ is a continuous map on a compact metric space X whose inverse has a finite number of continuous branches $\gamma = \{\gamma_i\}_{i=1}^N$ that are proper contractions and X is self-similar for γ. In other words, X is covered by compact sets Δ_i, $i = 1, \ldots, N$, such that $\varphi : \Delta_i \to X$ is an expansive homeomorphism ($\gamma_i = \varphi|_{\Delta_i}^{-1}$). As in [28] we assume *the open set condition* for γ, which in terms of φ says that there is a non-empty open set $V \subseteq X$, such that $\varphi^{-1}(V) \subseteq V$ and $\varphi^{-1}(V) \cap \Delta_i \cap \Delta_j = \emptyset$ for $i \neq j$. Then $\varphi^{-1}(V)$ is necessarily an open dense set in X not intersecting the set of branching points $B = \bigcup_{i \neq j} \{x \in \Delta_i \cap \Delta_j\}$ and we have $\Delta_{\text{reg}} = X \setminus B$, cf. [28, Proposition 2.6]. Using this one infers that each of the sets $\varphi^n(B)$ has empty interior. Thus $X \setminus \bigcup_{n=0}^{\infty} \varphi^n(B)$ is dense in X by Baire theorem. For any $x \in X \setminus \bigcup_{n=0}^{\infty} \varphi^n(B)$ its negative orbit $\bigcup_{n=0}^{\infty} \varphi^{-n}(x)$ lies entirely in Δ_{reg}. Every negative orbit is dense in X. Indeed, $A := \overline{\bigcup_{n=0}^{\infty} \varphi^{-n}(x)}$ is a closed set with $\varphi^{-1}(A) \subseteq A$, which implies $A = X$ by the uniqueness of the self-similar set X, see [26]. Using expansiveness of φ we conclude that every neighbourhood of $x_0 \in X \setminus \bigcup_{n=0}^{\infty} \varphi^n(B)$ contains a non-empty contracting open set. Minimality is clear. Hence $C(X) \rtimes L$ is a unital Kirchberg algebra by Corollary 9.6. This recovers [28, Theorem 3.8] when the systems of contractive maps form inverse branches of a continuous map.

Recall that the *Hutchinson measure* μ^H is the unique regular probability measure such that $\mu^H(A) = 1/N \sum_{i=1}^{N} \mu^H(\gamma_i(A))$ for all Borel $A \subseteq X$. Its support is X and so we may identify $C(X)$ with operators of multiplication on $L_2(\mu^H)$. If $\mu^H(B) = 0$ (which is automatic when $X \subseteq \mathbb{R}^d$ and γ_i's are similitudes, see [58]), then the composition operator T_φ is an isometry on $L_2(\mu^L)$ satisfying $L(a) = 1/N \cdot T_\varphi^* a T_\varphi$, see [25]. Thus using Proposition 3.12 one sees that $(id, N^{-1/2} T_\varphi)$ is a covariant representation of $L : C(J_R) \to C(J_R)$ and therefore $C(X) \rtimes L$ is isomorphic to the C^*-subalgebra of $B(L_2(\mu^H))$ generated by $C(X)$ and the composition operator T_φ. This recovers the main result of [25].

Example 9.9. (Expanding local homeomorphisms) Assume $\varphi : X \to X$ is an open continuous expanding map on a compact metric space, cf. [1,3] and references therein. Then any continuous $\varrho : X \to (0, \infty)$ defines a transfer operator L for φ and $C(X) \rtimes L \cong C^*(\mathcal{G})$, by Theorem 6.4. By [3, Lemma 7.4], φ is topologically free if and only if X has no isolated periodic points. Clearly, L is minimal if and only if φ is *minimal*, i.e. there is no non-trivial open set U with $\varphi^{-1}(U) = U$. Thus assuming X is infinite, by Theorem 8.11, we get that

$$C(X) \rtimes L \text{ is simple if and only if } \varphi \text{ is minimal.}$$

Assume now that there are no wandering points in X, or equivalently that periodic points are dense in X. Then by spectral decomposition, cf. [3, Theorem 2.5], φ is minimal iff φ is topologically transitive iff for every non-empty open $U \subseteq X$ there is $N \in \mathbb{N}$ such that $\bigcup_{k=1}^{N} \varphi^k(U) = X$. Thus if φ is minimal, then every negative orbit $\bigcup_{n=0}^{\infty} \varphi^{-n}(x)$ is dense in X and every non-trivial open subset $V \subsetneq X$ is contracting, so in particular L is contracting. Hence by Corollary 9.6 we get

$$C(X) \rtimes L \text{ is a Kirchberg algebra, if } \varphi \text{ is minimal and has no wandering points.}$$

This last statement improves [1, Proposition 4.2] (in the minimal case) and implies [19, Proposition 4.2]. If there are no wandering points, then by [3, Proposition 3.8] there exists a φ-invariant Borel probability measure μ with support X such that identifying $C(X)$ with operators of multiplication on $L_2(\mu)$ we have $L(1)^{-1} L(a) = T_\varphi^* a T_\varphi$, for $a \in C(X)$, where $T_\varphi \in B(L_2(\mu))$ is the composition operator with φ. Also $(id, L(1)^{-1/2} T_\varphi)$ is a covariant representation of L. Thus is if X has no isolated periodic points, then $C(X) \rtimes L$ is isomorphic to the C^*-subalgebra of $B(L_2(\mu))$ generated by $C(X)$ and the composition operator T_φ. If in addition φ is minimal (topologically transitive) and $\ln \varrho$ is Hölder continuous, then the above measure μ is unique and it is the *Gibbs measure* for φ and $\ln \varrho$.

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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Received: July 18, 2023.
Revised: July 25, 2024.
Accepted: August 1, 2024.